Open, closed and continuous function in bi-pre-supra topological space

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Abstract
In this paper we constructed a new space called bi-pre-supra topological space. Many concepts \((\mathcal{T}, \mathcal{PT})\)-open set, \((\mathcal{T}, \mathcal{P}^*\mathcal{T})\)-open set, bi-open set) were introduced. At last through this paper we introduced a new class of functions (open, closed and continuous) in bi-pre-supra topological space. We study and investigate some properties and characterization of above concepts.

Keywords: \((\mathcal{T}, \mathcal{PT})\)-open function, \((\mathcal{T}, \mathcal{P}^*\mathcal{T})\)-open function, bi-open function \((\mathcal{T}, \mathcal{PT})\)-closed function, \((\mathcal{T}, \mathcal{P}^*\mathcal{T})\)-closed function, bi-closed function, \((\mathcal{T}, \mathcal{PT})\)-continuous function, \((\mathcal{T}, \mathcal{P}^*\mathcal{T})\)-continuous function, bi-continuous function.

1-Introduction
In 1963 Kelley J. C. [5] was first introduced the concept of bi-topological spaces, where \(X\) is a non-empty set and \(\mathcal{T}_1, \mathcal{T}_2\) are topologies on \(X\). In 1982 Almashhor [1] introduced the concept of pre-open sets in topological space. By using this concept, several authors’ [4], [6], [7] defined and studies stronger or weaker types of topological concept.

In this paper, we introduced the concepts of bi-pre-supra topological space, via \((\mathcal{T}, \mathcal{PT})\)-open set, \((\mathcal{T}, \mathcal{P}^*\mathcal{T})\)-open set and bi-open set in bi-pre-supra topological space, and we study their basic properties and relationships with other concepts of sets. At last through this paper we introduced a new class of functions (open, closed and continuous) in bi-pre-supra topological space. We study and investigate some properties and characterization of above concepts.

2-Preliminaries
Definition 2.1 [1] A subset \(A\) of a space \((X, \mathcal{T})\) is called pre-open, if \(A \subseteq \text{int (cl}(A))\). The complement of pre-open set is said to be pre-closed.

Definition 2.2 [2] A subfamily \(\mathcal{T}\) of a family of subset of \(X\) is said to be a supra topology on \(X\) if:
1) \(X, \emptyset \in \mathcal{T}\)
2) If \(A_i \in \mathcal{T}\) for all \(i \in I\) then \(\bigcup A_i \in \mathcal{T}\)
\((X, \mathcal{T})\) is called a supra topological space. The element of \(\mathcal{T}\) are called supra open set in \((X, \mathcal{T})\) and complement of a supra open set is called a supra closed set.
Definition 2.3 [7] Let \((X, \mathcal{T}_1, \mathcal{T}_2)\) be a bi-topological space, and let \(G\) be a subset of \(X\). Then \(G\) is said to be \((i,j)\)-open set if \(G = A \cup B\) where \(A \in \mathcal{T}_1\) and \(B \in \mathcal{T}_2\). The complement of \((i,j)\)-open set is called \((i,j)\)-closed set.

Remark 2.4 [7] Notice that \((i,j)\)-open set need not necessarily form a topology.

Definition 2.4 [3] A subset \(A\) of a bi-topological space \((X, \mathcal{T}_1, \mathcal{T}_2)\) is called \((i,j)\)-neighborhood of a point \(x\) in \(X\) if there exists an \((i,j)\)-open set \(G\) such that \(x \in G \subseteq A\). And denoted \((i,j)\)-nbd.

Definition 2.5 [3] Let \(A\) be a subset of bi-topological space \((X, \mathcal{T}_1, \mathcal{T}_2)\). A point \(x\) in \(X\) is said to \((i,j)\)-limit point of \(A\) if for each \((i,j)\)-open set \(G\) containing \(x\) such that \(A \cap (G \setminus \{x\}) \neq \emptyset\). The set of all \((i,j)\)-limit point of \(A\) is called \((i,j)\)-derived set of \(A\) and denoted by \((i,j)\)-d\((A)\).

Definition 2.6 [7] Let \(A\) be a subset of bi-topological space \((X, \mathcal{T}_1, \mathcal{T}_2)\). Then the \((i,j)\)-closure of \(G\) denoted by \((i,j)\)-cl\((A)\), is defined by \(\bigcap\{F : A \subseteq F \text{ and } F \text{ is } (i,j)\text{-closed set}\}\).

Definition 2.7 [7] Let \(A\) be a subset of bi-topological space \((X, \mathcal{T}_1, \mathcal{T}_2)\). Then the \((i,j)\)-interior of \(A\) denoted by \((i,j)\)-int\((A)\), is defined by \(\bigcup\{G : G \subseteq A \text{ and } F \text{ is } (i,j)\text{-open set}\}\).

Definition 2.8 [8] A function \(f:(X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)\) is called open function if the image of every open set is open.

Definition 2.9 [8] A function \(f:(X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)\) is called closed function if the image of every closed set is closed.

Definition 2.10 [8] A function \(f:(X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)\) is called continuous function if the inverse image of any \(\mathcal{T}_Y\)-open set \(G\) is \(\mathcal{T}_X\)-open set.

3-Bi-pre-supra topological spaces

Definition 3.1 Let \(X\) be a non-empty set, let \(\mathcal{T}\) be a topology on \(X\) and let \(\mathcal{PT}\) is the set of all pre-open subset of \(X\) (for short \(\text{Po}(X)\)), then We say that \((X, \mathcal{T}, \mathcal{PT})\) is a bi-pre-supra topological space.

Now the deference between bi-topological space [Kelly] and bi-pre-supra topological space \(\mathcal{PT}\) is supra topology not topology.

