Abstract — Let $G = (V, E)$ be a simple graph. A set $D \subseteq V$ is a dominating set of $G$, if every vertex in $V - D$ is adjacent to at least one vertex in $D$. Let $K_n$ be complete graph with order $n$. Let $K_n^i$ be the family of dominating sets of a complete $K_n$ with cardinality $i$, and let $d(K_n^i, i) = |K_n^i|$. In this paper, we construct $K_n$ and obtain a recursive formula for $d(K_n^i, i)$. Using this recursive formula, we consider the polynomial $D(K_n, x) = \sum_{i=1}^{n} d(K_n^i, i)x^i$, which we call domination polynomial of complete graphs and obtain some properties of this polynomial.

I. INTRODUCTION

Let $G = (V, E)$ be a simple graph of order $|V| = n$. A set $D \subseteq V$ is a dominating set of $G$, if every vertex in $V - D$ is adjacent to at least one vertex in $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. For a detailed treatment of this parameter, the reader is referred to [6]. It is well known and generally accepted that the problem of determining the dominating sets of an arbitrary graph is a difficult one (see [5]). Alikhani and Peng found the dominating set and domination polynomial of cycles and certain graph [1],[2]. Gehet, Khalaf and Hasni found the dominating set and domination polynomial of stars and wheels [3][4]. Let $G_n$ be graph with order $n$ and let $G_n^i$ be the family of dominating sets of a graph $G_n$ with cardinality $i$ and let $d(G_n^i, i) = |G_n^i|$. We call the polynomial $D(G_n, x) = \sum_{i=\gamma(G)}^{n} d(G_n^i, i)x^i$, the domination polynomial of graph $G$ [2]. Let $K_n^i$ be the family of dominating sets of a complete graph $K_n$ with cardinality $i$ and let $d(K_n^i, i) = |K_n^i|$. We call the polynomial $D(K_n, x) = \sum_{i=1}^{n} d(K_n^i, i)x^i$, the domination polynomial of complete graph. In the next section we construct the families of dominating sets of $K_n$ with cardinality $i$ by the families of dominating sets of $K_{n-1}$ with cardinality $i$ and $i-1$. We investigate the domination polynomial of complete graphs in section 3. And we study dominating sets and domination polynomial of some cases of complete graphs with missing edges in section 4.

As usual we use $\binom{n}{i}$ for the combination $n$ to $i$, and we denote the set $\{1, 2, \ldots, n\}$ simply by $[n]$, and we denote $\rho(v)$ to degree of the vertex $v$, and let $\Delta(G) = \max\{\rho(v) | \forall v \in V(G)\}$ and $\delta(G) = \min\{\rho(v) | \forall v \in V(G)\}$.

II. DOMINATING SETS OF COMPLETE GRAPHS

Let $K_n, n \geq 3$, be the complete graph with $n$ vertices $V(K_n) = [n]$ and $E(K_n) = \{(v, u): \forall v, u \in V(K_n)\}$. Let $K_n^i$ be the family of dominating sets of $K_n$ with cardinality $i$. We shall investigate dominating sets of complete graph. To prove our main results we need the following lemmas:

Lemma 1 [3] The following properties hold 8 graph $G$. 

$(i) |G_n^1| = 1$  

$(ii) |G_n^{n-1}| = n$  

$(iii) |G_n^i| = 0$ if $i > n$ 

$(iv) |G_n^i| = 0$
Lemma 2 [3] The following properties are hold by definition of combination \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \) for all \( n \in \mathbb{Z}^+ \).

(i) \( \binom{n}{0} = 1 \)  
(ii) \( \binom{n}{n-1} = n \)  
(iii) \( \binom{n}{n} = n \)  
(iv) \( \binom{n}{i} = 0 \) if \( i > n \)

Theorem 1 Let \( K_n \) be complete graph with order \( n \), then \( d(K_n, i) = \binom{n}{i} \) \( \forall \ n \in \mathbb{Z}^+, \text{ and } i = 1, 2, ..., n \)

Proof.
Let \( K_n \) be a complete graph, since every vertex \( v \in K_n \) it is adjacent with every other vertex \( u \in K_n \) then every subset of \( K_n \) with cardinality \( i \) \( \forall 1 \leq i \leq n \) is dominating sets of \( K_n \), therefore \( d(K_n, i) = \binom{n}{i} \).

Theorem 2 Let \( K_n \) be complete graph with order \( n \), then \( d(K_n, i) = d(K_{n-1}, i) + d(K_{n-1}, i-1) \) \( \forall i > 1, n > 1 \).

Proof.
(i) We have \( \binom{n}{i} = \frac{n(n-1)(n-2)...(n-i+1)(n-i)!}{i!(n-i)!} = \frac{(n-1)n-i)!}{(n-i)(n-i-1)!} \).

