Bayesian Fixed Sample Size Procedure for Selecting the Better of Two Poisson Populations With General Loss Function

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Abstract

In this paper an optimal (Bayesian) fixed sample size procedure for selecting the better of two Poisson populations is proposed and studied . Bayesian decision-theoretic approach with general loss function and Gamma priors are used to construct this procedure .

The numerical result of this procedure are given with different loss functions constant , linear and quadratic , in one equation we can obtain the Bayes risk for the three types of the loss functions : constant , linear and quadratic . in this paper the numerical results are given by using Math Works Matlab ver. 7.10.0 .

Keywords and phrases : selection procedure , Bayesian decision theoretic , Bayes risk

1- Introduction

The Poisson distribution is most commonly used to model the number of random occurrences of some phenomenon in the specified unit of space or time , for example the number of phone calls received by a telephone operator in 10-minutes or the term frequencies in a given document . [see 1]

A common problem that arises in practice is the selection of the better of two Poisson populations with unknown parameters .

Formally , we can state the problem as follows . Consider two independent Poisson populations Π_1, Π_2 with unknown occurrence means λ_1, λ_2 respectively . Let $\lambda_{[1]} \le \lambda_{[2]}$ be the ordered values of the parameters λ_1, λ_2 . It is assumed that the exact pairing between the ordered and unordered parameters is unknown . The population Π_i with $\lambda_i = \lambda_{[2]}$ (i=1,2) is called the better population .A correct selection is defined as the selection of the population associated with $\lambda_{[2]}$.

Many researchers have considered this problem under different types of formulations . Goel (1972) studied the problem of selecting a subset of k Poisson populations which includes the best , i.e. the one having the largest value of the parameter . Gupta and Nagel (1971) proposed a randomize selection rule for Poisson distribution . Alam and Thompson (1973) proposed a procedure to select simultaneously the population associated with largest parameter and estimate this parameter .

Gupta and Huang (1975) considered the selection from k Poisson populations a variable size subset including that population with the largest parameter when (equal) sample sizes are taken . Gupta and Wong (1977) discussed the problem of selecting a subset of k different Poisson processes including the best which is associated with the largest value of the mean rate .

Gupta , Leong and Wong (1979) considered the problem of selecting a subset of k Poisson populations including the best which is associated with the smallest value of the parameter .

Liang and Panchapakesan (1987) derived a Bayes rule having the isotonic property for selecting the Poisson populations superior to a control population under general loss function .

Gupta and Liang (1991) proposed an empirical Bayes method for Poisson selection problem , where the goal is to select all good populations and exclude all load populations .

Gupta and Liang (1999) studied the problem of selecting the most reliable Poisson population from among k competitors provided it is better than a control using nonparametric Bayes approach . Madhi and Hathoot (2005) proposed a Bayesian fixed sample size procedures for this problem .

2-Description of the Problem

Consider two independent Poisson populations Π_1, Π_2 with known probability density function

$$f(x;\lambda_i) = \frac{e^{-\lambda_i}\lambda_i^{\,x}}{x!}, x = 0,1,2,...; \lambda_i > 0$$

With unknown parameter λ_i (i=1,2) . We consider the problem : how to find the best population (i.e. the one associated with the largest parameter λ_i) . Let $\lambda_{[1]} \leq \lambda_{[2]}$ be the ordered values of the parameters λ_1, λ_2 . It is assumed that the exact pairing between the ordered and unordered parameters is unknown . The population Π_i with $\lambda_i = \lambda_{[2]}$ (i=1,2) is called the best population . A correct selection is defined as the selection of the population associated with $\lambda_{[2]}$.

The aim of this present paper is to derive approach for selecting the best of two Poisson populations , that is the one having the largest parameter $\lambda_{[2]}$ by using Bayesian decision – theoretic framework with Gamma prior and with general loss function .

3-Basic Definitions and Concepts

3-1-Statistical Decision Theory

(i) Basic Ideas [see 3]

Statistics December be consider as the science of decision making in the presence of uncertainty . The problems of statistical inferences can fit into the decision theory framework , for example , testing of a hypothesis H_o against a hypothesis H_1 December be regarded as a decision between two actions (i) accepting H_o or (ii) accepting H_1 .

In decision problems , the state of nature is unknown , but a decision maker must be made – a decision whose consequences depend on the unknown state of nature . Such a problem is a statistical decision problem when there are data that give partial information a bout the unknown state .