Example 3.2 Let \(X = \{1,2,3,4\}\)
\[
\mathcal{T} = \{\emptyset, X, \{1\}, \{2,3\}, \{1,2,3\}\}
\]
\[
\text{Po}(X) = \mathcal{PT} = \{\emptyset, X, \{1\}, \{2,3\}, \{1,2,3\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{1,2,4\}, \{1,3,4\}\}
\]
\((X, \mathcal{T}, \mathcal{PT})\) is bi-pre-supra topological space

Definition 3.3 Let \((X, \mathcal{T}, \mathcal{PT})\) be a bi-pre-supra topological space, and let \(G\) be a subset of \(X\). Then

i) \(G\) is said to be \((\mathcal{T}, \mathcal{PT})\)-open set if \(G = A \cup B\) where \(A \in \mathcal{T}\) and \(B \in \mathcal{PT}\).

ii) The complement of \((\mathcal{T}, \mathcal{PT})\)-open set is called \((\mathcal{T}, \mathcal{PT})\)-closed set.

iii) \(G\) is said to be \((\mathcal{T}, \mathcal{PT})^*\)-open set if \(G = A \cup B\) where \(A \in \mathcal{T}\), \(B \in \mathcal{PT}\) and \(B \notin \mathcal{T}\).

iv) The complement of \((\mathcal{T}, \mathcal{PT})^*\)-open set is called \((\mathcal{T}, \mathcal{PT})^*\)-closed set.
v) G is said to be bi-open set if \( G = A \) where \( A \in \mathcal{T} \) and \( A \in \mathcal{P} \mathcal{T} \).

vi) The complement of bi-open set is called bi-closed set.

**Proposition 3.4**

1) Every bi-open set is \((\mathcal{T}, \mathcal{P} \mathcal{T})\)-open set and every bi-closed set is \((\mathcal{T}, \mathcal{P} \mathcal{T})\)-closed set but the converse is not true.

2) Every \((\mathcal{T}, \mathcal{P} \mathcal{T})^*\)-open set is \((\mathcal{T}, \mathcal{P} \mathcal{T})\)-open set and every \((\mathcal{T}, \mathcal{P} \mathcal{T})^*\)-closed set is \((\mathcal{T}, \mathcal{P} \mathcal{T})\)-closed set but the converse is not true.

**Example 3.5**

Let \( X = \{1, 2, 3, 4\} \)
\[ \mathcal{T} = \{\emptyset, X, \{2\}, \{1,3\}, \{1,2,3\}\} \]
\[ \mathcal{T}^c = \{\emptyset, X, \{1,3,4\}, \{2,4\}, \{4\}\} \]
\[ \mathcal{P} \mathcal{T} = \{\emptyset, X, \{4\}, \{1,3\}, \{1,3,4\}, \{1,2,4\}, \{2,3\}, \{2,4\}\} \]
\[(\mathcal{T}, \mathcal{P} \mathcal{T})\)-open sets = \{\emptyset, X, \{2\}, \{1,2,3\}, \{2,3\}, \{1,2\}, \{1,3\}, \{1,2,4\}, \{2,3,4\}\}
\[(\mathcal{T}, \mathcal{P} \mathcal{T})\)-closed sets = \{\emptyset, X, \{1,3,4\}, \{3\}, \{2,3\}, \{1,2\}\}
\[(\mathcal{T}, \mathcal{P} \mathcal{T})^*\)-open sets = \{\emptyset, X, \{4\}, \{1,3,4\}, \{1,2\}, \{2,4\}\}
\[(\mathcal{T}, \mathcal{P} \mathcal{T})^*\)-closed sets = \{\emptyset, X, \{1,4\}, \{3\}, \{1\}, \{2,4\}\}

**Definition 3.6** Let \((X, \mathcal{T}, \mathcal{P} \mathcal{T})\) be a bi-pre-supra topological space, and let \( A \) be a subset of \( X \). Then

i. The \((\mathcal{T}, \mathcal{P} \mathcal{T})\)-closure of \( G \) denoted by \((\mathcal{T}, \mathcal{P} \mathcal{T})\)-cl\((A)\), is defined by \( \cap \{ F : A \subseteq F \text{ and } F \text{ is } (\mathcal{T}, \mathcal{P} \mathcal{T})\text{-closed set} \} \)

ii. The \((\mathcal{T}, \mathcal{P} \mathcal{T})^*\)-closure of \( A \) denoted by \((\mathcal{T}, \mathcal{P} \mathcal{T})^*\)-cl\((A)\), is defined by \( \cap \{ F : A \subseteq F \text{ and } F \text{ is } (\mathcal{T}, \mathcal{P} \mathcal{T})^*\text{-closed set} \} \)

iii. The bi- closure of \( A \) denoted by bi-cl\((A)\), is defined by \( \cap \{ F : A \subseteq F \text{ and } F \text{ is bi-closed set} \} \)

