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Some Convergence Theorems for the Fixed Point in Banach Spaces

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Abstract

Let X be a uniformly smooth Banach space, T:X $\longrightarrow$ X be Φ-strongly quasi accretive (Φ-hemi contractive) mappings. It is shown under suitable conditions that the Ishikawa iteration sequence converges strongly to the unique solution of the equation Tx = f. Our main results is to improve and extend some results about Ishikawa iteration for type from contractive, announced by many others.

Keywords : Convergence Theorems ,Fixed Point , Banach Spaces

1. Introduction

Let X be an arbitrary Banach space with norm $\|\cdot\|$ and the dual space X*. The normalized duality mapping J:X $\longrightarrow 2^X$ is defined by

$$J(x)=\{f \in X^*:< x,f >=\| x\|\cdot\| f \|,\| f \|=\| x \|\}$$

where <·,·> denotes the generalized duality pairing. It is known that if X is uniformly smooth, then J is single valued and is uniformly continuous on any bounded subset of X.

Let T:D(T) ⊆ X $\longrightarrow$ X be an operator, where D(T) and R(T) denote the domain and range of T, respectively, and I denote the identity mapping on X.

We recall the following two iterative processes to Ishikawa and Mann, [1], [2]:

i- Let K be a nonempty convex subset of X, and T:K $\longrightarrow$ K be a mapping, for any given x0 ∈ K the sequence <xn> defined by

xn + 1 = (1 – αn)xn + αnTyn

yn = (1 – βn)xn + βnTxn (n ≥ 0)

is called Ishikawa iteration sequence, where <αn> and <βn> are two real sequences in [0,1] satisfying some conditions.

ii- In particular, if βn = 0 for all n ≥ 0 in (i), then <xn> defined by

x0 ∈ K, xn + 1 = (1 – αn)xn + αnTyn, n ≥ 0

is called the Mann iteration sequence.

Recently Liu [3] introduced the following iteration method which he called Ishikawa (Mann) iteration method with errors.

For a nonempty subset K of X and a mapping T:K $\longrightarrow$ K, the sequence <xn> defined for arbitrary x0 in K by

yn = (1 – βn)xn + βnTxn + vn,

xn + 1 = (1 – αn)xn + αnTyn + un for all n = 0, 1,2,…,

where <un> and <vn> are two summable sequences in X (i.e., $\sum_{n=0}^{\infty}\| u_n\| < \infty$ and $\sum_{n=0}^{\infty}\| v_n\| < \infty$), <αn> and <βn> are two real sequences in [0,1], satisfying suitable conditions, is called the Ishikawa iterates with errors. If βn = 0 and vn = 0 for all n, then the sequence <xn> is called the Mann iterates with errors.

The purpose of this paper is to define the Ishikawa iterates with errors to fixed points and solutions of Φ-strongly quasi accretive and Φ-hemi-contractive operators equations. Our main results improve and extend the corresponding results recently obtained by [3] and [4]. Via replaced the assumption summable sequences by the assumption bounded sequences, T need not be Lipschitz and the assumption that T is strongly accretive mapping is replaced by assumption that T is Φ-strongly quasi accretive and Φ-hemi-contractive, and main results improve the corresponding results recently obtained by [5]. Via replaced the assumption quasi-strongly accretive and quasi-strongly pseudo-contractive mappings by the assumption Φ-strongly quasi accretive and Φ-hemi-contractive operators.

1.1 Definition: [5], [6], [7]

A mapping T with domain D(T) and range R(T) in X is said to be strongly accretive if for any x, y ∈ D(T), there exists a constant k ∈ (0,1) and j(x – y) ∈ J(x – y) such that<Tx – Ty, j(x – y)> ≥ k $\|x - y\|^2$

The mapping T is called Φ-strongly accretive if there exists a strictly increasing function Φ : [0,∞] $\longrightarrow$ [0,∞] with Φ(0) = 0 such that the inequality

<Tx – Ty, j(x – y) ≥ Φ(||x – y||)·||x – y||

holds for all x, y ∈ D(T). It is well known that the class of strongly accretive mappings is a proper subclass of the class of Φ-strongly accretive mapping.

