Free Vibration Of Simply Supported Beam Subjected To Axial Force

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Abstract

This paper presents a theoretical investigation of the free transverse vibrations of a uniform beam that is simply supported at both ends subjected to a static axial force. The beam is considered to have different shapes of cross section (Rectangle section, Box section, I-section and T-section) with the same value of area for all and at different values of second moments of area. The problem is modeled and solved analytically. based on the analytical solution. The effect of static axial force (tension or compression) on the characteristics of vibration (natural frequency and mode shape) are studied. It was concluded that increasing the tensile axial force causes an increase in the natural frequency while increasing the compressive force resulting in decreasing the natural frequency.

It is found that the beam of rectangular cross section have natural frequency lower than the other types of cross sections for the same applied force, while the beam of T-section has the higher value of natural frequency compared with other types. The beam of rectangle section losses its stability by buckling with compression force less than the other types. The beams of the same shape of cross section and the same value of the area have lower value of natural frequency at which have smaller value of second moment of area.

Key words: Transverse vibration, buckling, natural frequency

الخلاصة

يتناول هذا البحث دراسة نظرية للاهتزازات الحرة المستعرضة للدعامة المنتظمة ذات النهايات المفصلية والخاضعة لقوة محورية ساكنة. الدعامة المدروسة تمتلك أشكال مختلفة لمقطعها العرضي (المقطع المستطيل، المقطع الصندوقي ، المقطـع بشكل حرف I والمقطع بشكل حرف T) ولنفس القيمة لمساحة المقطع العرضي بالنسبة لكل الأشكال ولقيم مختلفة للعزم الثاني للمساحات. التحليل النظري اعتمد على الطريقة التحليلية لدراسة تأثير القوة المحورية الساكنة(شد أو ضـغط) علـى خـصائص الاهتزازات والمتمثلة بالتردد الطبيعي وشكل النسق. زيادة قوة الشد المحورية يسبب زيادة التردد الطبيعي للدعامة لكن تأثير زيادة قوة الضغط ينتج انخفاض بالتردد الطبيعي.

وجد إن الدعامة ذات المقطع العرضي المستطيل الشكل ولنفس القوة المحورية المسلطة على بقية المقاطع تمتلك تردد طبيعي اقل مما عليه للدعامة ذات المقاطع العرضية الأخرى حيث إن الدعامة ذات المقطع العرضي بشكل حرف T تمتلك أعلـى قيمة للتردد الطبيعي عند مقارنتها مع الأنواع الأخرى. الدعامة ذات المقطع بشكل المستطيل تفقد الاستقرار بسبب الانبعاج عند قوة ضغط تكون اقل مما عليه بالنسبة للأشكال الأخرى من المقاطع. الدعامات ذات الشكل الواحد وعند نفس القيمة لمـساحة المقطـع تمتلك أوطأ تردد طبيعي وذلك عند اقل قيمة من عزم المساحات الثاني.

List Of Symbols

a	Cross section area (m^2).
C_r	Arbitrary Constant.
E	Modulus of elasticity (N/m^2).
h	Thickness of Beam (m)
I	2^{nd} Moment of area (m^4).
L	Length of beam (m).
M	Bending moment (N.m).
m	Mass of the beam per unit length (kg/m).
Q	Shear force (N).
r	Number of modes.
S	Axial force (N).
t	Time (s).
ω	Natural frequency of beam (rad/s).

1. Introduction

A beam cross section can be rectangular, circular, or annular, or it can be a rolled I-section or a built-up section. Beams are fabricated of steel, aluminum, concrete, wood, and composite materials. They are used in buildings, bridges, aircraft, machinery, and other types of structures.

Vlasov, 1961 developed the theory of constraint torsion of beam, the effect is obvious in term of dynamic and stability phenomena when the global characteristics of a structure are investigated such as frequency, mode shape, or critical load causing a loss of stability. **Jiriusek, 1981** used the finite element method to formulate a 4-node isoparametric beam element including transverse shear and saint-venant torsion theory to derive the frequency equation for free-free boundary conditions. **Chang, 1993** presented the random vibration response analysis of a model which simulates a robotic arm**.** The left end of beam is attached by both translational and rotational springs, and the right end is free and carrying a heavy tip mass. **Heppler, 1995** derived the equations of motion and boundary conditions for a free-free Timoshenko beam with rigid bodies attached at the end points. **Gurgoze, 1996** used the Lagrange multipliers method to derive the frequency equation of a clamped-free Euler-Bernoulli beam with tip mass where a spring-mass system is attached. **Liu, 1996** derived the non-dimensional governing equation and boundary conditions for the in-plane vibration of a uniform, free-free beam subject to constant tension. This beam can be used as an appropriate model for pipeline towing problem in ocean engineering. **Voros GM. 2004,** used numerical method to derive linear stiffness matrix and mass matrix to study the free vibration of beam where the displacement compatibility transformation takes into account the torsion-flexural coupling in beam. **Gabor, 2007** presented the development of the stiffener for plate is based on a general beam theory, which includes the constraint torsional warping effect and the second order term of finite rotations.

In this paper, frequency equation, mode shape are obtained in analytic form of beam which have different shapes of cross section at the same value of area and buckling force is studied by the present work and using Euler formula of buckling.

2. Theoretical Analysis

To derive the equation that governing the transverse vibration of a beam of length L, with the following properties at section x; m(x) is the mass per unit length, A(x) is the cross-section area, and I(x) is the second moment of area, assume small deflections y(x,t) and rotations $\partial y / \partial x$. Consider the lateral vibration of a beam, loaded axially , as shown in Fig.(1). Whenever a beam is compressed, there is concern about its buckling. As the axial load is increased, a critical value of stress is reached with a new (buckled) deformation configuration is possible. Since this configuration generally undesirable, structural failure occur at this critical load. Design codes generally assume a failure at some load less than the buckling load[**Benaroya, 1998**]. To formulate the problem, use the free body diagram of an arbitrary section of the beam with all external force has been drawn as shown in Fig.(2), noting that there is an additional moment term Sy due to the constant axial force S, where y is the deflection at the section under consideration. This is shown in Fig (2).