**Example 3.7** Let \( X = \{1, 2, 3, 4\} \)
\[ \mathcal{T} = \{\emptyset, X, \{4\}, \{1,3\}, \{1,3,4\}\} \]
\[ \mathcal{T}^c = \{\emptyset, X, \{1,3,4\}, \{2,4\}, \{4\}\} \]
\[ \mathcal{P} \mathcal{T} = \{\emptyset, X, \{4\}, \{1,3\}, \{1,3,4\}, \{1\}, \{3\}, \{1,4\}, \{3,4\}, \{1,2,4\}, \{2,3\}, \{2,3,4\}\} \]
\[(\mathcal{T}, \mathcal{P} \mathcal{T})\)-open sets = \{\emptyset, X, \{4\}, \{1,3,4\}, \{1,4\}, \{3,4\}, \{1,2,4\}, \{2,3,4\}, \{1,3\}\}
\[(\mathcal{T}, \mathcal{P} \mathcal{T})\)-closed sets = \{\emptyset, X, \{1,2,3\}, \{2\}, \{1,2\}\}
\[(\mathcal{T}, \mathcal{P} \mathcal{T})^*\)-open sets = \{\emptyset, X, \{1,4\}, \{3,4\}, \{1,2,4\}, \{2,3,4\}, \{1,3\}\}
\[(\mathcal{T}, \mathcal{P} \mathcal{T})^*\)-closed sets = \{\emptyset, X, \{3\}, \{1\}, \{2,4\}\}

bi-open sets = \{\emptyset, X, \{2\}, \{1,3\}\}
bi-closed sets = \{\emptyset, X, \{1,3,4\}, \{2,4\}\}

Take \( G = \{1,2\} \), \( H = \{1,2,3\} \)
\( (\mathcal{T}, \mathcal{P} \mathcal{T})\)-cl\((G)\) = \{1,2\}
\( \text{bi-cl}(G) = \{1,2,3\} \)
\( (\mathcal{T}, \mathcal{P} \mathcal{T})\)-cl\((H)\) = \{1,2,3\}
\( (\mathcal{T}, \mathcal{P} \mathcal{T})^*\)-cl\((H)\) = \{1,2,3\}
Definition 3.8 Let \((X, \mathcal{T}, \mathcal{P}_\mathcal{T})\) be a bi-pre-supra topological space, and let \(A\) be a subset of \(X\). Then:

(i) The \((\mathcal{T}, \mathcal{P}_\mathcal{T})\)-interior of \(A\) denoted by \((\mathcal{T}, \mathcal{P}_\mathcal{T})\)-int\(\{A\}\), is defined by \(\bigcup\{F : F \subseteq A \text{ and } F \text{ is } (\mathcal{T}, \mathcal{P}_\mathcal{T})\text{-open set}\}\).

(ii) The \((\mathcal{T}, \mathcal{P}_\mathcal{T})^*\)-interior of \(A\) denoted by \((\mathcal{T}, \mathcal{P}_\mathcal{T})^*\)-int\(\{A\}\), is defined by \(\bigcup\{F : F \subseteq A \text{ and } F \text{ is } (\mathcal{T}, \mathcal{P}_\mathcal{T})^*\text{-open set}\}\).

(iii) The bi-interior of \(A\) denoted by bi-int\(\{A\}\), is defined by \(\bigcup\{F : F \subseteq A \text{ and } F \text{ is bi-open set}\}\).

Example 3.9 Let \(X = \{1,2,3,4\}\)
\[\mathcal{T} = \{\emptyset, X, \{1\}, \{2,4\}, \{1,2,4\}\}\]
\[\mathcal{T}^c = \{\emptyset, X, \{2,3,4\}, \{1,3\}, \{3\}\}\]
\[\mathcal{P}_\mathcal{T} = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,4\}, \{1,2,3\}, \{1,3,4\}, \{2,4\}\}\]

\((\mathcal{T}, \mathcal{P}_\mathcal{T})\)-open sets = \(\{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}\}\)

\((\mathcal{T}, \mathcal{P}_\mathcal{T})^*\)-open sets = \(\{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,3,4\}, \{2,4\}\}\)

bi-open sets = \(\{\emptyset, X, \{1\}, \{2\}, \{1,2,4\}\}\)

Take \(G = \{1,2,3\}\)
\((\mathcal{T}, \mathcal{P}_\mathcal{T})\)-int\(\{G\}\) = \(\{1,2,3\}\)
\((\mathcal{T}, \mathcal{P}_\mathcal{T})^*\)-int\(\{G\}\) = \(\{1,2,3\}\)

bi-int\(\{G\}\) = \(\{1\}\)

4-Open and closed function in bi-pre-supra topological space

In this section we introduce a new class of open and closed function in bi-pre-supra topological space.

Definition 4.1 A function \(f:(X, \mathcal{T}_X, \mathcal{P}_\mathcal{T}_X) \to (Y, \mathcal{T}_Y, \mathcal{P}_\mathcal{T}_Y)\) is called

1- bi\((\mathcal{T}, \mathcal{P}_\mathcal{T})\)-open function if the image of every \((\mathcal{T}_X, \mathcal{P}_\mathcal{T}_X)\)-open set is \((\mathcal{T}_Y, \mathcal{P}_\mathcal{T}_Y)\)-open.

2- bi\((\mathcal{T}, \mathcal{P}_\mathcal{T})^*\)-open function if the image of every \((\mathcal{T}_X, \mathcal{P}_\mathcal{T}_X)^*\)-open set is \((\mathcal{T}_Y, \mathcal{P}_\mathcal{T}_Y)^*\)-open.

3- bi-open function if the image of every bi-open set is bi-open

Definition 4.2 A function \(f:(X, \mathcal{T}_X, \mathcal{P}_\mathcal{T}_X) \to (Y, \mathcal{T}_Y, \mathcal{P}_\mathcal{T}_Y)\) is called

1- bi\((\mathcal{T}, \mathcal{P}_\mathcal{T})\)-closed function if the image of every \((\mathcal{T}_X, \mathcal{P}_\mathcal{T}_X)\)-closed set is \((\mathcal{T}_Y, \mathcal{P}_\mathcal{T}_Y)\)-closed.

2- bi\((\mathcal{T}, \mathcal{P}_\mathcal{T})^*\)-closed function if the image of every \((\mathcal{T}_X, \mathcal{P}_\mathcal{T}_X)^*\)-closed set is \((\mathcal{T}_Y, \mathcal{P}_\mathcal{T}_Y)^*\)-closed.