An operator T: X $\longrightarrow$ X is quasi-strongly accretive if there exists a strictly increasing function $\Phi : [0,\infty] \longrightarrow [0,\infty]$ with $\Phi(0) = 0$ such that for any $x, y \in D(T)$

$$\mathrm{Re} \langle Tx - Ty, j(x - y) \rangle \geq \Phi(\|x - y\|)$$

An operator T: X $\longrightarrow$ X is called Φ-strongly quasi-accretive if there exist a strictly increasing function $\Phi:[0,\infty] \longrightarrow [0,\infty]$ with $\Phi(0) = 0$ such that for all $x \in D(T)$, $p \in N(T)$ there exist $j(x - p) \in J(x - p)$ such that

$$\langle Tx - Tp, j(x - p) \rangle \geq \Phi(\|x - p\|)\cdot\|x - p\|$$

where $N(T) = \{ x \in D(T):T(x) = 0\}$.

1.2 Remarks: [5], [6], [7]

1. A mapping T: X $\longrightarrow$ X is called strongly pseudo contractive if and only if (I – T) is strongly accretive.

2. A mapping T: X $\longrightarrow$ X is called Φ-strongly pseudo-contractive if and only if (I – T). Φ-strongly accretive.

3. A mapping T: X $\longrightarrow$ X is called quasi-strongly pseudo-contractive if and only if (I – T) is quasi-strongly accretive.

4. A mapping T: X $\longrightarrow$ X is called Φ-hemi-contractive if and only if (I – T) is Φ-strongly quasi-accretive.

The following lemma plays an important role in proving our main results.

1.1 Lemma: [1], [2]

Let X be a Banach space. Then for all $x, y \in X$ and $j(x + y) \in J(x + y)$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y,j(x+y)\rangle.$$

2. Main Results

Now, we state and prove the following theorems:

2.1 Theorem:

Let X be a uniformly smooth Banach space and let T:X$\longrightarrow$X be a Φ-strongly quasi-accretive operator.

Let $x_0 \in K$ the Ishikawa iteration sequence $\langle x_n \rangle$ with errors be defined by

$$y_n = (1 - \beta_n)x_n + \beta_n Sx_n + b_n v_n \qquad \ldots(1)$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sy_n + a_n u_n \text{ for all } n = 0, 1,2,\ldots . \qquad \ldots(2)$$

where $\langle\alpha_n\rangle$, $\langle\beta_n\rangle$, $\langle a_n\rangle$ and $\langle b_n\rangle$ are sequences in [0,1] satisfying

$$\lim_{n\to\infty}\alpha_n = 0 \text{ and } \lim_{n\to\infty}\beta_n = 0 \qquad \ldots(3)$$

$$\sum_{n=0}^{\infty}\alpha_n = \infty \qquad \ldots(4)$$

$$a_n \leq \alpha_n^{1+c}, c > 0, b_n > \beta_n \qquad \ldots(5)$$

and $\langle u_n\rangle$ and $\langle v_n\rangle$ are two bounded sequence in X. Define S: X$\longrightarrow$ X by $Sx = f + x - Tx$ for all $x \in X$, and suppose that R(S) is bounded, then $\langle x_n\rangle$ converges strongly to the unique solution of the equation Tx = f.

proof: Since T is Φ-strongly quasi-accretive, it follows that N(T) is a singleton, say {w}.
Let Tw = f, it is easy to see that S has a unique fixed point w, it follows from definition of S that

$$\langle Sx-Sy,j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|)\cdot\|x-y\| \ldots(6)$$

Setting y = w, we have

$$\langle Sx-Sw,j(x-w)\rangle \leq \|x-w\|^2 - \Phi(\|x-w\|)\cdot\|x-w\| \ldots(7)$$

We prove that $\langle x_n\rangle$ and $\langle y_n\rangle$ are bounded. Let

$$M_1 = \sup\{\|Sx_n - w\| + \|Sy_n - w\| : n \geq 0\} + \|x_0 - w\|$$

$$M_2 = \sup\{\|u_n\| + \|v_n\| : n \geq 0\}$$

$M = M_1 + M_2$

From (2) and (5), we get

$$\|x_{n+1} - w\| \leq (1-\alpha_n)\|x_n - w\| + \alpha_n\|Sy_n - w\| + a_n\|u_n\|$$
$$\leq (1-\alpha_n)\|x_n - w\| + \alpha_n M_1 + \alpha_n M_2$$