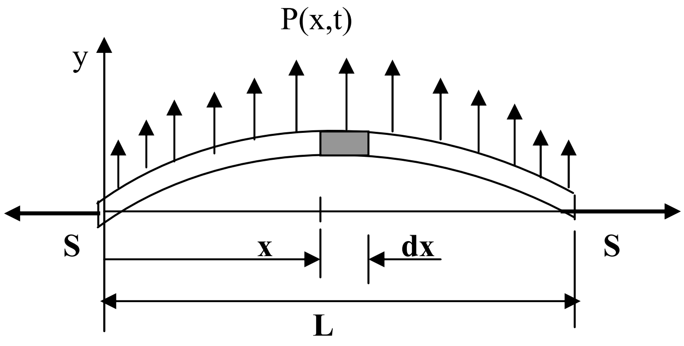


Fig.(1) : Transverse vibration of beam with axial force

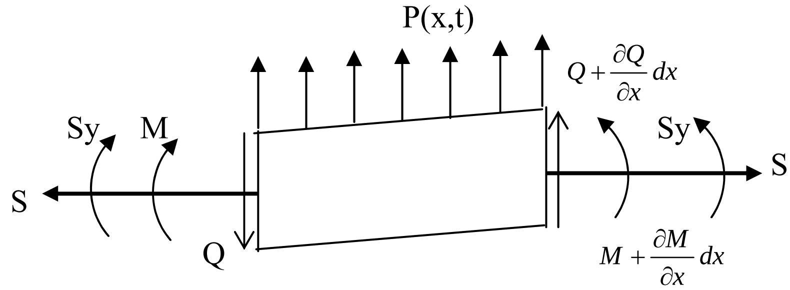


Fig.(2): Free body for the transverse vibration of beam with axial force

2.1 Eginevalues – Natural frequencies.

The equation of motion in the vertical direction remains the same (since S acts approximately perpendicular to y), the moment equation about the center of the cross-section now must include the moment due to S, is:

$$M + Sy = EI(x)\frac{d^2 y}{dx^2}, \qquad (1)$$

where, I = I(x), m = m(x)

$EI(x)\frac{d^2 y}{dx^2}$ the bending moment, $\frac{\partial}{\partial x}(EI(x)\frac{d^2 y}{dx^2})$ the shear force and $\frac{\partial^2}{\partial x^2}(EI(x)\frac{d^2 y}{dx^2})$ the load. [**Singer, 1981**]

while the governing equation of motion for y(x,t) [**Benaroya, 1998**],

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right] = -m(x)\frac{\partial^2 y}{\partial t^2}. \qquad (2)$$

The above equation becomes

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2} - Sy\right] + p(x,t) = -m(x)\frac{\partial^2 y}{\partial t^2}. \qquad (3)$$

Equation(3) can be solved by the method of separation of variables.

$$y(x,t)=Y(x)F(t). \qquad (4)$$

Substitute this equation into the governing equation (Eq. 3) gives:

$$\frac{d^2}{dx^2}\left[EIY''(x)F(t)-SY(x)F(t)\right]+p(x,t)=-m(x)Y(x)\ddot{F} \qquad (5)$$

Setting p(x,t)=0

$$\underbrace{\frac{1}{m(x)Y(x)}\frac{d^2}{dx^2}\left[EIY''(x)-SY(x)\right]}_{I}=\underbrace{\frac{\ddot{F}}{F}}_{II}=-\omega^2 . \qquad (6)$$

In the last equation, time-dependent variables and space-dependent variables have been placed on opposite sides of the equal; sign. Since part I is only a function of x, and part II is only a function of t, it must be that each is equal to the same constant, say $-\omega^2$. So that the solution in time is harmonic

$$\ddot{F}+\omega^2 F=0, \qquad (7)$$

This equation is solved, the result is simple harmonic motion with frequency ω.
Now Y(x) is governed by

$$\frac{d^2}{dx^2}\left[EIY''(x)-SY(x)\right]=-\omega^2 m(x)Y(x), \qquad (8)$$

and

$$EI\frac{d^4}{dx^4}\left[Y(x)\right]-S\frac{d^2}{dx^2}\left[Y(x)\right]=-\omega^2 m(x)Y(x), \qquad (9)$$

Define $\beta^4=\omega^2 m/EI$, and the eigenvalue equation becomes

$$Y''''(x)-\frac{S}{EI}Y''(x)+\beta^4(x)Y(x)=0 \qquad (10)$$

This is an eginvalue problem. Equation (7) required two initial conditions and equation (10) requires four boundary conditions for complete solution.

The eigenvalue problem must be solved for a particular set of boundary conditions, resulting in expressions for the eigenfunctions Y(x) and frequencies ω which the structure can accommodate in free vibration.

The effect of axial force S is significant. For a simply supported beam (setting p(x,t)=0 for the eigenvalue problem). we find

$$Y_r(x)=\sqrt{\frac{2}{mL}}\sin\frac{r\pi x}{L}, \quad r=1,2,3,\ldots, \qquad (11)$$

Differentiate equation (11) and substitute in to equation (10) to obtain

$$\omega_r=\left(\frac{r\pi}{L}\right)^2\sqrt{\frac{EI}{m}}\left(\sqrt{1+\frac{S}{EI}\left(\frac{L}{r\pi}\right)^2}\right). \qquad (12)$$

Note that for a tensile axial force +S, the effect is an increase in the frequencies of free vibration. Had a compressive force-S been applied, the frequencies would be given by

$$\omega_r=\left(\frac{r\pi}{L}\right)^2\sqrt{\frac{EI}{m}}\left(\sqrt{1-\frac{S}{EI}\left(\frac{L}{r\pi}\right)^2}\right), \qquad (13)$$

For r =1, the term $\frac{S}{EI}\left(\frac{L}{\pi}\right)^2$ is the ratio of S to the Euler buckling load. If $SL^2/EI\pi^2 \longrightarrow 1$, the lowest mode of vibration approaches a zero frequency and transverse buckling occur for $S=EI\,\pi^2/L^2$.

2.2 Eigenfunction – Modes shapes.

The mode shapes are given by the equation as shown below:

$Y_r(x)=C_r \sin \beta x$, [**Benaroya, 1998**] (14)

From define of $\beta^4=\omega^2 m/EI$ equation (14) becomes

$$Y_r(x)=C_r \sin \sqrt{\omega}\left(\frac{m}{EI}\right)^{\frac{1}{4}} x \qquad (15)$$

Substitute equation (12) into equation (15) and after simplification yields;

$$Y_r(x)=C_r \sin\left(\left(\frac{r\pi}{L}\right)\left(\sqrt{1+\left(\frac{S}{EI}\right)\left(\frac{L}{r\pi}\right)^2}\right)^{\frac{1}{4}}\right) x \qquad (16)$$

Where C_r is an arbitrary constant. To specify C_r, a normalization is carried out according to the rule

$$\int_0^L mY_r^2(x)dx = 1, \qquad r=1, 2, . \quad [\textbf{Benaroya, 1998}] \qquad (17)$$

Substitute equation (16) into equation (17) gives;

$$C_r^2 \int_0^L m \sin^2\left(\left(\frac{r\pi}{L}\right)\left(\sqrt{1+\left(\frac{S}{EI}\right)\left(\frac{L}{r\pi}\right)^2}\right)^{\frac{1}{4}}\right) x dx = 1 \qquad (18)$$