Let \( m = \frac{(n-1)(n-2)...(n-i+1)!}{i(n-1)(n-2)...(n-i+1)!} \), then \( d(K_n, i) = m \binom{n}{i} + d(K_{n-1}, i) + d(K_{n-1}, i-1) = mn - mi + mi = mn = d(K_n, i) \).

Using Theorem 1 and Theorem 2, we obtain the coefficients of \( D(K_n, x) \) for \( 1 \leq n \leq 15 \) in Table 1.

Let \( d(K_n, i) = |k_n^i| \). There are interesting relationships between the numbers \( d(K_n, i) \) (\( 1 \leq i \leq n \)) in the table.

### Table 1

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In the following theorem, we obtain some properties of \( d(K_n, i) \).

Theorem 3 The following properties hold for coefficients of \( D(K_n, x) \), \( \forall n \in \mathbb{Z}^+ \):

(i) \( d(K_n, 1) = n \).
(ii) \( d(K_n, i) = d(K_n, n-i) \).
(iii) If \( n \) is even number, then \( d(K_n, \frac{n}{2}) = d(K_n, \frac{n}{2}) \).
(iv) \( \gamma(K_n) = 1 \)

Proof.
Let \( K_n \) be a complete graph, then

(i) By Theorem 1 \( d(K_n, 1) = \binom{n}{1} = n \), then \( d(K_n, 1) = n \).

(ii) We have \( \binom{n}{i} = \binom{n}{n-i} \), then \( d(K_n, i) = d(K_n, n-i) \) (by Lemma 2 and Theorem 1)

(iii) It is hold from (ii)

(iv) since \( \{v\} \forall v \in V(K_n) \) is dominating set of \( K_n \), then \( \gamma(K_n) = 1 \).

### III. DOMINATION POLYNOMIAL OF A COMPLETE GRAPHS

In this section we introduce and investigate the domination polynomial of complete graphs.

**Definition**
Let \( k_n^i \) be the family of dominating sets of a complete \( K_n \) with cardinality \( i \), and let \( d(K_n, i) = |k_n^i| \), and since \( \gamma(K_n) = 1 \). Then the domination polynomial \( D(K_n, x) \) of \( K_n \) is defined as

\[
D(K_n, x) = \sum_{i=1}^{n} d(K_n, i)x^i
\]

Theorem 4 The following properties hold for all \( D(K_n, x) \) \( \forall n \geq 3 \):

(i) \( D(K_n, x) = D(K_{n-1}, x) + xD(K_{n-1}, x) + x \)

(ii) \( D(K_n, x) = \sum_{i=1}^{n} \binom{n}{i}x^i \)

Proof.

(i) from definition of the domination polynomial and Theorem 2, we have

\[
D(K_n, x) = \sum_{i=1}^{n} d(K_n, i)x^i = \sum_{i=1}^{n} d(K_{n-1}, i)x^i
\]
we have \( d(K_n,i) = 0 \) if \( i > n \) or \( i = 0 \) (Lemma1), then \( \sum_{i=1}^{n} d(K_{n-1}, i) x^i = \sum_{i=1}^{n-1} d(K_{n-1}, i) x^i = d(K_{n-1}, x) \) and (since \( d(K_{n-1}, i - 1) x^{i-1} = \binom{n-1}{i-1} = \binom{n-1}{0} = 1 \) if \( i = 1 \) and \( \sum_{i=2}^{n} d(K_{n-1}, i - 1) x^{i-1} = \sum_{i=1}^{n-1} d(K_{n-1}, i) x^i \)),

then \( \sum_{i=1}^{n} d(K_{n-1}, i - 1) x^i = x \sum_{i=1}^{n} d(K_{n-1}, i - 1) x^{i-1} = x \sum_{i=0}^{n-1} d(K_{n-1}, x) + x \)

therefore \( D(K_n, x) = D(K_{n-1}, x) + x D(K_{n-1}, x) + x \)

\((ii)\) \( D(K_n, x) = \sum_{i=1}^{n} d(K_n, i) x^i = \sum_{i=1}^{n} \binom{n}{i} x^i \) (by Theorem 1).  

**Example 1**

Let \( K_6 \) be complete graph with order 6,  
then \( \gamma(K_6) = 1 \) and \( D(K_6, x) = \sum_{i=1}^{5} \binom{6}{i} x^i = 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6 \) (see Fig-I).

**IV. DOMINATING SETS AND DOMINATION POLYNOMIAL OF COMPLETE GRAPH WITH MISSING EDGES**

Let \( K_n \) be complete graph with order \( n \), and let \( \varepsilon \) be number of missing edges from \( K_n \), we denoted by \( K_n(-\varepsilon) \) to complete graph with number of missing edges.

