Stability of Lorenz-like system (∐)

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Abstract

In this paper, we presented a new three dimensional Lorenz-like system. Nonlinear characteristic and basic dynamic properties of three dimensional autonomous system are studied by means of nonlinear dynamics theory, including the stability and we found that the parameters a, b are effected on the form of the roots where $b = -a/4$.

Keywords: New Lorenz-like system, stability**,** Routh-Hurwitz

1. Introduction:

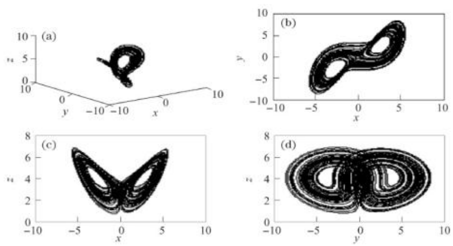
Nonlinear dynamics , commonly called the chaos theory, changes the scientific way of looking at the dynamics of natural and social systems, which has been intensively studied over the past two decades . Chaos is very interesting phenomenon closely related to nonlinear systems, which accurse so frequently that it has become important for workers in many disciplines to have a good grasp of the fundamental and basic tools of the science of chaotic dynamics[12].In 1963, Lorenz found the first classical chaotic attractor in three-dimension autonomous system[5]. In 1976, O.E. Rossler constructed several three-dimensional quadratic autonomous chaotic systems, which also have seven terms on the right-hand side , but with only one quadratic nonlinearity (xz). Obviously , any of these Rossler systems has a simpler algebraic structure as compared to the Lorenz system. It was believed that the Rossler systems might be the simplest possible chaotic flows , where the simplicity refers to the algebraic representation rather than the physical process described by the equations or the topological structure of the strange attractor.[3] In 1999 , Chen and Ueta found another similar but not topological equivalent chaotic attractor to Lorenz's[6]. In 2002 , Lu found the critical chaotic attractor between the Lorenz and Chen attractor [7]. In the same year , Lu et al. unified above three chaotic systems into a chaotic system which is called a unified chaotic system [11].In 2004, Liu etc. discovered another chaotic system by using physical electrical circuits and named it the Liu system [2,11]. In 2008, Tigan , reported a new three-dimensional differential system derived from the Lorenz system [10]. In 2009, Xian - Feng Li[a] , Yan - Dong Chu[a,*] , Jian - Gang Zhang[a,b], ying -Xiang Chang[a,b] is proposed a new Lorenz-like chaotic system derived from the Tigan system.[3] The nonlinear differential three-dimensional system is given by:

$$\dot{x} = a(y-x)$$
$$\dot{y} = abx - axz \qquad (1)$$
$$\dot{z} = -cz + xy$$

where $x = (x, y, z)^T \in R^3$ is the state variables of the system , a, b and c are constant coefficients assuming that $a \neq 0$ and $b > 0$. In 2010 , Gaoxiang Yang discovered another new Lorenz-like system[4]. The nonlinear differential three-dimensional system are:

$$x = a(y-x)$$
$$\dot{y} = bx - dxz \qquad (2)$$
$$\dot{z} = -cz + hx^2 + ky^2$$

Where $(x, y, z)^T \in R^3$ is the state variables of the system, a , b , c , d , h and k are parameters. It is three-dimensional autonomous system , which has seven terms on the right-hand side of the governing equations , also relies on three multipliers to introduce the nonlinearity necessary for folding trajectories , namely $xz, x^2 and\ \ y^2$.[4]



The chaotic attractor of the new Lorenz-like system(3)

2.Dynamical behavior of the new chaotic attractor:

The chaotic attractor of the new Lorenz-like system(2)

2.1. some basic properties

2.1.1. symmetry and invariance

First , it's noticed that the system has a symmetry S because the transformation $S:(x, y, z) \to (-x, -y, z)$,which permits system(3)is invariant for all values of the parameters a, b, c, d, h, k .Obviously , the $z - axis$ itself is an orbit, that is if $x = 0, y = 0$ at $t = 0$ then $x = 0, y = 0$ for all $t > 0$. Moreover, the orbit on the $z - axis$ tends to the origin as $t \to \infty$. And the transformation S indicates that the system are symmetrical on the $z - axis$, for instance , if φ is the solution of the system , and $S\varphi$ is too.[4]

2.1.2. Dissipative[4]

The system (2) can be dissipative system , because the divergence of thevector field , also called the trace of the Jacobian matrix is negative if and only if the sum of the parameters a and c is positive , that is

$$a+c>0,$$

$$div\vec{V}=\frac{\partial\dot{x}}{\partial x}+\frac{\partial\dot{y}}{\partial y}+\frac{\partial\dot{z}}{\partial z}=Tr(J)=-(a+c) \quad (3)$$

$$V(t)=V(0)e^{-(a+c)t}$$

So, the system will always be dissipative if and only if when $a+c>0$,

with an exponential: $\frac{d\vec{V}}{dt}=e^{-(a+c)}$.

