Bayes Estimators for the Reliability Function of Pareto Type I Distribution Under Squared-Log Error Loss Function

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Received 6/4/2014 – Accepted 29/9/2014

الخلاصة

أوجدنا في هذا البحث مقدرات بيز لدالة المعولية لتوزيع باريتو من النوع الأول تحت دالة الخسارة اللوغاريتمية التربيعية. وبغية الحصول على افضل فهم لتحليلنا البيزي فقد افترضنا حالة عدم وجود معلومات مسبقة عن معلمة الشكل باستخدام دالة جيفري للمعلومات كذلك وجود معلومات مسبقة متمثلة بالتوزيع الآسي. تمت مقارنة مقدر بيز لدالة المعولية لتوزيع باريتو من النوع الأول تحت دالة الخسارة اللوغاريتمية التربيعية مع بعض المقدرات التقليدية مثل مقدر الامكان الأعظم، المقدر المنتظم غير المتحيز ذو اقل تباين وأقل متوسط مربعات الخطأ. استناداً الى دراسة مونت-كارلو للمحاكاة فقد تمت مقارنة أداء هذه المقدرات بالاعتماد على متوسط مربعات الخطأ التكاملي (IMSE's) .

ABSTRACT

In this paper we obtained Basyian estimators of the reliability function of the Pareto type I distribution under Squared-Log error loss function. In order to get better understanding of our Bayesian analysis we consider non-informative prior for the shape parameter Using Jeffery prior Information as well as informative prior density represented by Exponential distribution. The Bayes estimator of the reliability function of the Pareto type I distribution under Squared-Log error loss function is compared with Some classical estimators such as, the Maximum Likelihood Estimator (MLE), the Uniformly Minimum Variance Unbiased Estimator (UMVUE), and the Minimum Mean Squared Error (MinMSE) estimator according to Monte-Carlo simulation study. The performance of these estimators is compared depending on the Integrated mean squared errors (IMSE's).

Key words: Pareto distribution, MLE, Bayes estimator, Squared-Log error loss function, Jeffery prior and Exponential prior.

INTRODUCTION

The Pareto distribution is named after the economist Vilfredo Pareto (1848-1923), this distribution is first used as a model for distributing incomes of model for city population within a given area, failure model in reliability theory [2], and a queuing model in operation research [6].

A random variable X, is said to follow the two parameters of Pareto distribution if its pdf is given by:

$$f(x;\alpha,\theta)=\frac{\theta\alpha^{\theta}}{x^{\theta+1}} \qquad ; \quad x\geq\alpha\,,\alpha>0\,,\theta>0 \qquad (1)$$

Where α and θ are the scale and shape parameters respectively. The corresponding cumulative distribution function (CDF) is given by

$$F(x;\alpha,\theta)=1-\left(\frac{\alpha}{x}\right)^{\theta} \quad , \quad x\geq\alpha; \quad \alpha,\theta>0 \qquad (2)$$

So, the reliability function is:

$$R(t)=\left(\frac{\alpha}{t}\right)^{\theta} \qquad (3)$$

SOME CLASSICAL ESTIMATION

In this section, we obtain some classical estimators of the shape parameter for the Pareto distribution represented by Maximum likelihood estimator, Uniformly Minimum Variance Unbiased Estimator and Minimum mean square error estimator.

Given $x_1, x_2, \ldots, x_n$ a random sample of size n from Pareto distribution, we consider estimation using method of Maximum likelihood as follows:

$$L(x_1, \ldots, x_n|\theta) = \prod_{i=1}^{n} f(x_n; \theta)$$

$$L(x_1, \ldots, x_n|\theta) = \theta^n \alpha^{n\theta} e^{-(\theta+1)\sum_{i=1}^{n} ln\, x_i}$$

The Log- likelihood function is given by

$$ln\, L(x_1, \ldots, x_n|\theta) = n\, ln\, \theta + n\theta ln\alpha - (\theta+1)\sum_{i=1}^{n} ln\, x_i$$