The basic elements of a statistical decision problem can be formalized mathematically as follows:

A set A , the action space , consisting of all possible actions , $a \in \mathrm{A}$, available to the decision maker ;

a set Ω , the parameter space , consisting of all possible 'state of the nature' , $\theta \in \Omega$, one and only one of which obtains or will obtain (this 'true' state being unknown to the decision-maker) ;

a function L , the loss function , having domain $\Omega \times \mathrm{A}$ (the set of all ordered pairs of consequences $(\theta, a), \theta \in \Omega, a \in \mathrm{A}$) and codomain R ;

a set R_x , the range of X , consisting of all the possible realizations , $x \in R_x$ of a random variable X , having a distribution whose probability function (pf) belongs to a specified family $\{f(x;\theta); \theta \in \Omega\}$;

A set D , the decision space, consisting of all possible decisions , $d \in D$, each such decision function d having domain R_x and codomain A .

(ii) The Risk Function [see 3]

For given (θ, a) the loss function depends on the outcome x and thus a random variable . Its expected value , i.e. its average over all possible outcomes is called the risk function and is denoted by

$$R(\theta, d) = \int_{R_x} L(\theta, d(x)) f(x;\theta) dx \quad (\text{X continues})$$

$$R(\theta, d) = \sum_{x \in R_x} L(\theta, d(x)) f(x;\theta) \qquad (\text{X discrete})$$

(iii) Minimax and Bayes Decision Functions [see 3]

The decision function d^* that minimizes $\mathrm{M(d)} = \max \mathrm{R}(\theta, \mathrm{d})$ is the minimax decision function . Similarly , the function d^{**} that minimizes the Bayes risk of a decision d is a Bayes decision function .

$$B(d) = E[R(\theta,d)] = \int_{\Omega} R(\theta,d)\pi(\theta)d\theta \quad (\Theta \text{continuous})$$

or

$$B(d) = \sum_{\Omega} R(\theta, d)\pi(\theta) \qquad (\Theta \ \text{discrete})$$

where $\pi(\theta)$ represents the distribution of degree of belief over Θ .

4- Solution of the Problem

We term our problem as a two-decision problem and represent it symbolically as

d_1 : population Π_1 *is the best if* $\lambda_1 \geq \lambda_2$
and
d_2 : population Π_2 *is the best if* $\lambda_1 < \lambda_2$

For parameter $\underline{\lambda}$ and action a , the loss function is defined as :

$$L_i(\lambda_i, a_i) = k_i(\lambda_i - a_i)^r \quad , \mathrm{i} = 1,2.$$

For r=0 , we have a constant loss function , for r=1 , we have a linear loss function and for r=2 , we have a quadratic loss function , k_1,k_2 give decision losses in units of costs .

Let us suppose that $\underline{X}_i = (x_{i1}, x_{i2}, ..., x_{in})$ be a random sample of size n arising from population Π_i. It follows that the likelihood function is

$$P_o(\underline{X}|\lambda_i) = \frac{\mathrm{e}^{-\sum_{j=1}^{\mathrm{n}} \mathrm{x}_\mathrm{j}}}{\prod_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{x}_\mathrm{j}!}, \quad \lambda_\mathrm{i} > 0 \, , \mathrm{x} > 0$$

Our first task in the Bayesian approach is the specification of a prior p.d.f g(λ) . we take the prior distribution to be a member of the conjugate class of Gamma priors $Gamma(\alpha_i, \beta_i)$, where a member of this class has density function

$$g(\lambda_i) = \frac{(\beta_i)^{\alpha_i}}{\Gamma(\alpha_i)} \lambda_i^{\alpha_i - 1} e^{-\beta_i \lambda_i} \quad , \alpha_\mathrm{i} > 0, \beta_i > 0, \lambda_i > 0$$

By Baye's theorem the posterior probability function of θ is given by

$$g(\lambda_i | \underline{X}_i) = \frac{(\beta_\mathrm{i}')^{\alpha_i'}}{\Gamma(\alpha_i')} \mathrm{e}^{-\beta_\mathrm{i}' \lambda_i} \lambda_\mathrm{i}^{\alpha_\mathrm{i}' - 1}$$

Where

$$\beta_i' = \beta_i + n \quad , \mathrm{i} = 1,2 \quad \text{and} \quad \alpha_i' = \alpha_i + \sum_{j=1}^{n} x_{ij}$$