After solution equation (18) obtain:

$$C_r^2 = \left(\frac{2}{mL}\right)\left(\left(\sqrt{1+\left(\frac{S}{EI}\right)\left(\frac{L}{r\pi}\right)^2}\right)^{-\frac{1}{4}}\right) \qquad (19)$$

Substitute C_r in equation (16) yield the modes to have the specific form for the tensile force as below :

$$Y_r(x) = \left(\frac{2}{mL}\right)^{\frac{1}{2}}\left(\left(\sqrt{1+\left(\frac{S}{EI}\right)\left(\frac{L}{r\pi}\right)^2}\right)^{-\frac{1}{8}}\right) \sin\left(\left(\sqrt{1+\left(\frac{S}{EI}\right)\left(\frac{L}{r\pi}\right)^2}\right)^{\frac{1}{4}}\frac{r\pi x}{L}\right) \qquad (20)$$

And for the compression force as

$$Y_r(x) = \left(\frac{2}{mL}\right)^{\frac{1}{2}}\left(\left(\sqrt{1-\left(\frac{S}{EI}\right)\left(\frac{L}{r\pi}\right)^2}\right)^{-\frac{1}{8}}\right) \sin\left(\left(\sqrt{1-\left(\frac{S}{EI}\right)\left(\frac{L}{r\pi}\right)^2}\right)^{\frac{1}{4}}\frac{r\pi x}{L}\right) \qquad (21)$$

3. Results and Discussion:

Table(1) shows the properties and dimensions of beam which have different cross sections and table(2) shows shapes of different cross section and its mechanical properties.

Table(3) shows the values of buckling force for different shape of cross section of beam by using Euler equation ($S = EI\,\pi^2/L^2$) directly and the present work (graphically where the natural frequency approach to minimum value at the buckling force). It can be seen that the difference between the two methods is very small, and the table show the maximum tensile force for beam after obtained the value of ultimate strength of structural steel from strength of material is 830 Mpa, therefore the allowable stress is 415 Mpa , where maximum tensile force = (allowable stress*cross section area of beam). Table (4) shows the effect of change second moment of area on the range of buckling force for all shapes at the same area by using the present work.

Figures (3 , 4) show the natural frequency of the first mode and second mode of vibration respectively as a function of the tensile force for second moment of area (I_1). It is shown that there is an increase in natural frequency with increasing the tensile force. This is attributed to the increase of stiffens of the beam at the same mass for all. It is seen that the beam of rectangular cross section has natural frequency lower than the others type because of it has the smallest value for second moment of area which cause decreased the stiffness of beam.

Figures (5 , 6) show the natural frequency of the first mode and second mode of vibration respectively as a function of the compressive force at second moment of area (I_1). It is shown that the natural frequency decreased with increasing the compressive force. This is attributed to the decrease of stiffens of the beam where all beams at the same mass. Also the beam of T–section bears the range of compressive force higher than the others before occurred the buckling because of it has the largest value of second moment of area causes increasing its stiffness.

Figures (7 , 8) show the natural frequency of the first mode of vibration as a function of the tensile and compressive force respectively at moment of inertia (I_2) and at the same value of cross section area in (I_1) means the same mass for all beams. It is shown that the natural frequency increase with increasing the second moment of area resultant to increased the stiffness of beam.

The main features of the mode shapes associated with the first two natural frequencies as a function of length for beam at simply support ends are shown in Fig.(9&10) for a tensile and a compressive force respectively under effect of different value of axial force. It can be note that for tensile force the amplitude is an decreased with increasing the force resultant to increased the natural frequency of beam, while for compressive force the amplitude is an increased with increasing the force resultant to decreased the natural frequency. The beam of rectangle section losses stability when the force approaches to buckling force can be seen that in the difference of amplitude when compare with the types at the same force.

4. Conclusions:

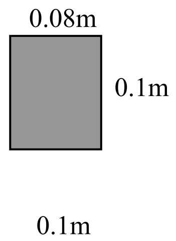
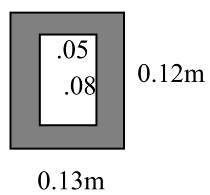
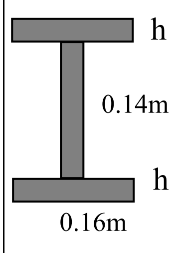
The main conclusions of the present work can be summarized as:

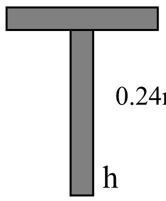
1- Natural frequency of beam for rectangular cross sectional area is lower than the others cross section at the same area for both tensile force and compressive force.
2- Natural frequency of beam is an increased with increasing second moment of area at the same area of cross section.
3- Beam with T-section bears high range of compressive force before occurred buckling .
4- The stiffness of beam is an increased with increasing second moment of area at the same area of cross section.

Table (1) : Specifications of the tested models

Parameter	Symbol	Value	Units
Length	**L**	**3**	**m**
Thickness	**h**	**0.02**	**m**
Modulus of elasticity	**E**	**200**	**Gpa**
Density	**ρ**	**7800**	**kg / m^3**

Table(2) : shapes of cross section

Cross section	shape	area (a) m^2	Second moment of area $(I_1)m^4$
Rect.-section		**$80*10^{-4}$**	**$6.667*10^{-6}$**
Box section		**$80*10^{-4}$**	**$12.27*10^{-6}$**
I-section		**$80*10^{-4}$**	**$38.18*10^{-6}$**

T-section		**80*10^{-4}**	**55.6*10^{-6}**

Table(3) : Buckling Force

Shape of cross section	Maximum tensile force MN	Buckling force by Euler Equation MN	Buckling force by Present work MN	Difference with respect to Euler equation %
Rect. section	**3.32**	**1.4634**	**1.45**	**0.915 %**
Box section	**3.32**	**2.693**	**2.64999**	**1.59 %**
I-section	**3.32**	**8.3805**	**8.350014**	**0.363%**
T-section	**3.32**	**12.204**	**12.20003**	**0.0325%**

Difference = (Euler Eq. – Present woke)/Euler Eq.*100%

Table(4) : Moment of Inertia with Buckling Force

Shape of cross section	Second moment of area (m^4)	Buckling Force (MN) Present	Second moment of	Buckling Force (MN)