We mine trail if \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_k \) missing edges, then edge \( \varepsilon_2, \varepsilon_2 \) adjacent \( \varepsilon_3, \ldots, \) and \( \varepsilon_k - 1 \) adjacent \( \varepsilon_k \)

To prove our main results in following theorems we need the following proposition

**Proposition 1**

Let \( G_n = K_n(-\varepsilon) \) the following properties hold for all graph \( G_n \). The following properties are clear see figure (2,3,4,5).

\( \gamma \( G \) = n - 2 \) if all missing edges are independent

\( \gamma \( G \) = n - 3 \) if all missing edges are trail

\( \gamma \( G \) = n - 1 - \varepsilon \) if all missing edges are adjacent.

\( n - 1 - \delta(G) \leq \varepsilon \)

\( \varepsilon \) Number of vertices with degree \( < n - 1 \) is \( \varepsilon + 1 \) if all missing edges are adjacent

\( \varepsilon \) Number of vertices with degree \( < n - 1 \) is \( \varepsilon + 1 \) if all missing edges are trail

\( \varepsilon \) Number of vertices with degree \( < n - 1 \) is \( 2\varepsilon \) if all missing edges are independent

\( \varepsilon \) Number of vertices with degree \( n - 2 \) is \( 2\varepsilon \) if all missing edges are independent

\( \varepsilon \) Number of vertices with degree \( n - 2 \) is \( \varepsilon \) and number of vertices with degree \( n - 2 \) is \( 2 \) if all missing edges are trail

\( \varepsilon \) Number of vertices with degree \( n - 2 \) is \( \varepsilon \) and number of vertices with degree \( n - 1 - \varepsilon \) is \( 1 \) if all missing edges are adjacent.

**Theorem 5** Let \( G_n = K_n(-\varepsilon) \), then the following properties hold for \( G \):

\( \gamma(G, i) = \binom{n}{i} \) \( \forall \ i > n - 1 - \delta(G) \)

\( \gamma(G, i) = \binom{n}{i} - \binom{n}{i-1} \) \( \forall \ i \leq \varepsilon \) if all missing edges are adjacent

\( \gamma(G, i) = \binom{n}{i} \) \( \forall \ i > \varepsilon \)

\( \gamma(G, i) = \binom{n}{i} \) \( \forall \ i > 1 \) if all missing edges are independent

\( \gamma(G, 1) = n - 2\varepsilon \) if all missing edges are independent

\( \gamma(G, 1) = n - (\varepsilon + 1) \) if all missing edges are adjacent

\( \gamma(G, 2) = \binom{n}{2} - (\varepsilon - 1) \) if all missing edges are trail

\( \gamma(G, 1) = \binom{n}{i} \) \( \forall \ i > 2 \) if all missing edges are trail

\( \gamma(G, 1) = n - \varepsilon \) if all missing edges are trail such that
formation $\mathcal{C}_g$

$|\{\text{vertices}\}| \quad d(G_n,2) = \binom{n}{2} - \varepsilon$ if all missing edges are trail such that formation $\mathcal{C}_g$

**Proof.**

By the properties in Proposition 1, we prove the following proposition:

(i) Let $v$ be vertex with degree $\delta(G)$, then it has $n - 1 - \delta(G)$ of missing edges, then we have $n - 1 - \delta(G)$ vertices are nonadjacent with $v$, hence every subset of $G_n$ with cardinality $i \geq n - 1 - \delta(G)$ is dominating set of $G_n$, then $d(G_n,i) = \binom{n}{i} - \binom{i}{i}$ (see Fig-2.3.4.5)

(ii) since all missing edges are adjacent, then we have $\varepsilon$ vertices are nonadjacent with one vertex, then we have only $\binom{n}{i} \forall \ i \leq \varepsilon$ are not dominating set of $G_n$, therefore $d(G_n,i) = \binom{n}{i} - \binom{i}{i}$ (see Fig-2.3.4.5)

(iii) We have $\varepsilon \geq n - 1 - \delta(G)$ and since $i \geq \varepsilon$ then $i \geq n - 1 - \delta(G)$ and $d(G_n,i) = \binom{n}{i}$ $\forall \ i \geq \varepsilon$ by (i) (see Fig-2.3.4.5)

(iv) since all missing edges are independent, then

$\delta(G) = n - 2$ then $d(G_n,i) = \binom{n}{i}$ $\forall \ i > 1$ by (i) (see Fig-3)

(v) since all missing edges are independent, and we have $\varepsilon$ number of missing edges, then we have $2\varepsilon$ vertices with degree $n - 2$, and since $\{v\} \forall \ v$ vertices with degree $n - 2$ is not dominating set of $G_n$, therefore $d(G_n,1) = n - 2\varepsilon$ (see Fig-3)