We derive the stopping (Baye's) risks of decision d_1 and d_2 for general loss function given above and the stopping risk (the posterior expected looses) of making decision d_i denoted by $R_i(\theta_1, \theta_2; d_i)$

$$R_1(d_1) = k_1 \left[\sum_{i=0}^{r} \frac{r!(-1)^{r-i} \Gamma(\alpha_2' + r - i) \Gamma(\alpha_1' + i)}{i!(r-i)!\Gamma(\alpha_1')\Gamma(\alpha_2')(\beta_2')^{r-i}(\beta_1')^i} - \sum_{i=0}^{r} \sum_{j=0}^{\alpha_2' + r - i} \frac{r!(-1)^{r-i}(\beta_1')\Gamma(\alpha_2' + r - i)\Gamma(\alpha_1' + \alpha_2' + r - j)(\beta_2')^{\alpha_2' - j}}{i!(r-i)!\Gamma(\alpha_1')\Gamma(\alpha_2')(\alpha_2' + r - i - j)!(\beta_1' + \beta_2')^{\alpha_1' + \alpha_2' + r - j}} \right]$$

$$R_2(d_2) = k_2 \left[\sum_{i=0}^{r} \frac{r!(-1)^{r-i} \Gamma(\alpha_1' + r - i) \Gamma(\alpha_2' + i)}{i!(r-i)!\Gamma(\alpha_1')\Gamma(\alpha_2')(\beta_1')^{r-i}(\beta_2')^{\circ}} - \sum_{i=0}^{r} \sum_{j=0}^{\alpha_1' + r - i} \frac{r!(-1)^{r-i}(\beta_2')\Gamma(\alpha_1' + r - i)\Gamma(\alpha_1' + \alpha_2' + r - j)(\beta_1')^{\alpha_2' - j}}{i!(r-i)!\Gamma(\alpha_1')\Gamma(\alpha_2')(\alpha_1' + r - i - j)!(\beta_1' + \beta_2')^{\alpha_1' + \alpha_2' + r - j}} \right]$$

If we take r=0 we find from the above equations the posterior expected looses for constant loss function for the two decisions d_1 and d_2 , if we take r=1 we find from the above equations the posterior expected looses for linear loss function for the two decisions , if we take r=2 we find from the above equations the posterior expected looses for quadratic loss function for two decision d_1 and d_2 .

For the two – decision problem considered a above , the Bayesian selection procedure is given as follows :

Make decision d_1 that is selecting Π_1 as the best population if $R_1(\theta_1, \theta_2; d_1) \leq R_2(\theta_1, \theta_2; d_2)$

and

Make decision d_2 that is selecting Π_2 as the best population if $R_1(\theta_1, \theta_2; d_1) > R_2(\theta_1, \theta_2; d_2)$

4- Numerical Results and Discussions

This section contains some numerical result about this procedure , we take various sample size n and various priors . We write a program for this procedure from which we give three types of Risk for three types of loss functions (constant , linear and quadratic) . from this numerical result we note that :

1-the procedure is well defined , as we seen in table (1) and table(2) .

2-as sample size n increase , the Bayes risk decreases for all loss function .

3-The Bayes risk for quadratic loss function is less than the Bayes risk for linear and constant loss functions .

4-we generate a random sample from each population by using the function (poissrnd) in Math Works Matlab ver. 7.10.0 .

Conclusions

In this paper we derives a procedure for selecting the best of two Poisson populations employing a decision – theoretic Bayesian frame work with general loss function with Beta prior . From this paper we note that :

