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THE ANALYSIS OF BIFURCATION SOLUTIONS BY BOUNDARY SINGULARITIES

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ABSTRACT

This paper introduces a study of bifurcation of extremals of functions of codimensions eight and twenty four at the origin. We have used the boundary singularities of smooth maps to study the bifurcation analysis of these functions. We have found the parametric equation of the caustic with the geometric description of this caustic. In addition, we have found bifurcation spreading of the critical points by introducing an application on one of our results.

Keywords: *Bifurcation solutions, Boundary Singularities, Caustic.*

1. INTRODUCTION

The nonlinear problems which occur in mathematics and physics may be formed in the form of operator equation,

$$f(x, \lambda) = b, x \in O, b \in Y, \lambda \in \mathbb{R}^n, \quad (1.1)$$

in which f is a smooth Fredholm map whose index is zero and X, Y are Banach spaces and O is an open subset of X . The method of reduction for these problems to the finite-dimensional equation,

$$\Theta(\xi, \lambda) = \beta, \xi \in M, \beta \in N, \quad (1.2)$$

may be used, where M and N are smooth finite-dimensional manifolds. Equation (1.1) can reduce to equation (1.2) by Lyapunov-Schmidt method in which equation (1.2) has all the topological and analytical properties of equation (1.1) (multiplicity, bifurcation diagram, etc), as such information can be found in [5], [7], [8], [12]. In initial years, the study of

singularities of smooth maps and its applications to the BVPs took an important character in the works of Saponov and his group. For example, in [11] Shvyreva (2002) studied the boundary singularities of the function,

$$\begin{aligned} \tilde{W}(\eta, \gamma) &= \eta_1^4 + (c\eta_1 + \eta_2)^2 - 2\varepsilon_1\eta_1^2 \\ &\quad + 2\varepsilon_2\eta_1^2\eta_2 + 2\varepsilon_3\eta_1\eta_2 \\ &\quad + 2\varepsilon_4\eta_1 + 2\varepsilon_5\eta_2, \quad \eta \\ &= (\eta_1, \eta_2), \quad \gamma \\ &= (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5), \quad \eta_1, \eta_2 \\ &\geq 0, \end{aligned}$$

and considered the functional

$$V(u, \lambda) = \int_0^\pi \left(\frac{(u')^2}{2} + \lambda(\cos(u(x)) - 1) \right) dx,$$

with the extra condition

$$g_1(u) = \int_0^1 \sin u \, dx \geq$$

$$0, \quad g_2(u) = -\int_0^{1/2} \sin u \, dx \geq 0, \text{ as an}$$

application of her results, and in [1] Abdul Hussain (2005) has studied the following

problem,

$$\frac{d^4u}{dx^4} + \alpha \frac{d^2u}{dx^2} + \beta u + u^2 = 0,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

with the extra condition

$$u(x_1) \geq 0, \quad u(x_2) \geq 0, \quad x_1, x_2 \in [0,1]$$

,by considering the following functional energy,

$$V(u, \lambda) = \int_0^1 \left(\frac{(u'')^2}{2} - \alpha \frac{(u')^2}{2} + \beta \frac{u^2}{2} + \frac{u^3}{3} \right) dx,$$

which is reduced to the study of the following key function with boundaries,

$$\begin{aligned} W(\xi, \gamma) &= \frac{\xi_1^3}{3} + \xi_1\xi_2^2 + \delta\xi_2^2 + \beta\xi_1 \\ &\quad + o(|\xi|^3) + O(|\xi|^3)O(\gamma) \\ \xi_1 - a\xi_2 &\geq 0, \quad \xi_1 + b\xi_2 \geq 0. \end{aligned}$$

This method supposes that $f: \Omega \subset E \rightarrow F$ is a smooth nonlinear Fredholm map of index zero. The map f has variational property, if there exists a functional $V: \Omega \subset E \rightarrow \mathbb{R}$ such that $f = \text{grad}_H V$ or equivalently,

$$\frac{\partial V}{\partial x}(x, \lambda)h = \langle f(x, \lambda), h \rangle_H, \quad \forall x \in \Omega, h \in E,$$

where $\langle \cdot, \cdot \rangle_H$ is the scalar product in Hilbert space H. Also it assumes that $E \subset F \subset H$. The solutions of equation $f(x, \lambda) = 0$ are the critical points of functional $V(x, \lambda)$. The method of finite-dimensional reduction (Lyapunov-Schmidt method) can reduce the problem,