Differentiating the log likelihood with respect to θ :

$$\frac{\partial[ln\, L(x_1, \ldots, x_n|\theta)]}{\partial\theta} = \frac{n}{\theta} + \mathrm{nln}\alpha - \sum_{i=1}^{n} ln\, \mathrm{x_i}$$

Setting this expression to zero and solving the equation yields the Maximum likelihood estimator of θ:

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} ln\, \mathrm{x_i} - nln\alpha}$$

$$\hat{\theta}_{\mathrm{ML}} = \frac{\mathrm{n}}{\mathrm{T} - \mathrm{nln}\alpha} \quad , \; where\;\; T = \sum_{i=1}^{n} ln\, \mathrm{x_i} \qquad (4)$$

Using the invariance property, the MLE for R(t) denoted by $\hat{R}_{ML}(t)$ may be obtained by replacing θ by its MLE $\hat{\theta}_{\mathrm{ML}}$ in (3) [7]

$$\hat{R}_{ML}(t) = \left(\frac{\alpha}{t}\right)^{\hat{\theta}_{ML}} \qquad (5)$$

Here, we obtain the Uniformly Minimum Variance Unbiased Estimator (UMVUE) of θ. Since the family of density (1) belongs to the Exponential family, therefore, statistic T is a complete sufficient statistic for θ.

We have:

$\ln\left(\frac{t_i}{\alpha}\right) \sim Exponential(\theta) \Rightarrow \mathrm{T} = \sum_{i=1}^{n} ln\left(\frac{t_i}{\alpha}\right) \sim$ Gamma (n, θ), with the density:

$$g(t) = \frac{\theta^n}{\Gamma(n)} t^{n-1} e^{-\theta t} \quad , \; t \geq 0 \;, \theta > 0$$

Thus,

$$E[\hat{\theta}_{\mathrm{ML}}] = nE\left(\frac{1}{T}\right)$$

$$E\left(\frac{1}{T}\right)=\int_0^\infty \frac{\theta^n t^{n-2}e^{-\theta t}}{\Gamma(n)}dt$$

$$=\frac{\theta}{n-1}\int_0^\infty \frac{\theta^{n-1} t^{n-2}e^{-\theta t}}{\Gamma(n-1)}dt$$

$$E\left(\frac{1}{T}\right)=\frac{\theta}{n-1} \qquad (6)$$

$$E[\hat{\theta}_{\mathrm{ML}}]=\frac{n\theta}{n-1} \qquad (7)$$

Recall that, T is a complete sufficient statistics for θ . Thus, $\hat{\theta}_{UMVUE}$ is a Uniformly minimum variance unbiased estimator (UMVUE) of θ.[19]

$$\hat{\theta}_{UMVUE}=\frac{n-1}{T} \qquad (8)$$

We can find the Minimum Mean Squared Error (MinMSE) estimator in the class of estimators of the form $\frac{c}{T}$. Therefore

$$MSE_\theta\left(\frac{c}{T}\right)=E\left[\left(\frac{c}{T}-\theta\right)^2\right]=c^2E\left(\frac{1}{T^2}\right)-2c\theta E\left(\frac{1}{T}\right)+\theta^2$$

Taking the derivative with respect to c on the both sides leads to

$$\frac{\partial}{\partial c}\mathrm{MSE}_\theta\left(\frac{c}{T}\right)=2cE\left(\frac{1}{T^2}\right)-2\theta E\left(\frac{1}{T}\right)$$

Let $\frac{\partial}{\partial c}\mathrm{MSE}_\theta\left(\frac{c}{T}\right)=0$, so

$$c=\frac{\theta E\left(\frac{1}{T}\right)}{E\left(\frac{1}{T^2}\right)}=\frac{\frac{\theta^2}{n-1}}{\frac{\theta^2}{(n-1)(n-2)}}=n-2$$

Therefore, we get c = n-2 and hence

$$\hat{\theta}_{MinMSE}=\frac{n-2}{T} \qquad (9)$$

Is the Minimum mean square error estimator for θ.