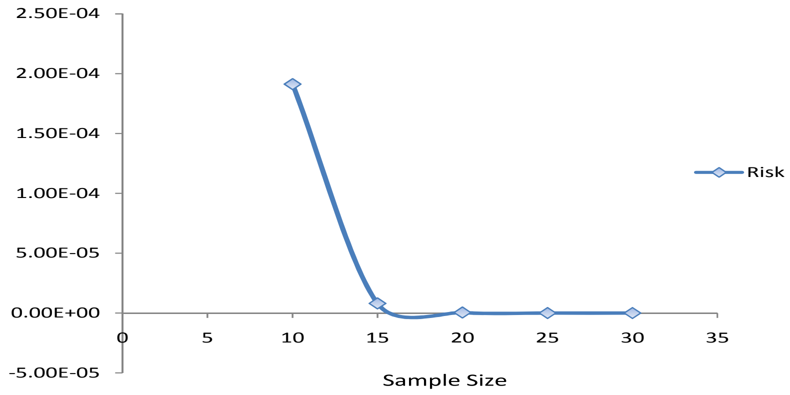
1- In this paper we derive approach for selecting the best of two Poisson populations by using Bayesian decision theory with general loss function .
2- In this procedure we can have Bayes risk for three loss function (constant, linear and quadratic) by using one equation .
3- the Bayes risk for quadratic loss function is less than the Bayes risk for linear and constant loss function .
4- from the numerical results in table(1) and (2) we saw that the procedure is well defined .
5- if we increase sample size , the Bayes risk will decreases for all loss functions as we saw in figure(1) , figure(2) and figure(3) .

Table (1) : The effect of sample size n on Bayes Risk (R_1 and R_2) , for fixed values of $\underline{\lambda} = (\lambda_1, \lambda_2)$ when $k_1=k_2=3$ and different prior

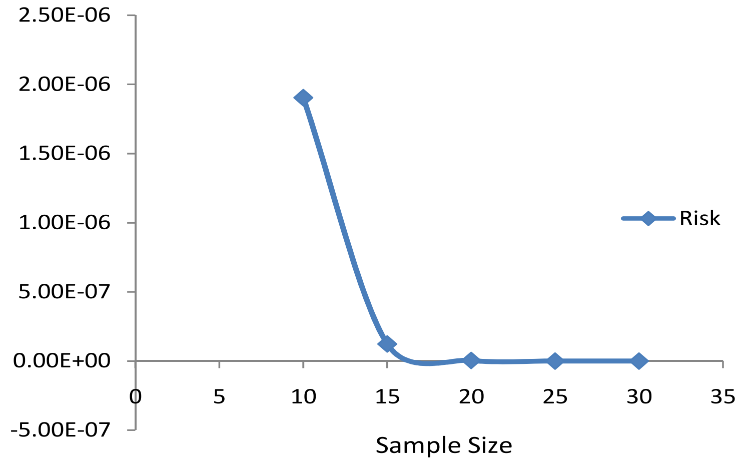
$\lambda_1 = 9$, $\lambda_2 = 2$, $k_1=k_2=3$, $prior = (2,3;4,3)$				
Bayes Risk	**n**	**Constant Loss**	**Linear Loss**	**Quadratic Loss**
$R_1(d_1)$	**10**	**1.0129E-004**	**1.9032E-006**	**8.5076E-008**
	15	**7.9995E-006**	**1.2330E-007**	**3.6477E-009**
	20	**3.8006E-007**	**4.467E-009**	**1.0142E-010**
	25	**1.2613E-008**	**1.1712E-010**	**2.1110E-012**
	30	**1.3293E-009**	**1.0829E-011**	**1.7145E-013**
$R_2(d_2)$	**10**	**2.9999**	**1.5380**	**0.8752**
	15	**3.0000**	**1.4143**	**0.7251**
	20	**3.0000**	**1.3632**	**0.6621**
	25	**3.0000**	**1.3384**	**0.6306**
	30	**3.000**	**1.3047**	**0.5952**
$\lambda_1 = 9$, $\lambda_2 = 2$, $k_1=k_2=3$, $prior = (4,7;6,5)$				
$R_1(d_1)$	**10**	**5.8666E-005**	**1.2258E-006**	**4.8907E-008**
	15	**4.4652E-006**	**6.8452E-008**	**2.0173E-009**
	20	**2.0350E-007**	**2.3801E-009**	**5.3830E-011**
	25	**6.4763E-009**	**5.9868E-011**	**1.0752E-012**
	30	**6.8915E-010**	**5.5884E-012**	**8.8429E-014**
$R_2(d_2)$	**10**	**2.9999**	**1.5800**	**0.9197**
	15	**3.0000**	**1.4543**	**0.7623**
	20	**3.0000**	**1.3686**	**0.6935**
	25	**3.0000**	**1.3686**	**0.6577**
	30	**3.0000**	**1.3309**	**0.6180**

Table (2) : The effect of sample size n on Bayes Risk (R_1 and R_2) , for fixed values of $\underline{\lambda} = (\lambda_1, \lambda_2)$ when $k_1=k_2=3$ and different prior