$$V(x, \lambda) \rightarrow \text{extr } x \in E, \lambda \in \mathbb{R}^n$$

into equivalent problem

$$W(\xi, \lambda) \rightarrow \text{extr } \xi \in \mathbb{R}^n,$$

where $W(\xi, \lambda)$ is called the key function. If we let $N = \text{span}\{e_1, \dots, e_n\}$ is a subspace of E , where e_1, \dots, e_n is an orthonormal set in H , then the key function $W(\xi, \lambda)$ may be defined by the form of

$$W(\xi, \lambda) = \inf_{x: \langle x, e_i \rangle = \xi_i} V(x, \lambda), \quad \xi =$$

(ξ_1, \dots, ξ_n) . The function W has all the topological and analytical properties of the

functional V (multiplicity, bifurcation diagram, etc) [7]. The study of bifurcation solutions of functional V is equivalent to the study of bifurcation solutions of key function. If f has a variational property, then the equation

$$\Theta(\xi, \lambda) = \text{grad}W(\xi, \lambda) = 0$$

is called the bifurcation equation.

Now, we will introduce some basic concepts:

Definition 1.1 [6]: A germ or function-germ at a point p is an equivalence class of germ-equivalent maps where germ-equivalent maps mean: Two maps $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^p$ are said to be germ-equivalent at $p \in \mathbb{R}^n$ if p is in the domain of both and there is a neighborhood U of p such that the restrictions coincide.

Definition 1.2 [6]: For a germ $f \in \mathcal{E}_n$ (the set of all germs at the origin of smooth functions on \mathbb{R}^n) define the Jacobian ideal to be the ideal generated by the partial derivatives of f , $I_f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$.

Definition 1.3 [6]: A germ in \mathcal{E}_n is said to be of finite codimension if the Jacobian ideal is of finite codimension in \mathcal{E}_n . If f has a critical point at the origin then $f \in m_n^2$ (where $m_n = \{f \in \mathcal{E}_n : f(0) = 0\}$ is the only maximal ideal in \mathcal{E}_n) and in this case $I_f \subset m_n$ and one says its codimension is $\text{cod}(f) := \dim\left(\frac{m_n}{I_f}\right)$. This number is finite if and only if f is of finite codimension.

Definition 1.4 [6]: A smooth λ -parameter family of functions on \mathbb{R}^n is a smooth map $F : \mathbb{R}^n \times \mathbb{R}^\alpha \rightarrow \mathbb{R}$ where \mathbb{R}^α is parameter space. It called an unfolding or deformation of a given germ $f \in \mathcal{E}_n$ if $f(x) = F(x, 0)$.

Remark 1.1: Two germs f and g are said to be contact equivalent if their zero - sets ($f^{-1}(0)$ and $g^{-1}(0)$) are diffeomorphic.

Remark 1.2: The multiplicity of the critical point of f at the origin which is denoted by μ is defined as follows $\mu = \text{cod}(f) + 1$.

Definition 1.5 [4]: The set of all λ for which the function $f(x, \lambda)$ has degenerate critical points is called Caustic and denoted by Σ ,

$$\Sigma = \left\{ \lambda \in \mathbb{R} : \frac{\partial f}{\partial x} = 0, \text{Det}(d^2(f_\lambda)_x) = 0 \right\},$$

where $d^2(f_\lambda)_x$ is the second derivative of f with respect to x and $\text{Det}()$ denotes the determinant of a matrix.

2. BOUNDARY SINGULARITIES OF FREDHOLM FUNCTIONAL [10]

To investigate the behavior of a Fredholm functional in a neighborhood of an angular singular point, one uses the reduction to an analogous problem

$$W(x) \rightarrow \text{extr},$$

where

$$\begin{aligned} x &\in C, \\ C &= \{x \\ &= (x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_{m+t}, \dots, x_n)^T \\ &\in \mathbb{R}^n; \end{aligned}$$

$$\exists \text{ integers } t, m \ni 0 \leq t \leq n - m \text{ and } x_m \geq 0, x_{m+1} \geq 0, \dots, x_{m+t} \geq 0\}.$$

We say that a point $a \in C$ is conditionally critical for a smooth function W in \mathbb{R}^n if $\text{grad}W(a)$ is orthogonal to the least face of C containing a and the set $\{(x_m, x_{m+1}, \dots, x_{m+t})^T \in \mathbb{R}^n : x_m \geq 0, x_{m+1} \geq 0, \dots, x_{m+t} \geq 0\}$ is called the m -hedral angle.