	I_1	work MN	area (m^4) I_2	Present work MN
Rect. section	**$6.667*10^{-6}$**	**1.45**	**$8.066*10^{-6}$**	**1.7499**
Box section	**$12.27*10^{-6}$**	**2.64999**	**$13.867*10^{-6}$**	**2.999**
I-section	**$38.18*10^{-6}$**	**8.350014**	**$45.867*10^{-6}$**	**10.05**
T-section	**$55.6*10^{-6}$**	**12.20003**	**$60.03*10^{-6}$**	**13.10003**

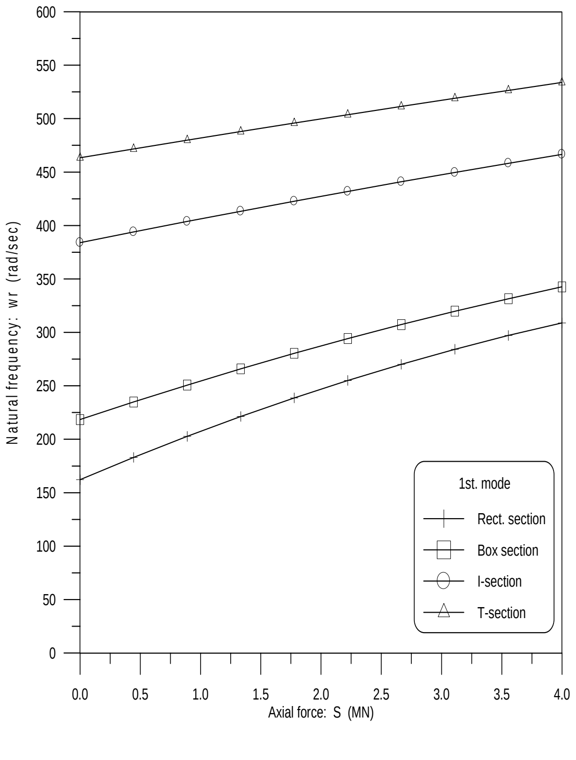


Fig. (3): Natural frequency as a function of a tensile axial force for 1st mode at (I_1)

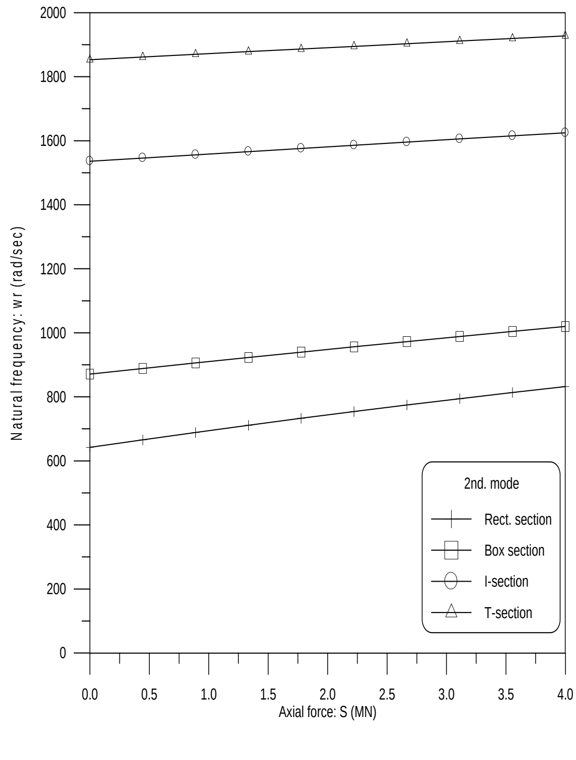
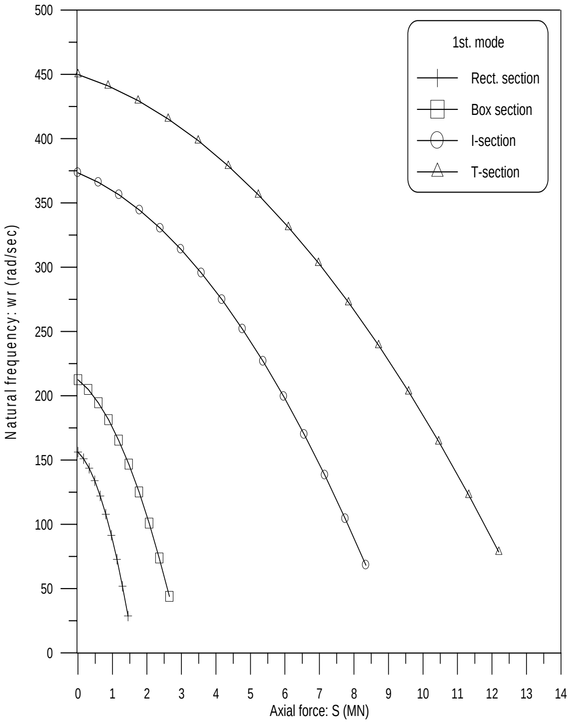


Fig. (4): Natural frequency of a function of a tensile axial force for 2nd mode at (I_1)



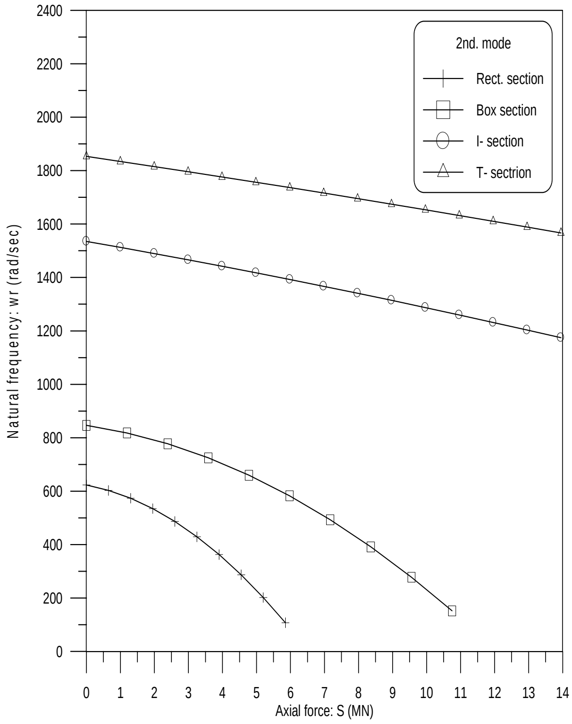
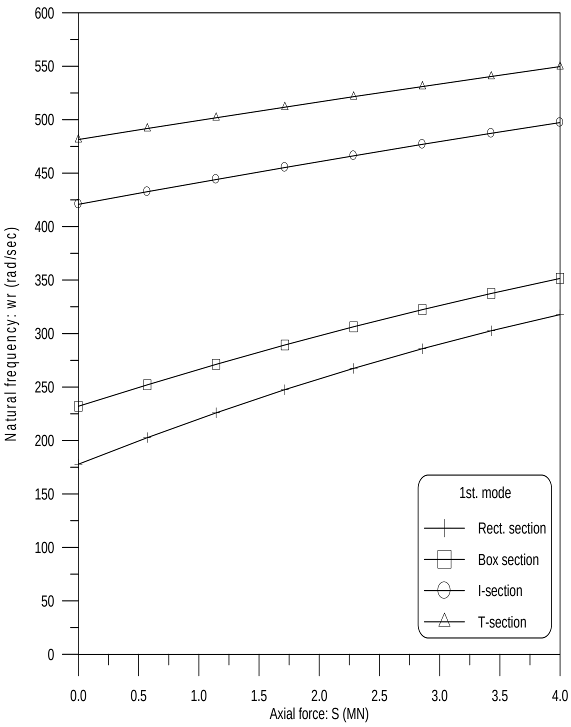