$\lambda_1 = 2$, $\lambda_2 = 9$, $k_1=k_2=3$, $prior = (2,3;4,3)$				
Bayes Risk	**n**	**Constant Loss**	**Linear Loss**	**Quadratic Loss**
$R_1(d_1)$	**10**	**2.9976**	**1.1989**	**0.5580**
	15	**2.9998**	**1.1837**	**0.5198**
	20	**3.0000**	**1.1884**	**0.5103**
	25	**3.0000**	**1.1982**	**0.5100**
	30	**3.0000**	**1.1867**	**0.4957**
$R_2(d_2)$	**10**	**0.0024**	**1.1837**	**0.5198**
	15	**1.7817E-004**	**3.1868E-006**	**1.0795E-007**
	20	**8.6710E-006**	**1.1481E-007**	**2.9099E-009**
	25	**3.0370E-007**	**3.1122E-009**	**6.1520E-011**
	30	**3.0127E-008**	**2.6738E-010**	**4.5968E-012**
$\lambda_1 = 2$, $\lambda_2 = 9$, $k_1=k_2=3$, $prior = (4,7;6,5)$				
$R_1(d_1)$	**10**	**2.9971**	**1.1257**	**0.4894**
	15	**2.9998**	**1.1365**	**0.4777**
	20	**3.0000**	**1.1547**	**0.4806**
	25	**3.0000**	**1.1724**	**0.4875**
	30	**3.0000**	**1.1652**	**0.4773**
$R_2(d_2)$	**10**	**0.0029**	**7.5685E-005**	**3.6792E-006**
	15	**2.1005E-004**	**3.8144E-006**	**1.3125E-007**
	20	**9.9418E-006**	**1.3319E-007**	**3.4171E-009**
	25	**3.3794E-007**	**3.4955E-009**	**6.9772E-011**
	30	**3.3787E-008**	**3.0232E-010**	**5.2420E-012**



Figure(1) : The influence of the sample size on the posterior expected loss for constant loss function



Figure(2) : The influence of the sample size on the posterior expected loss for Linear loss function

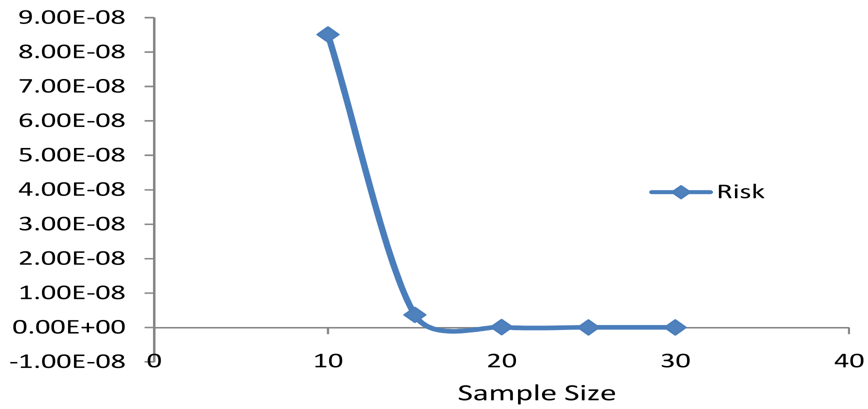


Figure (3) : The influence of the sample size on the posterior expected loss for quadratic loss function

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إجراءات بيزيه بحجم عينة ثابت مع دالة خسارة مشتركة لاختيار أفضل مجتمع من بين مجتمعيين بواسونيين

الخلاصة

يتضمن هذا البحث طريقة ذات حجم عينة ثابت لاختيار أفضل مجتمع من بين مجتمعيين يتبعان التوزيع البواسوني. وقد استخدم أسلوب القرار البيزي مع دالة خسارة مشتركة مع توزيع سابق للتجربة متمثل بتوزيع كاما لبناء هذه الطريقة .

وتضمن البحث أيضا نتائج عددية لهذا الاجراء مع دوال خسارة مختلفة ثابتة ، خطية وتربيعية حيث من معادلة واحدة بالإمكان الحصول على الخطورة البيزية لدوال الخسارة الثلاثة : الثابتة ، الخطية والتربيعية . النتائج العددية لهذا الأجراء تم ايجادها باستخدام نظام Matlab ver. 7.10.0