The multiplicity of the conditionally critical point a (and is denoted by $\hat{\mu}$) is the

dimension of the quotient algebra denotes the set,

$$\hat{Q} = \frac{\prod_a (\mathbb{R}^n)}{I},$$

where $\prod_a (\mathbb{R}^n)$ is the ring of germs of smooth functions on \mathbb{R}^n at the point a and

$$I := \begin{cases} \left(\frac{\partial W}{\partial x_1}, \dots, x_m \frac{\partial W}{\partial x_m}, \dots \right. & \text{for all } m > 1, \\ \left. x_{m+t} \frac{\partial W}{\partial x_{m+t}}, \dots, \frac{\partial W}{\partial x_n} \right) \\ \left(x_m \frac{\partial W}{\partial x_m}, \dots, x_{m+t} \frac{\partial W}{\partial x_{m+t}} \right. & \text{for } m = 1, \\ \left. \dots, \frac{\partial W}{\partial x_n} \right) \end{cases}$$

is the angular Jacobi ideal in $\prod_a (\mathbb{R}^n)$. The multiplicity $\hat{\mu}$ of a conditionally critical point a is equal to the sum of multiplicities $\mu + \mu_0$, where μ is the (usual) multiplicity of W on \mathbb{R}^n , while μ_0 is the (usual) multiplicity of the restriction $W|_{\partial C}$ (where ∂C is the boundary of the set C).

Then, we reduce the space of $W(x)$, $x \in \mathbb{R}^n$ to the space C as follows: let $\{e_1, \dots, e_n\}$ be an orthonormal set in H . By Lyapunov-Schmidt method one can write any element $z \in E$ as the form, $z = u + v$ where $u = \sum_{i=1}^n x_i e_i$, $v \perp e_i$, $x_i \in \mathbb{R}$, $i = 1, 2, \dots, n$.

In addition, let \mathbb{R}^n be a space defined by map $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$\pi(y) = \begin{cases} (y_1^2, y_2, \dots, y_n) & \text{if } y = (y_1, y_2, \dots, y_n), \\ (y_1, \dots, y_s, y_{s+1}^2, y_{s+1}, \dots, y_n) & \text{for some } 1 < s \leq n. \end{cases}$$

Then the function $W(x)$, $x \in \mathbb{R}^n$ lifts to the covering space \mathbb{R}^n by the relation $\widehat{W}(\pi(y)) = \widehat{W}(y)$. The function \widehat{W} is

invariant with respect to the natural involution

$$J(y_1, \dots, y_s, \dots, y_n) = (y_1, \dots, -y_s, \dots, y_n).$$

From the definition of function π we obtain $x_s = y_s^2$, so we have $x_s \geq 0$. Also, we require that the coordinates $\{x_s\}_{s \in \exists 1 \leq s \leq n}$ are equal to the coordinates of m -hedral angle. From above we conclude the domain of $W(x)$, $x \in \mathbb{R}^n$ may reduce to the space C .

If a critical point is "usual," then spreadings of bifurcating extremals (bif-spreadings) are represented by the row (c_0, c_1, \dots, c_n) , where c_i is the number of critical points of the Morse index i . If we are dealing with an angular (or boundary) critical point, then bif-spreadings are represented by the following matrix of order $m + 1 \times n + 1$:

$$\begin{pmatrix} c_0^1 & c_1^1 & \dots & \dots & c_n^1 \\ c_0^2 & c_1^2 & \dots & \dots & c_n^2 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ c_0^{m+t} & c_1^{m+t} & \dots & \dots & c_n^{m+t} \\ c_0 & c_1 & \dots & \dots & c_n \end{pmatrix}.$$

Here c_i^j is the number of the angular critical points of index i (for $j = 1, 2, \dots, m + t$), while c_i is the number of usual (situated inside C) critical points of index i . In this paper, we let $(n = 2, m = 1, t = 0, s = 1)$ and $(n = 2, m = 1, t = 1, s = 1, 2)$ for problems in section 3 and section 4, respectively.

3. SINGULARITIES OF THE FUNCTION OF CODIMENSION EIGHT

In this section, we consider the function that has codimension eight at the origin [2] defined by

$$W(z, \rho) = \frac{x_1^4}{4} + \frac{x_2^4}{4} + x_1^2 x_2 + \lambda_1 x_1^2 + \lambda_2 x_2, \tag{3.1}$$

where $z = (x_1, x_2)$, $\rho = (\lambda_1, \lambda_2)$.