2.1.3.Critical points[4]:

We now turn to analyzing the fixed points of the system (2). The equilibrium satisfy $a(y-x)=0, bx-dxz=0, -cz+hx^2+ky^2=0$.

Therefore system (3) has three equilibria if and only if $ac(b^2+d^2(h+k)^2)\neq 0, and (bcd(h+k))>0$:

$P_0(0,0,0), P_{\mp}(\mp\sqrt{bc/d(h+k)}, \mp\sqrt{bc/d(h+k)}, b/d)$.

In this paper , we study the stability of a New Lorenz-like system by using Routh-Hurwitz conditions to determine the stability and we found the cases that the critical pointes is stable or unstable.

3.Helping Results:

Remark 1 (Routh-Hurwitz Test) [1]:

All roots of the indicated polynomial have negative real parts precisely when the given conditions are:

$$\lambda^3+a\lambda^2+b\lambda+c \;:$$

$$a>0, c>0, ab-c>0 \quad (4)$$

Remark 2 (Critical Cases) [1]:

Critical cases in the theory of stability for differential equation means, that cases when the real part of all roots of the characteristic equation have no positive with the real part of at least one root being zero other express which is neither stable nor unstable.

4.Main Results:

Theorem1: The solution of system (2)at the critical Point $P_0(0,0,0)$ has the following cases:

1- asymptotically stable if $a,c>0, b<0$

2- unstable if $a,b,c>0$

3- critical case if $a,c>0, b=0$

proof: At the critical point $P_0(0,0,0)$, system(2) is linearized , the Jacobian matrix is defined as:

$$J_0=\begin{bmatrix}-a & a & 0\\ b & 0 & 0\\ 0 & 0 & -c\end{bmatrix}$$

And, the characteristic polynomial is:

$$f(\lambda)=\lambda^3+(a+c)\lambda^2+a(c-b)\lambda-abc=0 \quad (5)$$

Let $A=a+c, B=a(c-b), C=-abc$

By Routh-Hurwitz criteria , the roots of (6) have strictly negative real parts if and only if $A>0, C>0$ and $AB>C$, and this satisfied when $a,c>0, b<0$, hence the system (2) is asymptotically stable . While if $a,b,c>0$, then one of the Routh-Hurwitz condition does not satisfied ,therefore the system (2) is unstable . Finally if $a,c>0, b=0$, then system(2) is critical case. The proof is complete.

Thearem2: In (6), the relation between the parameters a,b and the form of the roots has the following cases , when $a>0, b<0$:

1- if $b\in(-\infty,-a/4)$, we get

$$\lambda_1=-c, \lambda_{2,3}=-k\mp im$$

2- if $b=-a/4$, we get

$$\lambda_1=-c, \lambda_2=\lambda_3=-k$$

3- if $b\in(-a/4,0)$, we get

$$\lambda_1=-c, \lambda_2=-k_1, \lambda_3=-k_2$$

where k, k_1, k_2 and m are positive constant.

Proof: : By analyzing equation (6) we get

$$-(c+\lambda)(\lambda^2+a\lambda-ab)=0 \quad (6)$$

and hence $\lambda_1=-c$ and

$\lambda_{2,3}=\frac{-a+\sqrt{a^2+4ab}}{2}$, therefore if $b\in(-\infty,-a/4)$, we get $\lambda_1=-c$ and $\lambda_{2,3}=-k\mp im$, while if $b=-a/4$, we get $\lambda_1=-c$ and $\lambda_2=\lambda_3=-k$, finally if $b\in(-a/4,0)$, we get $\lambda_1=-c$ and $\lambda_2=-k_1, \lambda_3=-k_2$, hence the system (3) is asymptotically stable and the proof is complete.