$$MSE(\hat{\theta}_{MLE})=E\left[\left(\frac{n}{T}-\theta\right)^2\right]=var\left(\frac{n}{T}\right)+\left[E\left(\frac{n}{T}\right)-\theta\right]^2 \qquad (10)$$

$$var\left(\frac{n}{T}\right)=n^2 var\left(\frac{1}{T}\right)$$

$$var\left(\frac{1}{T}\right)=E\left[\left(\frac{1}{T}\right)^2\right]-\left[E\left(\frac{1}{T}\right)\right]^2 \qquad (11)$$

$$E\left[\left(\frac{1}{T}\right)^2\right]=\frac{\theta^2}{(n-1)(n-2)}\int_0^\infty \frac{\theta^{n-2}t^{n-3}e^{-\theta t}}{\Gamma(n-2)}dt$$

$$=\frac{\theta^2}{(n-1)(n-2)} \qquad (12)$$

Substituting (6) and (12) into (11)

$$var\left(\frac{1}{T}\right) = \frac{\theta^2}{(n-1)(n-2)} - \left[\frac{\theta}{n-1}\right]^2$$

$$var\left(\frac{1}{T}\right) = \frac{\theta^2}{(n-1)^2(n-2)} \qquad (13)$$

Hence, $var\left(\frac{n}{T}\right) = \frac{n^2\theta^2}{(n-1)^2(n-2)}$

$$\left[E\left(\frac{n}{T}\right) - \theta\right]^2 = \left[E\left(\frac{n}{T}\right)\right]^2 + 2\theta E\left(\frac{n}{T}\right) + \theta^2$$

$$= \left[\frac{n\theta}{n-1}\right]^2 - 2\theta\left[\frac{n\theta}{n-1}\right] + \theta^2$$

$$= \frac{n^2\theta^2 - 2n\theta^2(n-1) + \theta^2(n-1)^2}{(n-1)^2}$$

$$\left[E\left(\frac{n}{T}\right) - \theta\right]^2 = \frac{\theta^2}{(n-1)^2} \qquad (14)$$

Substituting (13) and (14) into (10)

$$MSE\left(\hat{\theta}_{MLE}\right) = \frac{n^2\theta^2}{(n-1)^2(n-2)} + \frac{\theta^2}{(n-1)^2} = \frac{\theta^2(n+2)(n-1)}{(n-1)^2(n-2)}$$

$$MSE\left(\hat{\theta}_{MLE}\right) = \frac{\theta^2(n+2)}{(n-1)(n-2)} \qquad (15)$$

$$MSE\left(\hat{\theta}_{UMVUE}\right) = E\left[\left(\frac{n-1}{T} - \theta\right)^2\right]$$

$$MSE\left(\hat{\theta}_{UMVUE}\right) = var\left(\frac{n-1}{T}\right) + \left[E\left(\frac{n-1}{T}\right) - \theta\right]^2 \qquad (16)$$

$$var\left(\frac{n-1}{T}\right) = (n-1)^2 var\left(\frac{1}{T}\right) \qquad (17)$$

Substituting (13) into (17), we get

$$var\left(\frac{n-1}{T}\right) = \frac{(n-1)^2\theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{(n-2)} \qquad (18)$$

We have,

$$E\left(\frac{n-1}{T}\right) = \theta \qquad (19)$$

Substituting (18) and (19) into (16)

$$MSE\left(\hat{\theta}_{UMVUE}\right) = \frac{\theta^2}{(n-2)} \qquad (20)$$

$$MSE\left(\hat{\theta}_{MinMSE}\right) = E\left[\left(\frac{n-2}{T} - \theta\right)^2\right]$$

$$MSE\left(\hat{\theta}_{MinMSE}\right) = var\left(\frac{n-2}{T}\right) + \left[E\left(\frac{n-2}{T}\right) - \theta\right]^2 \qquad (21)$$

$$var\left(\frac{n-2}{T}\right) = (n-2)^2 var\left(\frac{1}{T}\right) \quad (22)$$