3- bi-closed function if the image of every bi-closed set is bi-closed

Example 4.3
\[X = \{1,2,3,4\}\]
\[\mathcal{T}_X = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}\}\]
\[\mathcal{T}_X^c = \{\emptyset, X, \{2,3,4\}, \{1,3\}, \{3\}\}\]
\[\mathcal{P}_\mathcal{T}_X = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,4\}\}\]

\((\mathcal{T}_X, \mathcal{P}_\mathcal{T}_X)\)-open sets = \(\{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}\}\)

\((\mathcal{T}_X, \mathcal{P}_\mathcal{T}_X)^*\)-open sets = \(\{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,3,4\}\}\)

\((\mathcal{T}_X, \mathcal{P}_\mathcal{T}_X)^*\)-open sets = \(\{\emptyset, X, \{1,2\}, \{1,2,3\}, \{1,4\}\}\)

\((\mathcal{T}_X, \mathcal{P}_\mathcal{T}_X)^*\)-closed sets = \(\{\emptyset, X, \{4\}, \{3\}\}\)
\[ Y = \{a, b, c, d\} \]
\[ T_Y = \{\emptyset, Y, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\} \]
\[ T_Y^c = \{\emptyset, Y, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \{d\}\} \]
\[ \mathcal{P}_T = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\} \]

- \((T_Y, \mathcal{P}_T)\)-open sets = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}
- \((T_Y, \mathcal{P}_T)\)-closed sets = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}
- \((T_Y, \mathcal{P}_T)\)-open sets = \{\emptyset, Y, \{a\}, \{a, b\}, \{a, c\}, \{a, d\\}\}
- \((T_Y, \mathcal{P}_T)\)-closed sets = \{\emptyset, Y, \{a\}, \{b\}, \{c\}\}

Let \(f : (X, T_X, \mathcal{P}_T_X) \rightarrow (Y, T_Y, \mathcal{P}_T_Y)\) defined by \(f(1) = a\), \(f(2) = b\), \(f(3) = c\), \(f(4) = d\). Then all types of function in def.[4.1],[4.2] are holding.

**Diagram 4.4**

The following diagram is valid

- \(bi-(T, \mathcal{P}_T)\)-open function
- \(bi-(T, \mathcal{P}_T)\)-closed function
- \(bi\)-open function

**Example 4.5**

\[ X = \{1, 2, 3, 4\} \]
\[ T_X = \{\emptyset, X, \{1\}\} \]
\[ T_X^c = \{\emptyset, X, \{2, 3, 4\}\} \]
\[ \mathcal{P}_T_X = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\} \]
- \((T_X, \mathcal{P}_T_X)\)-open sets = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}
- \((T_X, \mathcal{P}_T_X)\)-closed sets = \{\emptyset, X, \{2, 3, 4\}, \{1, 3, 4\}, \{3, 4\}\}

Let \(f : (Y, T_Y, \mathcal{P}_T_Y) \rightarrow (X, T_X, \mathcal{P}_T_X)\) defined by \(f(1) = a\), \(f(2) = b\), \(f(3) = c\), \(f(4) = d\). Then \(f\) is \(bi-(T, \mathcal{P}_T)\)-open (closed) function not bi-open (closed) function.

**Example 4.6**

\[ X = \{1, 2, 3, 4\} \]
\[ T_X = \{\emptyset, X, \{1\}, \{2\}\} \]
\[ T_X^c = \{\emptyset, X, \{2, 3, 4\}, \{1, 3, 4\}\} \]
\[ \mathcal{P}_X = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{4\}, \{1,2,3\}, \{1,2,4\}\} \]
\[ (\mathcal{T}_X, \mathcal{P}_X) \text{-open sets} = \{\emptyset, X, \{1\}, \{2\}, \{1,2,3\}, \{1,2,4\}\} \]
\[ (\mathcal{T}_X, \mathcal{P}_X) \text{-closed sets} = \{\emptyset, X, \{2,3,4\}, \{3,4\}, \{1,3,4\}, \{4\}, \{3\}\} \]
\[ (\mathcal{T}_X, \mathcal{P}_X)^* \text{-open sets} = \{\emptyset, X, \{1,2\}, \{1,2,4\}\} \]
\[ (\mathcal{T}_X, \mathcal{P}_X)^* \text{-closed sets} = \{\emptyset, X, \{4\}, \{3\}, \{4\}\} \]
\[ Y = \{a, b, c, d\} \]
\[ \mathcal{T}_Y = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,d\}\} \]
\[ \mathcal{T}_Y^* = \{\emptyset, Y, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\} \]
\[ \mathcal{P}_Y = \{\emptyset, Y, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\} \]
\[ (\mathcal{T}_Y, \mathcal{P}_Y) \text{-open sets} = \{\emptyset, Y, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\} \]
\[ (\mathcal{T}_Y, \mathcal{P}_Y) \text{-closed sets} = \{\emptyset, Y, \{b,c,d\}, \{a,c,d\}, \{c,d\}, \{d\}, \{c\}\} \]
\[ (\mathcal{T}_Y, \mathcal{P}_Y)^* \text{-open sets} = \{\emptyset, Y, \{a,b,c\}\} \]
\[ (\mathcal{T}_Y, \mathcal{P}_Y)^* \text{-closed sets} = \{\emptyset, Y, \{d\}\} \]

Let \( f : (X, \mathcal{T}_X, \mathcal{P}_X) \rightarrow (Y, \mathcal{T}_Y, \mathcal{P}_Y) \) defined by
\[
\begin{align*}
f(1) & = a, \quad f(2) = b, \quad f(3) = c, \quad f(4) = d. \end{align*}
\]
Then \( f \) is bi-(\( \mathcal{T}_X, \mathcal{P}_X \))-open (closed) function not bi-
\( \mathcal{T}, \mathcal{P}^* - \mathcal{T} - \mathcal{P} \)-open (closed) function.