Fig. (5): Natural frequency as a function of compressive axial force for 1st mode at (I_1)

Fig. (6): Natural frequency of a function of a compressive axial force for 2nd mode at (I_1)



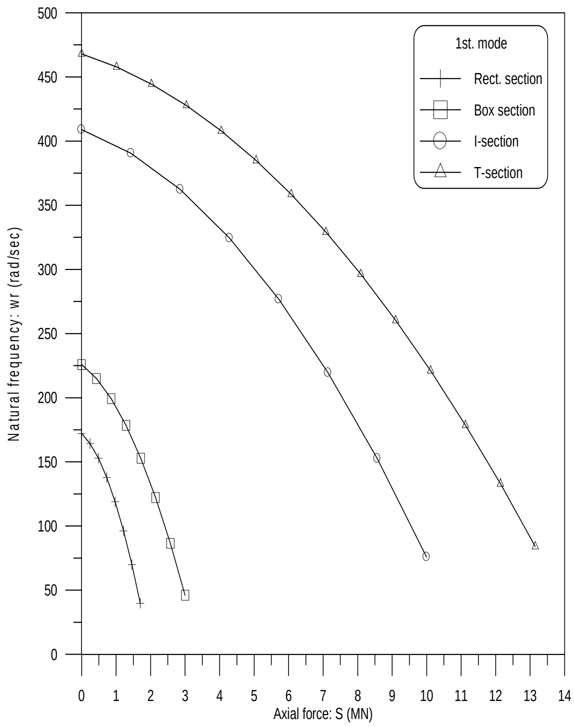


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Fig. (7): Natural frequency as a function of a tensile axial force for 1st mode at (I_2)

Fig. (8): Natural frequency as a function of a compressive axial force for 1st mode at (I_2)

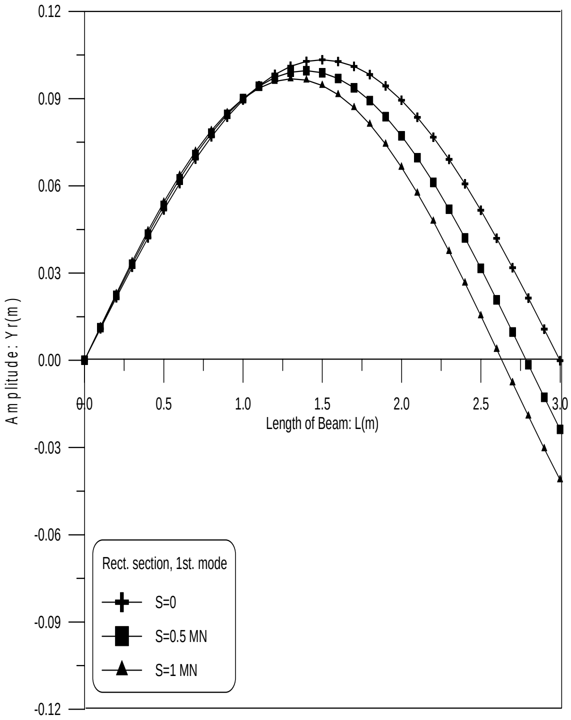


Fig.(9-a)

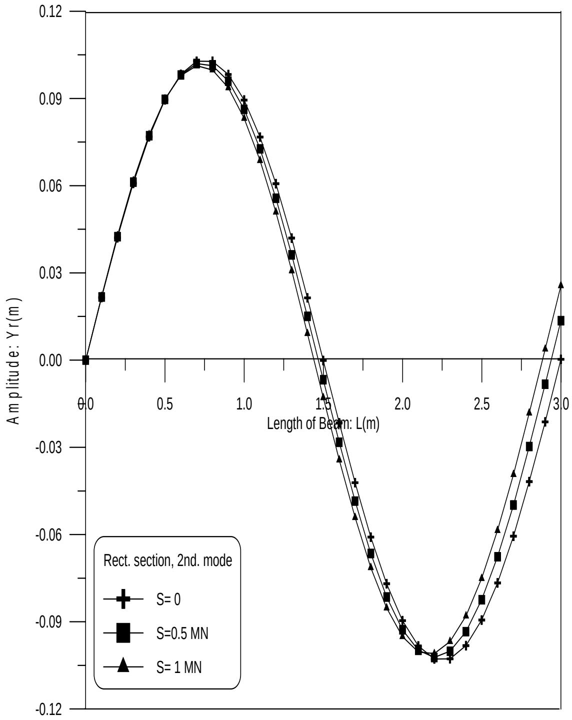
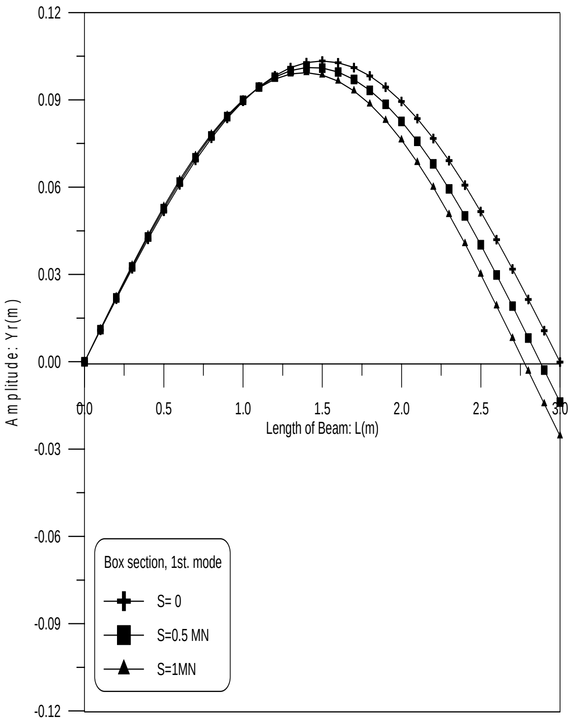