and hence

$$\|x_{n+1} - w\| \leq (1-\alpha_n)\|x_n - w\| + \alpha_n M \qquad \ldots(8)$$

Now, from (1) and (5), we have

$$\|y_n - w\| \leq (1-\beta_n)\|x_n - w\| + \beta_n\|Sx_n - w\| + b_n\|v_n\|$$
$$\leq (1-\beta_n)\|x_n - w\| + \beta_n M_1 + \beta_n M_2$$

and hence

$$\|y_n - w\| \leq (1-\beta_n)\|x_n - w\| + \beta_n M \qquad \ldots(9)$$

$$\|x_n - w\| \leq M \qquad \ldots(10)$$

Now, we show by induction that for all $n \geq 0$. For $n = 0$ we have $\|x_0 - w\| \leq M_1 \leq M$, by definition of M_1 and M.

Assume now that $\|x_n - w\| \leq M$ for some $n \geq 0$.

Then by (8), we have

$$\|x_{n+1} - w\| \leq (1-\alpha_n)\|x_n - w\| + \alpha_n M$$
$$\leq (1-\alpha_n)M + \alpha_n M = M.$$

Therefore, by induction we conclude that (10) holds substituting (10) into (9), we get

$$\|y_n - w\| \leq M$$

From (9), we have

$$\|y_n - w\|^2 \leq (1-\beta_n)^2\|x_n - w\|^2 + 2\beta_n(1-\beta_n)M\|x_n - w\| + \beta_n^2 M^2$$

Since $1 - \beta_n \leq 1$ and $\|x_n - w\| \leq M$, we get

$$\|y_n - w\|^2 \leq \|x_n - w\|^2 + 2\beta_n M^2 \qquad \ldots(12)$$

Using lemma (1.1), we get

$$\begin{aligned}\|x_{n+1}-w\|^2 &\le \|(1-\alpha_n)(x_n-w)+a_nu_n+\alpha_n(Sy_n-w)\|^2\\ &\le \|(1-\alpha_n)(x_n-w)+a_nu_n\|^2+\\ &\quad 2\alpha_n<Sy_n-w,j(x_{n+1}-w)>\\ &\le (1-\alpha_n)^2\|x_n-w\|^2+\\ &\quad 2(1-\alpha_n)a_n\|x_n-w\|\|u_n\|+a_n^2\|u_n\|^2+\\ &\quad 2\alpha_n<Sy_n-w,j(y_n-w)>+\\ &\quad 2\alpha_n<Sy_n-w,j(x_{n+1}-w)-j(y_n-w)>\end{aligned}$$

Hence, using (6) and definition of M, we get

$$\begin{aligned}\|x_{n+1}-w\|^2 &\le \|x_n-w\|^2-2\alpha_n\|x_n-w\|^2+\alpha_n^2\|x_n-w\|^2+\\ &\quad 2(1-\alpha_n)a_nM^2+a_n^2M^2+2\alpha_n\|y_n-w\|^2-\\ &\quad 2\alpha_n\Phi(\|y_n-w\|)\cdot\|y_n-w\|+2\alpha_nc_n\end{aligned}$$

where

$$c_n=<Sy_n-w,j(x_{n+1}-w)-j(y_n-w)> \quad \ldots(13)$$

By (10) and (12) and using that $a_n\le\alpha_n\alpha_n^c$, we obtain

$$\begin{aligned}\|x_{n+1}-w\|^2 &\le \|x_n-w\|^2-2\alpha_n\|x_n-w\|^2+\alpha_n^2M^2+\\ &\quad 2\alpha_n\alpha_n^cM^2+2\alpha_n\|x_n-w\|^2+\\ &\quad 4\alpha_n\beta_nM^2-2\alpha_nk_x\|y_n-w\|^2+2\alpha_nc_n\end{aligned}$$

and hence

$$\|x_{n+1}-w\|^2\le\|x_n-w\|^2-2\alpha_n\Phi(\|y_n-w\|)\cdot\|y_n-w\|+\alpha_n\lambda_n \quad \ldots(14)$$

where $\lambda_n=(\alpha_n+2\alpha_n^c+4\beta_n)M^2+2c_n$.