Next, we consider the stability of equilibrium $P\mp$, under the condition

$ac(b^2+d^2(h+k)^2)\neq 0, and (bcd(h+k))>0$

. As system(3) has the symmetry properties , P_+, and, p_- have the same stability . For the equilibrium p_+ , we can transform system(3) into the following form by

$$x=X+\sqrt{bc/d(h+k)}, y=Y+\sqrt{bc/d(h+k)}, z=Z+b/d ,$$

$$\begin{aligned}\dot{X}&=-a(Y-X)\\ \dot{Y}&=-d\sqrt{bc/d(h+k)}Z-dXZ\\ \dot{Z}&=2h\sqrt{bc/d(h+k)}X+2k\sqrt{bc/d(h+k)}Y-cZ+hX^2+kY^2\end{aligned} \quad (7)$$

and the characteristic polynomial of the system (7) at the critical point $P_o^{'}(0,0,0)$ is :

$$f(\lambda)=\lambda^3+(a+c)\lambda^2+((ac(h+k)+2kbc)/h+k)\lambda+2abc=0 \quad (8)$$

Theorem3:The solution of system (7) at the critical point $P_o^{'}(0,0,0)$ is:

1- asymptotically stable if

$b<(a(a+c)(h+k))/2(ah-ck), and, a,c,h,k>0, and, ah>ck$

2- unstable if

$b>(a(a+c)(h+k))/2(ah-ck), and, a,c,h,k>0, and, ah>ck$

3- critical case if

$b=(a(a+c)(h+k))/2(ah-ck), and, a,c,h,k>0, and, ah>ck$

proof: By using Routh-Hurwitz criteria , equation (8) has all roots with negative real parts if and only if the condition are satisfied as follows:

$A>0, C>0$,and $AB>C$, since $A=a+c, C=2abc, and, a,b,c,h,k$ are positive parameters . Consequently , $A>0$ always ,and $C>0$ also.

We must proof that $AB>C$, therefore

$$((ac(h+k)+2kbc)/(h+k))>2abc.$$

Which implies

$b<(a(a+c)(h+k))/2(ah-ck), and, ah>ck$, the proof of first condition is complete.

While if , the $b>(a(a+c)(h+k))/2(ah-ck), and, ah>ck$n one of the Routh- Hurwitz condition are not satisfied. Consequently, the system(7) is unstable. Finally , if $b=(a(a+c)(h+k))/2(ah-ck), and, ah>ck$, then satisfied remark 2 , hence the system (7) is critical case. The proof is complete.

5. Illustrative Examples:

Example 1: Investigate for stability of system (2)

$$\dot{x}=4(y-x)$$
$$\dot{y}=-x-3xz$$
$$\dot{z}=-3z+2x^2+y^2$$

Where $a=4, b=-1, c=3, d=2, h=2, k=1$

Solution: from equation (5) we get

$$\Rightarrow f(\lambda)=\lambda^3+7\lambda^2+16\lambda+12=0$$

And by using Routh- Hurwitz conditions we get $a_1=7>0, a_3=12>0, a_1a_2-a_3=100>0$, and system (2)is asymptotically stable. While if $a=4, b=1, c=3, d=h=2, k=1$, we get the system (2) is unstable. And if $a=4, b=0, c=3, d=h=2, k=1$, then the system (2) is critical case.

Example 2 : Investigate for stability of system (2) at the critical point p_+

Where $a=8, b<66, c=3, k=2, h=d=1$, let $b=60$

Solution: from equation (8) we get

$$f(\lambda)=\lambda^3+11\lambda^2+264\lambda+2880=0$$

and by using Routh- Hurwitz conditions we get

$$a_1=11>0, a_3=2880>0, a_1a_2-a_3=24>0$$

Hence the system (2)is asymptotically stable. While if $b=67$, we get $a_1=11>0, a_3=3216>0, a_1a_2-a_3=-4<0$, hence the system(2) is unstable.

Finally, if $b=66$ we get

$$a_1=11>0, a_3=3168>0 \text{ and}$$
$$a_1a_2-a_3=0$$

Hence the system(2) is critical case.

6.Conclusions:

In this paper , a new three dimensional Lorenz-like system is proposed. Nonlinear characteristic and basic dynamic properties of this system are considered by means of nonlinear dynamics theory , and we deduced that the stability of this system depended on the parameters a, b.

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استقرارية نظام شبيه لورنز(II)

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الملخص

في هذا البحث قدمنا دراسة لنظام شبيه لورنز ثلاثي الأبعاد ، وقد تم استخدام النظرية الديناميكية اللاخطية لدراسة النظام الذاتي ثلاثي الأبعاد والتي بضمنها الاستقرارية وتم التوصل إلى أن قيمة المؤثرين aو b لهما تأثير على شكل الجذورحيث أن $b=-a/4$.