Substituting (13) into (22)

$$var\left(\frac{n-2}{T}\right) = \frac{(n-2)\theta^2}{(n-1)^2} \quad (23)$$

$$E\left(\frac{n-2}{T}\right) = \frac{(n-2)\theta}{(n-1)} \quad (24)$$

Substituting (23) and (24) into (21)

$$MSE(\hat{\theta}_{MinMSE}) = \frac{(n-2)\theta^2}{(n-1)^2} + \frac{\theta^2}{(n-1)^2} = \frac{\theta^2}{(n-1)} \quad (25)$$

From (15), (20) and (25), we find that:

$MSE(\hat{\theta}_{MinMSE}) \leq MSE(\hat{\theta}_{UMVUE}) \leq MSE(\hat{\theta}_{ML})$

New, the MinMSE estimator of the Reliability function, denoted by $\hat{R}(t)_{\text{MinMSE}}$ approximately, will be:

$$\hat{R}(t)_{\text{MinMSE}} = \left(\frac{\alpha}{t}\right)^{\hat{\theta}_{MinMSE}} \quad (26)$$

BAYES ESTIMATOR UNDR SQUARED-LOG ERROR LOSS FUNCTION

Bayes estimators for the shape parameter θ and Reliability function were considered under squared-log error loss function with non-Informative prior which is represented by Jeffrey prior and informative loss function is represented by Exponential prior where the squared-log error loss function is of the form: [4]

$$L_2(\theta, \delta_1) = (\ln \delta_1 - \ln \theta)^2 = \left(\ln \frac{\delta_1}{\theta}\right)^2$$

Which is balanced with $\lim L_2(\theta, \delta_1) \to \infty \; as \; \delta_1 \to 0 \; or \; \infty.$ A balanced loss function takes both error of estimation and goodness of fit into account but the unbalanced loss function considers only error of estimation. This loss function is convex for $\frac{\delta_1}{\theta} \leq e$ and concave otherwise, but its risk function has a unique minimum with respect to δ_1.[4]

According to the above mentioned loss functions, we drive the corresponding Bayes estimators for θ using Risk function $R(\hat{\theta} - \theta)$, which minimizes the posterior risk

$$R(\hat{\theta} - \theta) = E\left[L(\hat{\theta}, \theta)\right] = \int_0^\infty (\ln \hat{\theta} - \ln \theta)^2 h(\theta|x_1 \dots \dots \dots .. x_n) d\theta$$

$$R(\hat{\theta} - \theta) = (\ln \hat{\theta})^2 - 2(ln\hat{\theta})\, E(\ln \theta|x) + E((\ln \theta)^2|x)$$

$$\frac{\partial \text{Rsik}}{\partial \hat{\theta}} = 2(\ln \hat{\theta})\frac{1}{\hat{\theta}} - \frac{2}{\hat{\theta}} E((\ln \theta)|x)$$

By letting $\frac{\partial R_2(\hat{\theta}-\theta)}{\partial\hat{\theta}} = 0$

The Bayes estimator for the parameter θ of Pareto distribution under the squared-log error loss function is:

$\hat{\theta} = \text{Exp}[\text{E}(\ln\theta|\text{x})]$ (27)

According to the Squared-Log error loss function, the corresponding Bayes estimator for the reliability function will be:

$\hat{R}(t) = Exp[E(lnR(t)|t)]$ (28)

$$E(lnR(t)|t) = \int_0^\infty lnR(t)\ h(\theta|x_1,\dots,x_n)d\theta$$

We have $R(t) = \left(\frac{\alpha}{t}\right)^\theta$

Hence,

$E[lnR(t)] = \ln\left(\frac{\alpha}{t}\right)E[\theta]$ (29)

Substituting (29) into (28), we get:

$\hat{R}(t) = Exp\left[\ln\left(\frac{\alpha}{t}\right)E[\theta]\right]$ (30)