Fig.(9-b)



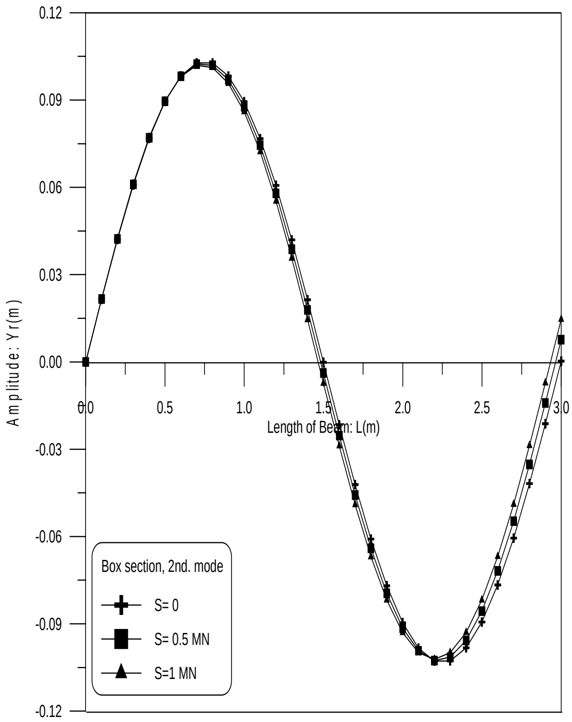
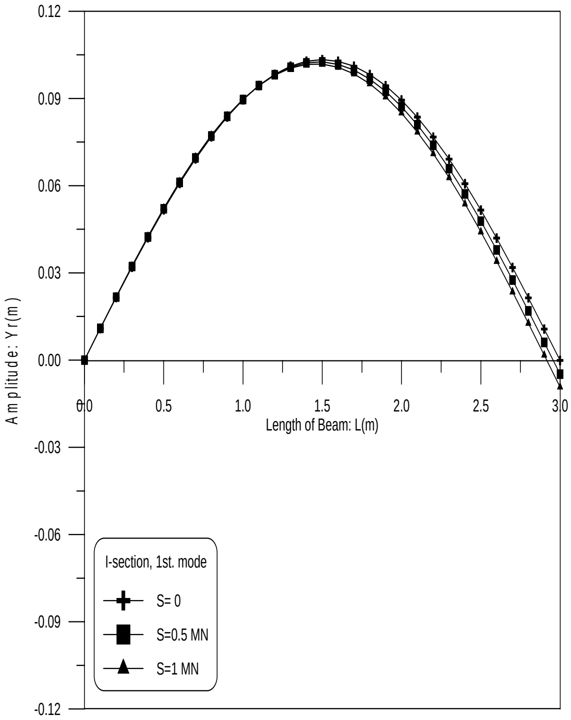


Fig.(9-c) Fig.(9-d)



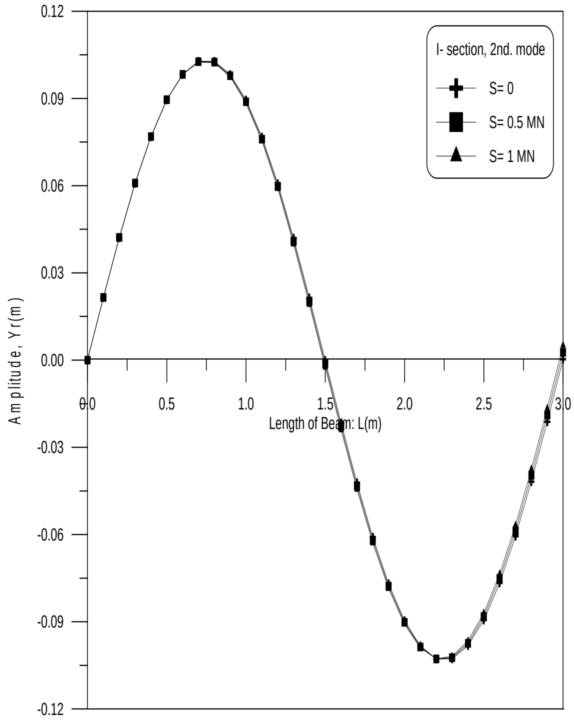
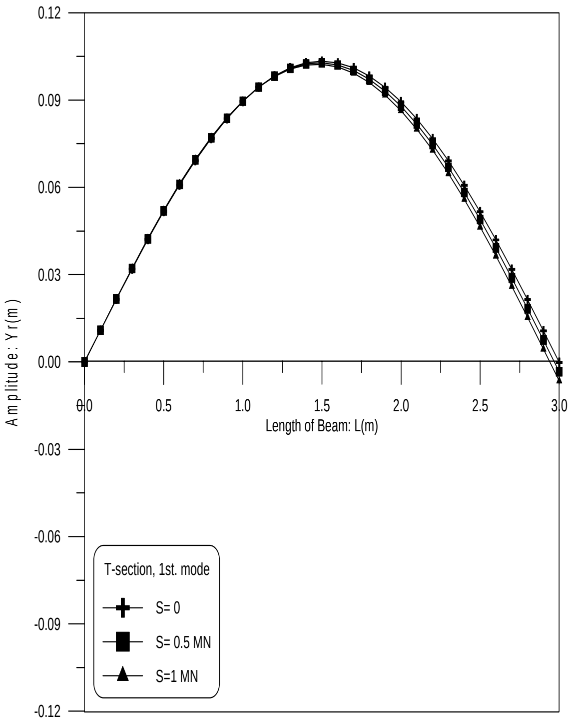


Fig.(9-e) Fig.(9-f)



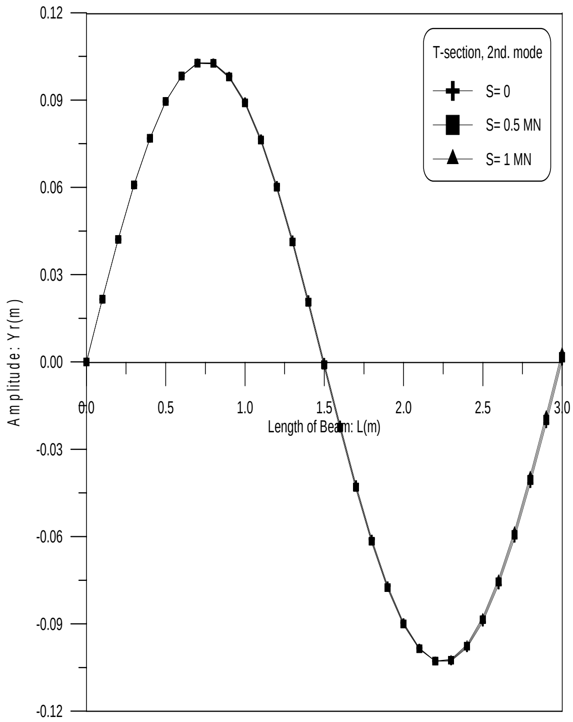


Fig.(9-g) Fig.(9-h)