**Theorem 4.7**

A function \( f : (X, \mathcal{T}_X, \mathcal{P}_X) \rightarrow (Y, \mathcal{T}_Y, \mathcal{P}_Y) \) is bi-(\( \mathcal{T}_X, \mathcal{P}_X \))-open iff \( f((\mathcal{T}_X, \mathcal{P}_X)-\text{int}(A)) \subseteq (\mathcal{T}_Y, \mathcal{P}_Y)-\text{int}(f(A)) \) for all \( A \subseteq X \)

**Proof:**

Let \( f \) be-(\( \mathcal{T}_X, \mathcal{P}_X \))-open function and \( A \subseteq X \)

Since \( (\mathcal{T}_X, \mathcal{P}_X)-\text{int}(A) \) is \( (\mathcal{T}_X, \mathcal{P}_X) \)-open set and \( f \) is bi-(\( \mathcal{T}_X, \mathcal{P}_X \))-open function then

\( f((\mathcal{T}_X, \mathcal{P}_X)-\text{int}(A)) \) is \( (\mathcal{T}_Y, \mathcal{P}_Y) \)-open set subset of \( Y \)

Since \( (\mathcal{T}_Y, \mathcal{P}_Y)-\text{int}(A) \subseteq A \) then:

\( f((\mathcal{T}_Y, \mathcal{P}_Y)-\text{int}(A)) \subseteq (\mathcal{T}_Y, \mathcal{P}_Y)-\text{int}(f(A)) \)

Conversely:

Suppose that the condition is true and \( A \) is \( (\mathcal{T}_X, \mathcal{P}_X) \)-open set subset of \( X \)

Now \( f(A) = f((\mathcal{T}_X, \mathcal{P}_X)-\text{int}(A)) \subseteq (\mathcal{T}_Y, \mathcal{P}_Y)-\text{int}(f(A)) \)

i.e \( f(A) = (\mathcal{T}_Y, \mathcal{P}_Y)-\text{int}(f(A)) \)

then \( f(A) \) is \( (\mathcal{T}_Y, \mathcal{P}_Y) \)-open

**Theorem 4.8**

A function \( f : (X, \mathcal{T}_X, \mathcal{P}_X) \rightarrow (Y, \mathcal{T}_Y, \mathcal{P}_Y) \) is bi-(\( \mathcal{T}_X, \mathcal{P}_X \))-closed iff \( (\mathcal{T}_X, \mathcal{P}_X)-\text{cl}(f(A)) \subseteq f((\mathcal{T}_Y, \mathcal{P}_Y)-\text{cl}(A)) \) for all \( A \subseteq X \).

**Proof:**

Let \( f \) be-(\( \mathcal{T}_X, \mathcal{P}_X \))-closed function and \( A \subseteq X \)

Since \( (\mathcal{T}_X, \mathcal{P}_X)-\text{cl}(A) \) is \( (\mathcal{T}_X, \mathcal{P}_X) \)-closed set and \( f \) is bi-(\( \mathcal{T}_X, \mathcal{P}_X \))-closed function then

\( f((\mathcal{T}_X, \mathcal{P}_X)-\text{cl}(A)) = (\mathcal{T}_Y, \mathcal{P}_Y)-\text{cl}(f((\mathcal{T}_X, \mathcal{P}_X)-\text{cl}(A))) \)

But \( A \subseteq (\mathcal{T}_Y, \mathcal{P}_Y)-\text{cl}(A) \)

This \( f(A) \subseteq f((\mathcal{T}_X, \mathcal{P}_X)-\text{cl}(A)) \)

\( \Rightarrow (\mathcal{T}_Y, \mathcal{P}_Y)-\text{cl}(f(A)) \subseteq (\mathcal{T}_Y, \mathcal{P}_Y)-\text{cl}(f((\mathcal{T}_X, \mathcal{P}_X)-\text{cl}(A))) \)

\( \Rightarrow (\mathcal{T}_Y, \mathcal{P}_Y)-\text{cl}(f(A)) \subseteq f((\mathcal{T}_X, \mathcal{P}_X)-\text{cl}(A)) \)
Conversely:
If the condition is true and $A \subseteq X$ closed set
Then $(\mathcal{T}, \mathcal{P}\mathcal{T})$-cl$(f(A)) \subseteq f((\mathcal{T}, \mathcal{P}\mathcal{T})$-cl$(A)) = f(A)$
i.e $(\mathcal{T}, \mathcal{P}\mathcal{T})$-cl$(f(A)) = f(A)$
Then $f(A)$ is $(\mathcal{T}, \mathcal{P}\mathcal{T})$-closed set subset of $Y$.

5- Continuous function in bi-pre-supra topological space
In this section we introduce a new class of continuous function in bi-pre-supra topological space.

Definition 5.1 A function $f : (X, \mathcal{T}_X, \mathcal{P}\mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y, \mathcal{P}\mathcal{T}_Y)$ is called
1- bi-$(\mathcal{T}, \mathcal{P}\mathcal{T})$-continuous function if the inverse image of any $(\mathcal{T}_Y, \mathcal{P}\mathcal{T}_Y)$-open set $G$ is $(\mathcal{T}_X, \mathcal{P}\mathcal{T}_X)$-open set.
2- bi-$(\mathcal{T}, \mathcal{P}\mathcal{T})^*$-continuous function if the inverse image of any $(\mathcal{T}_Y, \mathcal{P}\mathcal{T}_Y)^*$-open set $G$ is $(\mathcal{T}_X, \mathcal{P}\mathcal{T}_X)^*$-open set.
3- bi-continuous function if the inverse image of any bi-open set is bi-open.