PRIOR AND POSTERIOR DISTRIBUTIONS

In this paper we consider informative as well as non-informative prior density for θin order to get better understanding of our Bayesian analysis as follows:

(i)Bayes Estimator Using Jeffery Prior Information:

Let us assume that θ has non-informative prior density defined by using Jeffrey prior information $g(\theta)$ which given by:

$g_1(\theta) \propto \sqrt{I(\theta)}$

Where $I(\theta)$ represents Fisher information which defined as follows [1]:

$$I(\theta) = -nE\left(\frac{\partial^2 lnf}{\partial\theta^2}\right)$$

Hence,

$g_1(\theta) = c\sqrt{-nE\left(\frac{\partial^2 lnf}{\partial\theta^2}\right)}$, where c is a constant (31)

$$lnf(x;\theta) = ln\theta + n\theta ln\alpha - (\theta+1)lnx$$

$$\frac{\partial lnf}{\partial\theta} = \frac{1}{\theta} + nln\alpha - lnx$$

$$\frac{\partial^2 lnf}{\partial\theta^2} = -\frac{1}{\theta^2}$$

$$E\left(\frac{\partial^2 lnf}{\partial\theta^2}\right) = -\frac{1}{\theta^2}$$

After substitution into (10), we find that:

$$g_1(\theta) = \frac{c}{\theta}\sqrt{n}$$

$$h_1(\theta|x_1, \dots, x_n) = \frac{L(x_1, \dots, x_n|\theta)g_1(\theta)}{\int_0^\infty L(x_1, \dots, x_n|\theta)g_1(\theta)d\theta}$$

Where $L(x;\alpha,\theta) = \theta^n \alpha^{n\theta} e^{-(\theta+1)\sum lnx} = \theta^n e^{n\theta ln\alpha} e^{-(\theta+1)\sum lnx}$

$$h_1(\theta|x_1, \dots, x_n)$$

$$= \frac{\theta^n e^{n\theta ln\alpha} e^{-(\theta+1)\sum lnx} \frac{c}{\theta}\sqrt{n}}{\int_0^\infty \theta^n e^{n\theta ln\alpha} e^{-(\theta+1)\sum lnx} \frac{c}{\theta}\sqrt{n}\, d\theta} \quad where, \alpha\ is\ a\ constant$$

$$= \frac{c\sqrt{n}e^{\sum lnx}\theta^{n-1}e^{-\theta(\sum lnx - nln\alpha)}}{c\sqrt{n}e^{\sum lnx}\int_0^\infty \theta^{n-1}e^{-\theta(\sum lnx - nln\alpha)}d\theta}$$

$$= \frac{\theta^{n-1}e^{-\theta(\sum lnx - nln\alpha)}}{\int_0^\infty \theta^{n-1}e^{-\theta(\sum lnx - nln\alpha)}d\theta}$$

$$= \frac{\theta^{n-1}e^{-\theta(T - nln\alpha)}}{\frac{\Gamma(n)}{[T-nln\alpha]^n}\int_0^\infty \frac{[T-nln\alpha]^n}{\Gamma(n)}\theta^{n-1}e^{-\theta(T-nln\alpha)}d\theta}$$

$$= \frac{[T-nln\alpha]^n\theta^{n-1}e^{-\theta(T-nln\alpha)}}{\Gamma(n)}$$

This posterior density is recognized as the density of the Gamma distribution:

$\theta \sim Gamma(n, (T - nln\alpha))$, with:

$$E(\theta) = \frac{n}{T - nln\alpha} \quad , \quad ver(\theta) = \frac{n}{(T-nln\alpha)^2}$$

$$\mathrm{E}(\ln\theta|\mathrm{x}) = \frac{(T-nln\alpha)^{\mathrm{n}}}{\Gamma(\mathrm{n})}\int_0^\infty \ln\theta\ \theta^{\mathrm{n}-1}\mathrm{e}^{-\theta[T-nln\alpha]}\mathrm{d}\theta$$