(vi) since all missing edges are trail, and we have $\varepsilon$ of missing edges, then we have $\varepsilon + 1$ vertices with degree $< n - 1$, and since $\{v\} \forall \ v$ vertices with degree $< n - 1$ is not dominating set of $G_n$, therefore $d(G_n,1) = n - (\varepsilon + 1)$ (see Fig-4)

(vii) since all missing edges are adjacent, and we have $\varepsilon$ of missing edges, then we have $\varepsilon + 1$ vertices with degree $< n - 1$, and since $\{v\} \forall \ v$ vertices with degree $< n - 1$ is not dominating set of $G_n$, therefore $d(G_n,1) = n - (\varepsilon + 1)$ (see Fig-2)

(viii) $\forall \ v \in V(G)$ if $\rho(v) = n - 3$, then there exist set with cardinality $2$ is not dominating set of $G$, since all missing edges are trail, then we have $(\varepsilon - 1)$ vertices with degree $n - 3$, hence we have $(\varepsilon - 1)$ sets are not dominating set of $G$, therefore $d(G_n,2) = \binom{n}{2} - (\varepsilon - 1)$ (see Fig-4)

(ix) since all missing edges are trail, then $\delta(G) = n - 3$, therefore $d(G_n,i) = \binom{n}{i}$ $\forall \ i > 2$ by (i) (see Fig-4)

(x) Since all cycle $V(C) = E(C)$, then we have $\varepsilon$ vertices with degree $< n - 1$, therefore $d(G_n,1) = n - \varepsilon$ by (i) (see Fig-5)

(xi) $\rho(v) = n - 3$ $\forall \ v \in C$, then $d(G_n,2) = \binom{n}{2} - \varepsilon$ by (vi) (see Fig-5).

**Theorem 6** Let $G_n = K_n (\varepsilon)$, then the following properties hold for $G$:

(i) $D(K_n (\varepsilon), x) = (n - l - \varepsilon)x + \sum_{i=2}^{n} \binom{n}{i} - \binom{i}{i}$ if all missing edges are adjacent

(ii) $D(K_n (\varepsilon), x) = (n - 2\varepsilon)x + \sum_{i=2}^{n} \binom{n}{i}$ if all missing edges are independent

(iii) $D(K_n (\varepsilon), x) = (n - 1 - \varepsilon)x + \sum_{i=2}^{n} \binom{n}{i} - (\varepsilon - 1)$ if all missing edges are trail

(iv) $D(K_n (\varepsilon), x) = (n - 2\varepsilon)x + \sum_{i=2}^{n} \binom{n}{i}$ if all missing edges are trail such that formation $\mathcal{C}_g$

**Proof.**

(i) Since all missing edges are adjacent by Theorem 5 in (ii) and (vi) $D(K_n (\varepsilon), x) = d(K_n (\varepsilon), 1)x + \sum_{i=2}^{n} \binom{n}{i} = (n - 2\varepsilon)x + \sum_{i=2}^{n} \binom{n}{i}$

(ii) Since all missing edges are independent by Theorem 5 in (iv) and (v) $D(K_n (\varepsilon), x) = d(K_n (\varepsilon), 1)x + \sum_{i=2}^{n} \binom{n}{i} = (n - 1 - \varepsilon)x + \sum_{i=2}^{n} \binom{n}{i}$

(iii) Since all missing edges are trail by Theorem 5 in (vii), (viii) and (ix) $D(K_n (\varepsilon), x) = d(K_n (\varepsilon), 1)x + d(K_n (\varepsilon), 2)x^2 + \sum_{i=2}^{n} d(K_n (\varepsilon), i)x^i$
Since all missing edges are trail such that formation $C_8$ by theorem 5 in (ix),(x) and (vi) \( D(K_n (-\varepsilon), x) = d(K_n (-\varepsilon), 1) x + d(K_n (-\varepsilon), 2) x^2 + \sum_{i=3}^{n} d(K_n (-\varepsilon), i) x^i \) \( = (n - \varepsilon) x + \binom{n}{2} x^2 + \sum_{i=3}^{n} \binom{n}{i} x^i. \)

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**Example 2**

Let \( G_{-4} = K_8 \), be complete graph with four missing edges are adjacent, then Domination Polynomial of \( G \) is

\[
D(G_{-4}, x) = (3)x + \sum_{i=2}^{8} \left( \binom{8}{i} - \binom{4}{i} \right)x^i = 3x + 22x^2 + 52x^3 + 69x^4 + 56x^5 + 28x^6 + 8x^7 + x^8
\]

by Theorem 6 (see Fig-2)

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**REFERENCES**