First we show that $c_n\longrightarrow 0$ as $n\longrightarrow\infty$, observe that from (1) and (2), we have

$$\begin{aligned}\|x_{n+1}-y_n\| &\le \|(\beta_n-\alpha_n)(x_n-w)+\alpha_n(Sy_n-w)-\\ &\quad \beta_n(Sx_n-w)+a_nu_n-b_nv_n\| \quad \text{and}\\ &\le(\beta_n-\alpha_n)\|x_n-w\|+\alpha_n\|Sy_n-w\|+\\ &\quad \beta_n\|Sx_n-w\|+\alpha_n\|u_n\|+\beta_n\|v_n\|\end{aligned}$$

hence, by (10) and definition of M.

$$\|x_{n+1}-y_n\|\le(3\beta_n+\alpha_n)M \quad \ldots(15)$$

Therefore $\|x_{n+1}-w-(y_n-w)\|\longrightarrow 0$ as $n\longrightarrow\infty$.

Since $<x_{n+1}-w>$, $<y_n-w>$ and $<Sy_n-w>$ are bounded and j is uniformly continuous on any bounded subset of X, we have $j(x_{n+1}-w)-j(y_n-w)\longrightarrow 0$ as $n\longrightarrow\infty$, $c_n=<Sx_n-w,j(x_{n+1}-w)-j(y_n-w)>\longrightarrow 0$ as $n\longrightarrow\infty$. Thus $\lim_{n\to\infty}\lambda_n=0$,

$\inf\{\|y_n-w\|:n\ge 0\}=S\ge 0$.

We prove that S = 0. Assume the contrary, i.e., S > 0.

Then $\|y_n-w\|\ge S>0$ for all $n\ge 0$.

Hence

$\Phi(\|y_n-w\|)\ge\Phi(S)>0$.

Thus from (14)

$$\|x_{n+1}-w\|^2\le\|x_n-w\|^2-\alpha_n\Phi(S)\cdot S-\alpha_n[\Phi(S)\cdot S-\lambda_n] \quad \ldots(16)$$

for all $n\ge 0$. Since $\lim_{n\to\infty}\lambda_n=0$, there exists a positive integer n_0 such that $\lambda_n\le\Phi(S)\cdot S$ for all $n\ge n_0$.

Therefore, from (16), we have

$$\|x_{n+1}-w\|^2\le\|x_n-w\|^2-\alpha_n\Phi(S)\cdot S,$$

or

$\alpha_n\Phi(S)\cdot S\le\|x_n-w\|^2-\|x_{n+1}-w\|^2$ for all $n\ge n_0$.

Hence

$$\begin{aligned}\Phi(S)\cdot S\cdot\sum_{j=n_0}^{n}\alpha_j &= \|x_{n_0}-w\|^2-\|x_{n+1}-w\|^2\\ &\le\|x_{n_0}-w\|^2,\end{aligned}$$

which implies $\sum_{n=0}^{\infty}\alpha_n<\infty$, contradicting (4).

Therefore, S = 0.

From definition of S, there exists a subsequence of $<\|y_n-w\|>$, which we will denote by $<\|y_i-w\|>$, such that

$$\lim_{j\to\infty}\|y_j-w\|=0 \quad \ldots(17)$$

Observe that from (1) for all $n\ge 0$, we have

$$\begin{aligned}\|x_n-w\| &\le \|y_n-w+\beta_n(x_n-w)-\\ &\quad \beta_n(Sx_n-w)+b_nv_n\|\\ &\le\|y_n-w\|+\beta_n\|x_n-w\|+\\ &\quad \beta_n\|Sx_n-w\|+b_n\|v_n\|.\end{aligned}$$

Since $b_n\le\beta_n$, by definition of A, B and M we get

$\|x_n-w\|\le\|y_n-w\|+3\beta_nM$, for all $n\ge 0$ …(18)

Thus by (3), (17) and (18), we have

$$\lim_{j\to\infty}\|x_j-w\|=0 \quad \ldots(19)$$

Let $\varepsilon>0$ be arbitrary. Since $\lim_{n\to\infty}\alpha_n=0,\lim_{n\to\infty}\beta_n=0$ and $\lim_{n\to\infty}\lambda_n=0$, there exists a positive integer N_0 such that

$$\alpha_n\le\frac{\varepsilon}{3M},\ \beta_n\le\frac{\varepsilon}{3M},\lambda_n\le\Phi\left(\frac{\varepsilon}{3}\right)\cdot\frac{\varepsilon}{3} \quad \text{for all}$$

$n\ge N_0$.