Let $y = \theta(T - nln\alpha)$

$$\Rightarrow \theta = \frac{y}{T-nln\alpha}, \quad d\theta = \frac{dy}{T-nln\alpha}$$

Hence,

$$\mathrm{E}(\ln\theta|\mathrm{x}) = \frac{(\mathrm{e}^{-\mathrm{y}})^{\mathrm{n}}}{\Gamma(\mathrm{n})}\int_0^\infty \ln\left(\frac{\mathrm{y}}{T-nln\alpha}\right)\left(\frac{\mathrm{y}}{T-nln\alpha}\right)^{\mathrm{n}-1}\frac{\mathrm{e}^{-\mathrm{y}}}{T-nln\alpha}dy$$

$$= \frac{(T-nln\alpha)^n}{\Gamma(n)(T-nln\alpha)^n}\int_0^\infty [\ln y - \ln(T-nln\alpha)]y^{n-1}e^{-y}dy$$

$$= \int_0^\infty \frac{(lny)\,\mathrm{y}^{\mathrm{n}-1}\mathrm{e}^{-\mathrm{y}}}{\Gamma(n)}dy - \frac{ln(T-nln\alpha)}{\Gamma(n)}\int_0^\infty y^{n-1}e^{-y}dy$$

$$\mathrm{E}(\ln\theta|\mathrm{x}) = \varphi(n) - \ln[\mathrm{T} - \mathrm{n}\ln\alpha] \quad (32)$$

Where, $\varphi(n) = \frac{\Gamma(\acute{n})}{\Gamma(n)}$ is the digamma function [6]

Substituting (32) into (27), we get

$$\hat{\theta}_1 = \text{Exp}\left[\varphi(n) - ln(T - nln\alpha)\right] \qquad (33)$$

Now, using (9) to estimate Reliability function we reach to:

$$\hat{\text{R}}(\text{t})_{\text{BJ}} = \text{Exp}\left[\frac{n}{T - nln\alpha} \ln\left(\frac{\alpha}{t}\right)\right] \qquad (14)$$

We can notice that $\hat{\text{R}}(\text{t})_{\text{BJ}}$ is equivalent to the Maximum Likelihood Estimator for R(t).

***(ii)* Posterior Distribution Using Exponential Prior Distribution.**

Assuming that θ has informative prior as Exponential prior, which takes the following form:

$$g_2(\theta) = \frac{1}{\lambda} e^{-\frac{\theta}{\lambda}} \qquad , \qquad \theta\,, \lambda > 0$$

So, the posterior distribution for the parameter θ given the data $(x_1, x_2, \dots x_n)$ is:

$$h_2(\theta|x_1, x_2, \dots x_n) = \frac{\pi_{i=1}^{n} f(x_i|\theta) g_2(\theta)}{\int_0^\infty \pi_{i=1}^{n} f(x_i|\theta) g_2(\theta) d\theta}$$

Then the posterior distribution became as follows:

$$h_2(\theta|x_1, x_2, \dots x_n) = \frac{\left[T - nln\alpha + \frac{1}{\lambda}\right]^{n+1} \theta^n e^{-\theta\left[T - nln\alpha + \frac{1}{\lambda}\right]}}{\Gamma(n+1)} \qquad (15)$$

This posterior density is recognized as the density of the Gamma distribution where:

$\theta \sim Gamma\left(n+1, \frac{1}{\lambda} + \sum_{i=1}^{\text{n}} ln - nln\alpha\right)$, with:

$$E(\theta) = \frac{n+1}{\frac{1}{\lambda} + \sum_{i=1}^{\text{n}} ln\, \text{x}_\text{i} - nln\alpha} \qquad , \qquad ver(\theta) = \frac{n+1}{\left(\frac{1}{\lambda} + \sum_{i=1}^{\text{n}} ln\, \text{x}_\text{i} - nln\alpha\right)^2}$$