Example 5.2 $X = \{1, 2, 3, 4\}$
$\mathcal{T}_X = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$
$\mathcal{P}\mathcal{T}_X = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$
$\mathcal{T}_Y = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$
$\mathcal{P}\mathcal{T}_Y = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$

Let $f : (X, \mathcal{T}_X, \mathcal{P}\mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y, \mathcal{P}\mathcal{T}_Y)$ defined by $f(1) = a$ $f(2) = b$ $f(3) = c$ $f(4) = d$.

Diagram 5.3
The following diagram is valid

Example 5.4
$X = \{1, 2, 3, 4\}$
$\mathcal{T}_X = \{\emptyset, X, \{1\}\}$
$\mathcal{P}\mathcal{T}_X = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}, \{1, 4\}, \{1, 2, 4\}, \{1, 3, 4\}\}$
$\mathcal{T}_Y = \{\emptyset, Y, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}\}$
$\mathcal{P}\mathcal{T}_Y = \{\emptyset, Y, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}\}$
(\mathcal{T}_Y, \mathcal{P}_T)-open sets = \{\emptyset, Y, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\}

(\mathcal{T}_Y, \mathcal{P}_T)^*\text{-open sets} = \{\emptyset, Y, \{a\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\}

Let \(f:(Y, \mathcal{T}_Y, \mathcal{P}_T) \rightarrow (X, \mathcal{T}_X, \mathcal{P}_X)\) defined by

\[
\begin{align*}
f(a) &= 1 \\
f(b) &= 2 \\
f(c) &= 3 \\
f(d) &= 4
\end{align*}
\]

Then \(f\) is bi-(\(\mathcal{T}, \mathcal{P}_T\))-continuous function not bi-continuous function.

**Example 5.5**

\(X = \{1,2,3,4\}\)

\(\mathcal{T}_X = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}\}\)

\(\mathcal{P}_X = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}\}\)

\(\mathcal{T}_Y = \{\emptyset, Y, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}\)

\(\mathcal{P}_Y = \{\emptyset, Y, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}\)

Let \(f:(X, \mathcal{T}_X, \mathcal{P}_X) \rightarrow (Y, \mathcal{T}_Y, \mathcal{P}_Y)\) defined by

\[
\begin{align*}
f(1) &= a \\
f(2) &= b \\
f(3) &= c \\
f(4) &= d
\end{align*}
\]

Then \(f\) is bi-(\(\mathcal{T}, \mathcal{P}_T\))-continuous function not bi-(\(\mathcal{T}, \mathcal{P}_T^*\))-continuous function.

**Theorem 5.6**

Let the function \(f:(X, \mathcal{T}_X, \mathcal{P}_X) \rightarrow (Y, \mathcal{T}_Y, \mathcal{P}_Y)\) and \(g:(Y, \mathcal{T}_Y, \mathcal{P}_Y) \rightarrow (Z, \mathcal{T}_Z, \mathcal{P}_Z)\) be bi-(\(\mathcal{T}, \mathcal{P}_T\))-continuous. Then the composition function \(gof:(X, \mathcal{T}_X, \mathcal{P}_X) \rightarrow (Z, \mathcal{T}_Z, \mathcal{P}_Z)\) is also bi-(\(\mathcal{T}, \mathcal{P}_T\))-continuous.

**Proof:**

Let \(G\) be an \((\mathcal{T}, \mathcal{P}_T)\)-open subset of \(Z\).

Then \(g^{-1}(G)\) is \((\mathcal{T}, \mathcal{P}_T)\)-open in \(Y\) since \(g\) is continuous.

But \(f\) is also bi-(\(\mathcal{T}, \mathcal{P}_T\))-continuous, so \(f^{-1}[g^{-1}(G)]\) is \((\mathcal{T}, \mathcal{P}_T)\)-open in \(X\).

Now \((gof)^{-1}(G) = f^{-1}[g^{-1}(G)]\)

Thus \((gof)^{-1}(G)\) is \((\mathcal{T}, \mathcal{P}_T)\)-open in \(X\) for every \((\mathcal{T}, \mathcal{P}_T)\)-open subset \(G\) of \(Z\).

\(gof\) is continuous.

**Theorem 5.7**

A function \(f:(X, \mathcal{T}_X, \mathcal{P}_X) \rightarrow (Y, \mathcal{T}_Y, \mathcal{P}_Y)\) is bi-(\(\mathcal{T}, \mathcal{P}_T\))-continuous iff the inverse image of every \((\mathcal{T}, \mathcal{P}_T)\)-closed subset of \(Y\) is a \((\mathcal{T}, \mathcal{P}_T)\)-closed subset of \(X\).

**Proof:**

Suppose \(f:(X, \mathcal{T}_X, \mathcal{P}_X) \rightarrow (Y, \mathcal{T}_Y, \mathcal{P}_Y)\) is bi-(\(\mathcal{T}, \mathcal{P}_T\))-continuous. Let \(F\) be a \((\mathcal{T}, \mathcal{P}_T)\)-closed subset of \(Y\).

Then \(F^c\) is \((\mathcal{T}, \mathcal{P}_T)\)-open, and so \(f^{-1}(F^c)\) is \((\mathcal{T}, \mathcal{P}_T)\)-open in \(X\).