Fig.(9;a-h):Mode shapes associated with the first two natural frequency of beam for different cross section respect to tensile force

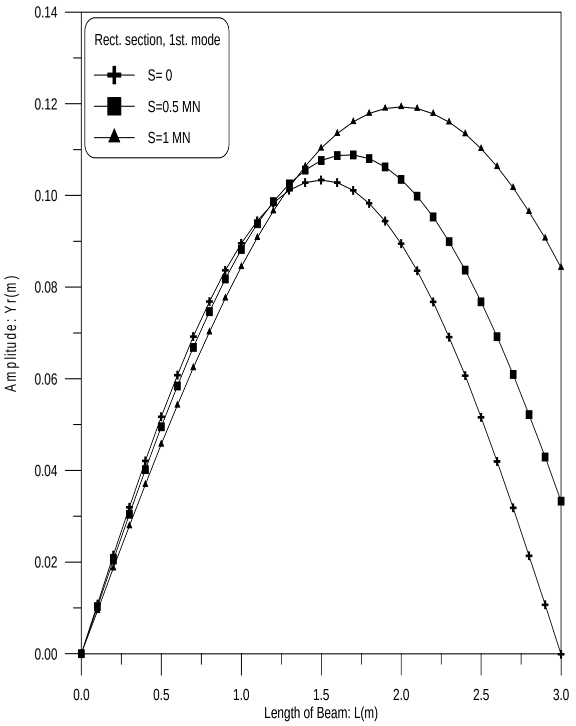


Fig.(10-a)

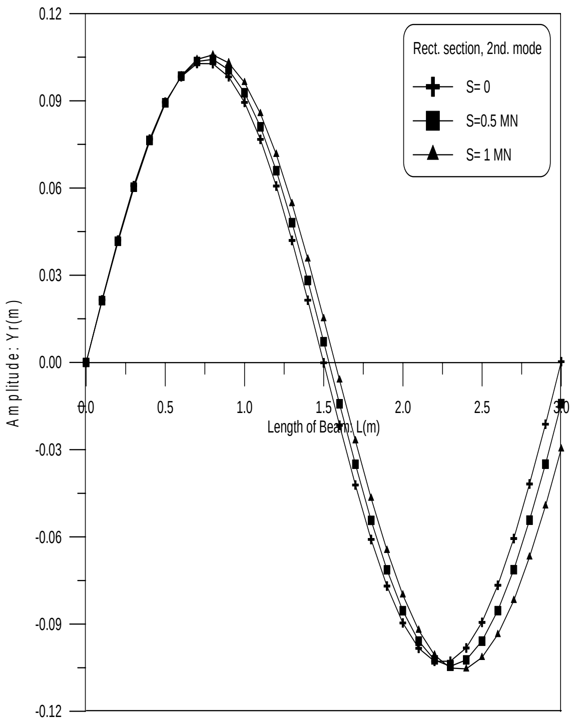


Fig.(10-b)

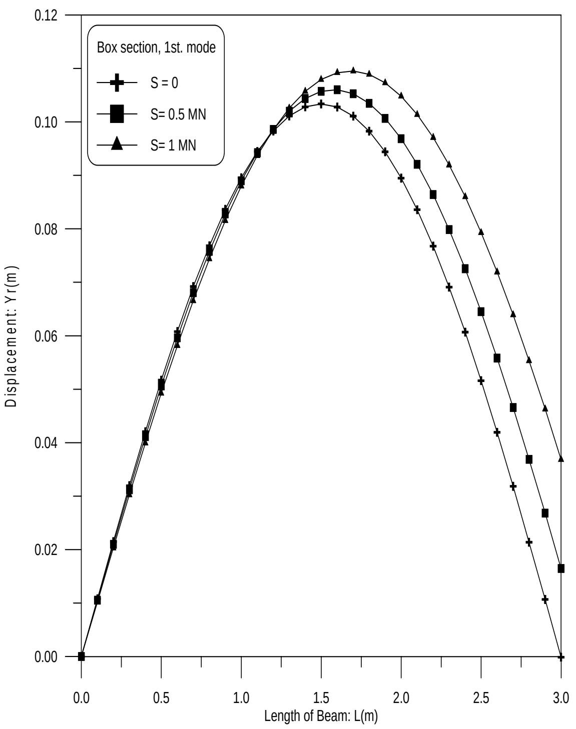


Fig.(10-c)

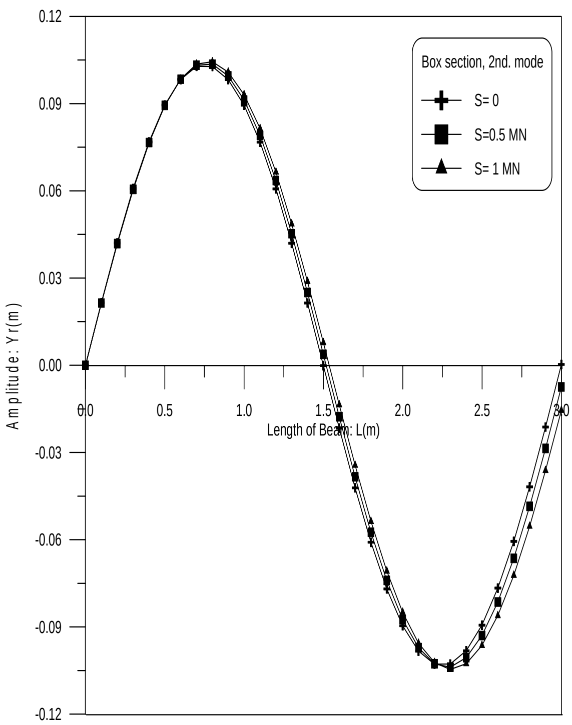


Fig.(10-d)