The Bayes estimator under Squared-Log error loss function will be:

$$\hat{\theta}_2 = \text{Exp}\left[\int_0^\infty ln\theta\, h_2(\theta|t) d\theta\right]$$

$$= \int_0^\infty ln\theta \frac{\left[T - nln\alpha + \frac{1}{\lambda}\right]^{n+1} \theta^n e^{-\theta\left[T - nln\alpha + \frac{1}{\lambda}\right]}}{\Gamma(n+1)} d\theta \qquad (16)$$

Let: $y = \theta\left[T - nln\alpha + \frac{1}{\lambda}\right]$

$$\Rightarrow \theta = \frac{y}{T - nln\alpha + \frac{1}{\lambda}} \quad , \quad d\theta = \frac{1}{T - nln\alpha + \frac{1}{\lambda}} dy$$

Substituting into (16), we have:

$E(ln\theta|x)$

$$= \frac{\left(T - nln\alpha + \frac{1}{\lambda}\right)^n}{\Gamma(n+1)} \int_0^\infty \ln\left(\frac{y}{T - nln\alpha + \frac{1}{\lambda}}\right)\left(\frac{y}{T - nln\alpha + \frac{1}{\lambda}}\right)^n e^{-y} dy$$

By simplification, we get:

$$E(ln\theta|x) = \varphi(n+1) + \ln\left(T - nln\alpha + \frac{1}{\lambda}\right)$$

Hence,

$$\hat{\theta} = ExP\left[\varphi(n+1) + \ln\left(T - nln\alpha + \frac{1}{\lambda}\right)\right] \qquad (17)$$

Now, the corresponding Bayes estimator for R(t) with posterior distribution (15) come out as:

$$\hat{R}_2(t) = ExP\left[\frac{(n+1)\ln\left(\frac{\alpha}{t}\right)}{T - nln\alpha + \frac{1}{\lambda}}\right] \qquad (18)$$

SIMULATION RESULTS

In our simulation study, we generated I = 2500 samples of size n = 20, 50, and100 from Pareto distribution to represent small, moderate and large sample size with the several values of shape parameter, θ =0.5, 1.5 and 2.5 , the scale parameter $\alpha = 1$, 1.4 and taking t = 1.5, 3. We chose two values of λ for the Exponential prior (λ =0.5, 3).

In this section, Monte-Carlo simulation study is performed to compare the methods of estimation using mean square Errors (MSE's) and integral mean squares error (IMSE), where

$$\mathrm{MSE}(\hat{\theta}) = \frac{\sum_{i=1}^{I}\left(\hat{\theta}_i - \theta\right)^2}{I}$$

The integrated mean squared error (IMSE) is an important global measure and it more accurate than MSE which is defined as distance between the estimate value of the reliability fonction and actual value of reliability fonction given by équation

$$\mathrm{IMSE}\big(\widehat{\mathrm{R}}(\mathrm{t})\big) = \frac{1}{\mathrm{I}}\sum_{\mathrm{i}=1}^{\mathrm{I}}\left[\frac{1}{\mathrm{n_t}}\sum_{\mathrm{j}=1}^{\mathrm{n_t}}\big(\widehat{\mathrm{R}}_{\mathrm{i}}(\mathrm{t_j}) - \mathrm{R}(\mathrm{t_j})\big)^2\right]$$

$$\mathrm{IMSE}\big(\widehat{\mathrm{R}}(\mathrm{t})\big) = \frac{1}{\mathrm{n_t}}\sum_{\mathrm{j}=1}^{\mathrm{n_t}}\mathrm{MSE}\big(\widehat{\mathrm{R}}_{\mathrm{i}}(\mathrm{t_j})\big)$$

Where $i = 1, 2, \ldots, L, n_t$ the random limits of t_i .

In this paper, we use $\mathrm{t} = 1.5, 1.8, 2.1, 2.4, 2.7, 3$

The results were summarized and tabulated in the following tables for each estimator and for all sample sizes.