But \(f^{-1}(F^c) = [f^{-1}(F)]^c\)

Therefore \(f^{-1}(F)\) is \((\mathcal{T}, \mathcal{P}_T)\)-closed.

**Conversely:**

Assume \(F\) is \((\mathcal{T}, \mathcal{P}_T)\)-closed in \(Y\) implies \(f^{-1}(F)\) is \((\mathcal{T}, \mathcal{P}_T)\)-closed in \(X\).

Let \(G\) be an \((\mathcal{T}, \mathcal{P}_T)\)-open subset of \(Y\).

Then \(G^c\) is \((\mathcal{T}, \mathcal{P}_T)\)-closed in \(Y\), and so \(f^{-1}(G^c) = [f^{-1}(G)]^c\) is \((\mathcal{T}, \mathcal{P}_T)\)-closed in \(X\).

Accordingly, \(f^{-1}(G)\) is \((\mathcal{T}, \mathcal{P}_T)\)-open and therefore \(f\) is bi-(\(\mathcal{T}, \mathcal{P}_T\))-continuous.
Theorem 5.8
A function $f: (X, \mathcal{T}_X, \mathcal{PT}_X) \to (Y, \mathcal{T}_Y, \mathcal{PT}_Y)$ is bi-$(\mathcal{T}, \mathcal{PT})$-continuous iff for every subset $G \subseteq X$, $f((\mathcal{T}, \mathcal{PT})-\text{cl}(G)) \subseteq (\mathcal{T}, \mathcal{PT})-\text{cl}(f(G))$.

Proof:
Suppose $f: (X, \mathcal{T}_X, \mathcal{PT}_X) \to (Y, \mathcal{T}_Y, \mathcal{PT}_Y)$ is bi-$(\mathcal{T}, \mathcal{PT})$-continuous
Now $f(G) \subseteq (\mathcal{T}, \mathcal{PT})-(f(G))$, so
$G \subseteq f^{-1}(f(G)) \subseteq f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(f(G)))$
But $(\mathcal{T}, \mathcal{PT})-\text{cl}(f(G))$ is $(\mathcal{T}, \mathcal{PT})$-closed .
And so $f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(f(G)))$ is also $(\mathcal{T}, \mathcal{PT})$-closed .
Hence $G \subseteq (\mathcal{T}, \mathcal{PT})-\text{cl}(G) \subseteq f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(f(G)))$
And therefore $f((\mathcal{T}, \mathcal{PT})-\text{cl}(G)) \subseteq (\mathcal{T}, \mathcal{PT})-\text{cl}(f(G))$
$(\mathcal{T}, \mathcal{PT})-\text{cl}(f(G))=f(f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(f(G)))$

Conversely:
Assume $f((\mathcal{T}, \mathcal{PT})-\text{cl}(G)) \subseteq (\mathcal{T}, \mathcal{PT})-\text{cl}(f(G))$ for any $G \subseteq X$, and let $F$ be a $(\mathcal{T}, \mathcal{PT})$-closed subset of $Y$ .
Set $G=f^{-1}(F)$, i.e $(\mathcal{T}, \mathcal{PT})-\text{cl}(G)=G$.
Now $f((\mathcal{T}, \mathcal{PT})-\text{cl}(G))=f((\mathcal{T}, \mathcal{PT})-\text{cl}(f^{-1}(F))) \subseteq (\mathcal{T}, \mathcal{PT})-\text{cl}(f(f^{-1}(F)))=(\mathcal{T}, \mathcal{PT})-\text{cl}(F)=F$
Hence $(\mathcal{T}, \mathcal{PT})-\text{cl}(G) \subseteq f^{-1}(f((\mathcal{T}, \mathcal{PT})-\text{cl}(G)) \subseteq f^{-1}(F)=G$
But $G \subseteq (\mathcal{T}, \mathcal{PT})-\text{cl}(G)$
So $(\mathcal{T}, \mathcal{PT})-\text{cl}(G)=G$ and $f$ is bi-$(\mathcal{T}, \mathcal{PT})$-continuous function .

Theorem 5.9
A function $f: (X, \mathcal{T}_X, \mathcal{PT}_X) \to (Y, \mathcal{T}_Y, \mathcal{PT}_Y)$ is bi-$(\mathcal{T}, \mathcal{PT})$-continuous iff for every subset $G \subseteq Y$, $(\mathcal{T}, \mathcal{PT})-\text{cl}(f^{-1}(G)) \subseteq f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(G))$.

Proof:
Let $f: (X, \mathcal{T}_X, \mathcal{PT}_X) \to (Y, \mathcal{T}_Y, \mathcal{PT}_Y) \text{ be bi-}(\mathcal{T}, \mathcal{PT})\text{-continuous function}$. To prove that $(\mathcal{T}, \mathcal{PT})-\text{cl}(f^{-1}(G)) \subseteq f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(G))$ for every subset $G \subseteq X$.
Since $G \subseteq (\mathcal{T}, \mathcal{PT})-\text{cl}(G)$, Then
$(\mathcal{T}, \mathcal{PT})-\text{cl}(f^{-1}(G)) \subseteq (\mathcal{T}, \mathcal{PT})-\text{cl}(f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(G))))$…..(1)
$(\mathcal{T}, \mathcal{PT})-\text{cl}(G)$ is $(\mathcal{T}, \mathcal{PT})$-closed in $Y$, $f$ is bi-$(\mathcal{T}, \mathcal{PT})$-continuous function
Implies $(\mathcal{T}, \mathcal{PT})-\text{cl}(f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(G)))$ is $(\mathcal{T}, \mathcal{PT})$-closed in $X$ .
Implies $(\mathcal{T}, \mathcal{PT})-\text{cl}(f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(G)))=f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(G))$.…..(2)
From (1) and (2) we get $(\mathcal{T}, \mathcal{PT})-\text{cl}(f^{-1}(G)) \subseteq f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(G))$