From (19), there exists $k\ge N_0$ such that

$$\|x_k-w\|<\varepsilon \quad \ldots(20)$$

We prove by induction that

$$\|x_{k+n}-w\|<\varepsilon \text{ for all } n\ge 0 \quad \ldots(21)$$

For n = 0 we see that (21) holds by (20).

Suppose that (21) holds for some $n\ge 0$ and that $\|x_{k+n+1}-w\|\ge\varepsilon$. Then by (15), we get

$$\varepsilon \le \|x_{k+n+1} - w\| = \|y_{k+n} - w + x_{k+n+1} - y_{k+n}\|$$
$$\le \|y_{k+n} - w\| + \|x_{k+n+1} - y_{k+n}\| \quad \text{Hence}$$
$$\le \|y_{k+n} - w\| + (\alpha_{k+n} + 3\beta_{k+n})M$$
$$\le \|y_{k+n} - w\| + \frac{2\varepsilon}{3}$$
$$\|y_{k+n} - w\| \ge \frac{\varepsilon}{3}$$

From (14), we get

$$\varepsilon^2 \le \|x_{k+n+1} - w\|^2 \le \|x_{k+n} - w\|^2 - 2\alpha_{k+n}\Phi\left(\frac{\varepsilon}{3}\right)\cdot\frac{\varepsilon}{3} + \alpha_{k+n}\Phi\left(\frac{\varepsilon}{3}\right)\cdot\frac{\varepsilon}{3}$$
$$\le \|x_{k+n} - w\|^2 < \varepsilon^2,$$

which is a contradiction. Thus we proved (21). Since ε is arbitrary, from (21), we have $\lim_{n\to\infty}\|x_n - w\| = 0$. ■

2.1 Remark:

If in theorem (2.1), $\beta_n = 0$, $b_n = 0$, then we obtain a result that deals with the Mann iterative process with errors.

Now, we state the Ishikawa and Mann iterative process with errors for the Φ-hemi contractive operators.

2.2 Theorem:

Let X be a uniformly smooth Banach space, let K be a non empty bounded closed convex subset of X and $T:K \longrightarrow K$ be a Φ-hemi-contractive operator. Let w be a fixed point of T and let for $x_0 \in K$ the Ishikawa iteration sequence $<x_n>$ be defined by

$$y_n = \overline{\beta_n}\, x_n + \beta_n T x_n + b_n v_n$$
$$x_{n+1} = \overline{\alpha_n}\, x_n + \alpha_n T y_n + a_n u_n, \; n \ge 0$$

where $<u_n>, <v_n> \subset K$, $<\alpha_n>$, $<\beta_n>$, $<a_n>$, $<b_n>$ are sequences as in theorem (2.1) and

$$\overline{\alpha_n} = 1 - \alpha_n - a_n,$$
$$\overline{\beta_n} = 1 - \beta_n - b_n.$$

Then $<x_n>$ converges strongly to the unique fixed point of T.

proof: Obviously $<x_n>$ and $<y_n>$ are both contained in K and therefore, bounded. Since T is Φ-hemi-contractive, then (I – T) is Φ-strongly quasi accretive. The rest of the proof is identical the proof of theorem 2.1 with y = w and T = S, and is therefore omitted. ■

2.2 Remark:

If in theorem (2.2), $\beta_n = 0$ and $b_n = 0$, then we obtaine the corresponding result for the Mann iteration process with errors.

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بعض نظريات التقارب للنقطة الصامدة في فضاءات بناخ

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الخلاصة :

ليكن X فضاء بناخ املس منتظم، T:X ⟶X تطبيقات Φ–القوية شبه المتزايدة (Φ–نصف انكماشية). بُرهن تحت شروط مدروسة إنه متتابعة التكرار اشيكاوا تقترب بقوة الى الحل الوحيد للمعادلة Tx = f . إحدى نتائجنا هي لتحسين وتوسيع بعض النتائج حول تكرار اشيكاوا لانواع من الانكماشية المعلنة لدى آخرين كثيرين.