Table 1: IMSE's of the Different Estimators for Pareto Distribution
Where $\theta = 0.5$, $R(t)_{\alpha=1} = 0.57745967$, $R(t)_{\alpha=1.4} = 0.68313$

Estimator \ n		20		50		100	
		$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1$	$\alpha = 1.4$
MinMSE		0.0037297	0.0018656	0.0014338	0.0007181	0.0007068	0.0003543
BJ(Sq. Log)		0.0040447	0.0021093	0.0014703	0.0007484	0.0007168	0.0003487
BE (Sq. Log)	$\lambda = 0.5$	0.0036041	0.0018706	0.0014089	0.0007166	0.0007021	0.0003547
	$\lambda = 3$	0.0044940	0.0023735	0.0015395	0.0007883	0.0007346	0.0003723
Best Estimator		BE(Sq. Log) $\lambda = 0.5$	MinMSE	BE(Sq. Log) $\lambda = 0.5$	BE(Sq. Log) $\lambda = 0.5$	BE(Sq. Log) $\lambda = 0.5$	BJ(Sq. Log)

Table 2: IMSE's of the Different Estimators for Pareto Distribution
Where $\theta = 1.5$, $R(t)_{\alpha=1} = 0.57745967$, $R(t)_{\alpha=1.4} = 0.68313$

Estimator \ n		20		50		100	
		$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1$	$\alpha = 1.4$
MinMSE		0.0069365	0.0054207	0.0025560	0.0002048	0.0012402	0.0010033
BJ(Sq. Log)		0.0060542	0.0052179	0.0024023	0.0002003	0.0012040	0.0009937
BE (Sq. Log)	$\lambda = 0.5$	0.0052546	0.0040936	0.0022760	0.0001823	0.0011691	0.0009456
	$\lambda = 3$	0.0058954	0.0051766	0.0023749	0.0001998	0.0011977	0.0009932
Best Estimator		BE(Sq. Log) $\lambda = 0.5$	BE(Sq. Log) $\lambda = 0.5$	BE(Sq. Log) $\lambda = 0.5$	BE(Sq. Log) $\lambda = 0.5$	BE(Sq. Log) $\lambda = 0.5$	BE(Sq. Log) $\lambda = 0.5$

Table 3: IMSE's of the Different Estimators for Pareto Distribution
Where $\theta = 2.5,\ R(t)_{\alpha=1} = 0.57745967,\quad R(t)_{\alpha=1.4} = 0.68313$

Estimator \ n		20		50		100	
		$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1$	$\alpha = 1.4$
MinMSE		0.0050510	0.0056479	0.00175183	0.0020622	0.0008306	0.0009972
BJ(Sq. Log)		0.0038279	0.0048701	0.0015517	0.0019273	0.0007823	0.0009652
BE (Sq. Log)	ℷ = 0.5	0.0053541	0.0056636	0.0018501	0.0020975	0.0008580	0.00100677
	ℷ = 3	0.0035196	0.0045048	0.0014976	0.0018669	0.0007684	0.0009501
Best Estimator		BE(Sq. Log) ℷ = 3	BE(Sq. Log) ℷ = 3	BE(Sq. Log) ℷ = 3	BE(Sq. Log) ℷ = 3	BE(Sq. Log) ℷ = 3	BE(Sq. Log) ℷ = 3

DISCUSSION

In general, the performance of Bayes estimator for reliability function of Pareto distribution under Squared-Log error loss function with exponential distribution was better in performance than using Jeffery prior (which equivalent to the Maximum likelihood estimator) for all sample sizes and with different values for θ. On the other hand, when θ < 2.5 the results show clearly, that, the increasing of the scale parameter α will decrease the values of IMSE for all estimators and with all cases.

Finally, the simulation results show that, with large values of shape parameter of Pareto distribution (θ ≥ 2.5), the performance of Bayes estimator under Squared-Log error loss function using exponential prior with large value of ℷ (ℷ=3) is more appropriate than using exponential prior with small value of ℷ (ℷ=0.5), and vice versa.

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