Conversely:
Suppose that $f: (X, \mathcal{T}_X, \mathcal{PT}_X) \to (Y, \mathcal{T}_Y, \mathcal{PT}_Y)$ is a function such that $(\mathcal{T}, \mathcal{PT})-\text{cl}(f^{-1}(G)) \subseteq f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(G))$ for every subset $G \subseteq X$.
To prove that $f$ is bi-$(\mathcal{T}, \mathcal{PT})$-continuous function
Let $F \subseteq Y$ be an arbitrary $(\mathcal{T}, \mathcal{PT})$-closed set then $(\mathcal{T}, \mathcal{PT})-\text{cl}(F)=F$
By hypothesis
$(\mathcal{T}, \mathcal{PT})-\text{cl}(f^{-1}(F)) \subseteq f^{-1}((\mathcal{T}, \mathcal{PT})-\text{cl}(F))=f^{-1}(F)$…..(3)
But $f^{-1}(F) \subseteq (\mathcal{T}, \mathcal{PT})-\text{cl}(f^{-1}(F))$ for every $F$…..(4)
From (3) and (4) we get $f^{-1}(F) = (\mathcal{T}, \mathcal{PT})-\text{cl}(f^{-1}(F))$
Then $f^{-1}(F)$ is $(\mathcal{T}, \mathcal{PT})$-closed .
So by theorem 3.2.17
$f$ is bi-$(\mathcal{T}, \mathcal{PT})$-continuous .
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Theorem 5.10
A function \( f:(X, \mathcal{T}_X, \mathcal{P}T_X) \to (Y, \mathcal{T}_Y, \mathcal{P}T_Y) \) is bi-(\( \mathcal{T}, \mathcal{P}T \))-continuous  iff for every subset \( G \subseteq X \), \( f^{-1}((\mathcal{T}, \mathcal{P}T)\text{-int}(G)) \subseteq (\mathcal{T}, \mathcal{P}T)\text{-int}(f^{-1}(G)) \).

Proof:
let \( f:(X, \mathcal{T}_X, \mathcal{P}T_X) \to (Y, \mathcal{T}_Y, \mathcal{P}T_Y) \) be bi-(\( \mathcal{T}, \mathcal{P}T \))-continuous function. To prove that \( f^{-1}((\mathcal{T}, \mathcal{P}T)\text{-int}(G)) \subseteq (\mathcal{T}, \mathcal{P}T)\text{-int}(f^{-1}(G)) \) for every subset \( G \subseteq Y \).

\( G \subseteq Y \) implies \( (\mathcal{T}, \mathcal{P}T)\text{-int}(G) \subseteq (Y, \mathcal{T}_Y, \mathcal{P}T_Y) \)

Implies \( f^{-1}((\mathcal{T}, \mathcal{P}T)\text{-int}(G)) \subseteq (X, \mathcal{T}_X, \mathcal{P}T_X) \) since \( f \) is bi-(\( \mathcal{T}, \mathcal{P}T \))-continuous function

Implies \( (\mathcal{T}, \mathcal{P}T)\text{-int}(f^{-1}((\mathcal{T}, \mathcal{P}T)\text{-int}(G))) \subseteq f^{-1}((\mathcal{T}, \mathcal{P}T)\text{-int}(G)). \)

(\( \mathcal{T}, \mathcal{P}T \))-int(G) \subseteq G implies \( f^{-1}((\mathcal{T}, \mathcal{P}T)\text{-int}(G)) \subseteq f^{-1}(G) \)

Implies \( (\mathcal{T}, \mathcal{P}T)\text{-int}(f^{-1}((\mathcal{T}, \mathcal{P}T)\text{-int}(G))) \subseteq (\mathcal{T}, \mathcal{P}T)\text{-int}(f^{-1}(G)) \)

Implies \( f^{-1}((\mathcal{T}, \mathcal{P}T)\text{-int}(G)) \subseteq (\mathcal{T}, \mathcal{P}T)\text{-int}(f^{-1}(G)) \)

Conversely:
Suppose that \( f^{-1}((\mathcal{T}, \mathcal{P}T)\text{-int}(G)) \subseteq (\mathcal{T}, \mathcal{P}T)\text{-int}(f^{-1}(G)). \)

To prove that \( f \) bi-(\( \mathcal{T}, \mathcal{P}T \))-continuous function.

Let \( G \) be an \( (\mathcal{T}, \mathcal{P}T) \)-open subset of \( Y \) and hence \( (\mathcal{T}, \mathcal{P}T)\text{-int}(G)=G \).

If we show \( f^{-1}(G) \) is \( (\mathcal{T}, \mathcal{P}T) \)-open in \( X \), the result will follow.

\( f^{-1}(G)=f^{-1}((\mathcal{T}, \mathcal{P}T)\text{-int}(G)) \subseteq (\mathcal{T}, \mathcal{P}T)\text{-int}(f^{-1}(G)) \) [by(2)]

Then \( f^{-1}(G) \subseteq (\mathcal{T}, \mathcal{P}T)\text{-int}(f^{-1}(G)) \) ....(3)

But \( (\mathcal{T}, \mathcal{P}T)\text{-int}(f^{-1}(G)) \subseteq f^{-1}(G) \) is always true .....(4)

From (3) and (4) we get that \( f^{-1}(G)=(\mathcal{T}, \mathcal{P}T)\text{-int}(f^{-1}(G)) \)

So that \( f^{-1}(G) \) is \( (\mathcal{T}, \mathcal{P}T) \)-open.

References