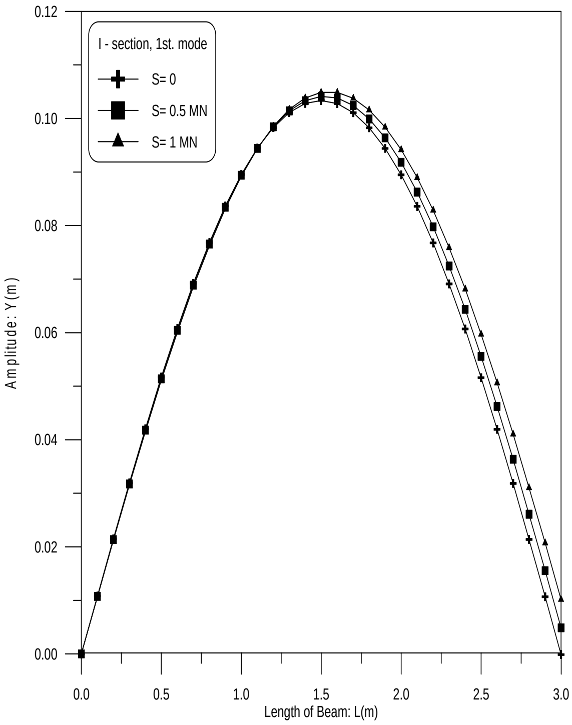


Fig.(10-e)

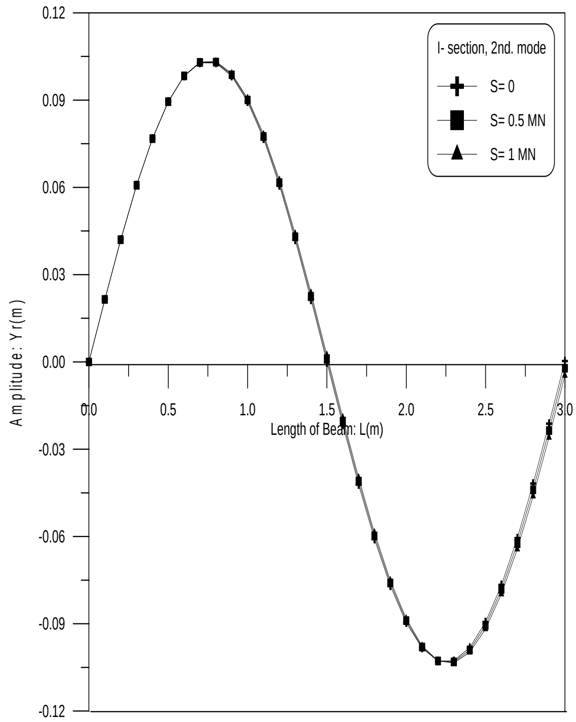


Fig.(10-f)

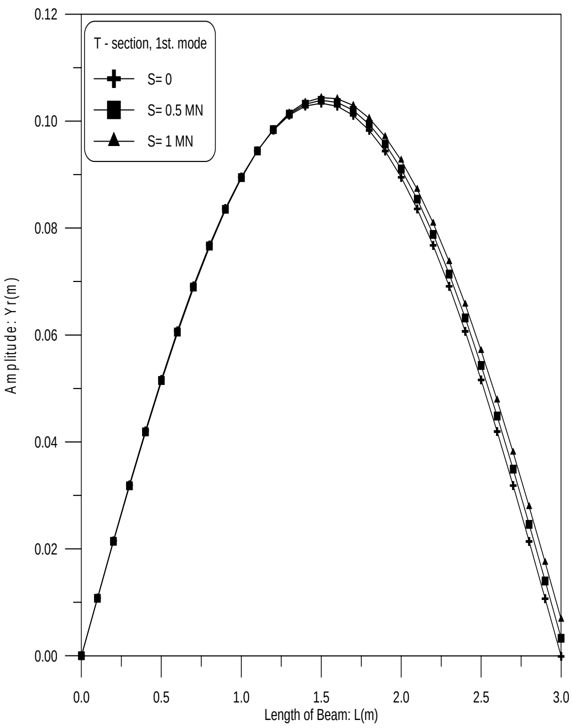


Fig.(10-g)

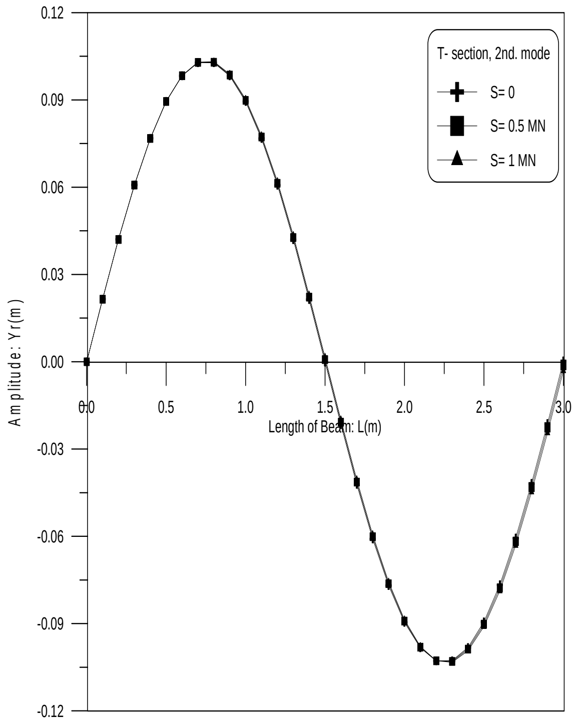


Fig.(10-h)

Fig. (10;a-h): Mode shapes associated with the first two natural frequency of beam for different cross section respect to compressive force

5. References

Benaroya, "Mechanical Vibration", Prentice – Hill, Inc., U.S.A., 1998.

Chang, 1993, "Forced Vibration of a Mass-Loaded Beam with Heavy Tip Body", J. of Sound and Vibration, Vol. 164, PP. 471 –484.

Heppler and M. White, 1995, "Vibration Modes and Frequencies of Timoshenko Beam with Attached Rigid Bodies", J. of App. Mech., Vol. 62, PP. 193 – 199.

Gabor, 2007, "Buckling and Vibration of Stiffened Plates", J. of (I.RE.M.E.)Vo., 1, n., 1.

Gurgoze, 1996, "On Eigenfrequencies of a Cantilever Beam with Attached Tip Mass and a Spring-Mass system", J. Sound and Vibration, Vol. 190, PP. 149 –162.

Jiroousik J., 1981, "A family of Variable Section Curved Beam and Thick Shell or Membrane Stiffening Isoperimetric Elements", Int J Numer MethodS Eng., Vol. 17, PP. 171 – 186

Liu, 1996, "Vibration of a Free-Free Beam under Tensile Axial Loads, J. of Sound and Vibration Vol. 190, PP. 273 –282.

Singer, "Strength of Materials" 3rd edition, 1981.

Vlasove VZ, 1961, "Thin Walled Elastic Beams", National Science Foundation.

Vorose GM, 2004, "Free Vibration of Thin Walled Beams", Periodica Polytechnica Ser Mech. Eng., Vol. 48, PP. 99 – 110.