Function (3.1) has multiplicity 9 and then it has codimension 8. The main purpose is to find a geometrical description (bifurcation diagram) of the caustic of function (3.1) and then to determine the spreading of the critical points of this function. To avoid some difficulties in the study of function (3.1), we assume the following $x_1^2 = x$ and $x_2 = y$. So the study of function (3.1) is equivalent to the study of the following function

$$W(z, \rho) = \frac{x^2}{4} + \frac{y^4}{4} + xy + \lambda_1 x + \lambda_2 y, \tag{3.2}$$

where $z = (x, y)$, $x \geq 0$ and $\rho = (\lambda_1, \lambda_2)$.

Since, the germ of function (3.2) is

$$W_0 = \frac{x^2}{4} + \frac{y^4}{4}.$$

So, from section 2 we have

$$I = \left(x \frac{\partial W_0}{\partial x}, \frac{\partial W_0}{\partial y} \right) = \left(\frac{x^2}{2}, y^3 \right) = (x^2, y^3).$$

Accordingly, the multiplicity of function (3.2) is $\hat{\mu} = 6$ where $\mu = \mu_0 = 3$. Since multiplicity is equal to the number of critical points [3], hence the number of critical points of function (3.2) is six, three points lie on the boundary $x = 0$ and three points lie in the interior, so the caustic of function (3.2) is the union of three sets,

$$\Sigma = \Sigma_{0,1}^{int} \cup \Sigma_{0,1}^{ext} \cup \Sigma_{1,1},$$

where $\Sigma_{0,1}^{int}$ and $\Sigma_{0,1}^{ext}$ are the subsets (components) of the caustic corresponding to the degeneration of boundary singularities along the boundary and along the normal, respectively, while $\Sigma_{1,1}$ is the component corresponding to the degeneration of interior (non-boundary) critical points.

3.1 Degeneration Along The Boundary (Internal Degeneration)

To determine the set $\Sigma_{0,1}^{int}$, we consider boundary critical points of function (3.2) such that the second-order partial derivative of this function with respect to y vanishes at these points, i.e, the following relations are valid:

$$\frac{\partial W(0,y,\lambda_1,\lambda_2)}{\partial y} = \frac{\partial^2 W(0,y,\lambda_1,\lambda_2)}{\partial y^2} = 0$$

or

$$\lambda_2 + y^3 = 3y^2 = 0.$$

From these relations, we easily obtain that the set $\Sigma_{0,1}^{int}$ is defined by the equation

$$\lambda_2 = 0.$$

3.2 Degeneration Along The Boundary (External Degeneration)

To determine the set $\Sigma_{0,1}^{ext}$ we consider boundary critical points of function (3.2) such that the first-order partial derivative of this function with respect to x vanishes at these points, i.e, the following relations are valid:

$$\frac{\partial W(0,y,\lambda_1,\lambda_2)}{\partial y} = \frac{\partial W(0,y,\lambda_1,\lambda_2)}{\partial x} = 0$$

or

$$\lambda_2 + y^3 = \lambda_1 + y = 0.$$

From these relations, it is easy to see that the set $\Sigma_{0,1}^{ext}$ is given by the equation

$$-(\lambda_2 - \lambda_1^3) = 0.$$

3.3 Degeneration Of Interior (Non-boundary)

To determine the set $\Sigma_{1,1}$, we consider the critical points of function (3.2) defined by the system,

$$\frac{\partial W(x,y,\lambda_1,\lambda_2)}{\partial x} = \frac{\partial W(x,y,\lambda_1,\lambda_2)}{\partial y} = 0, \quad x > 0$$

or

$$\lambda_1 + \frac{x}{2} + y = \lambda_2 + x + y^3 = 0. \quad (3.3)$$

Then, make the determinate of the Hessian matrix of function (3.2) equal to zero to get the equation,

$$\frac{3y^2}{2} - 1 = 0. \quad (3.4)$$

Solving relations (3.3) with the equation (3.4), we have, $x = \lambda_1 - \frac{3\lambda_2}{2}$ and $y = \frac{3\lambda_2 - 3\lambda_1}{4}$ (we use Maple 2016 soft program in solving). By substituting y in equation (3.4) we get the parametric equation of the set $\Sigma_{1,1}$,

$$108\lambda_1^2 - 108\lambda_2\lambda_1 + 27\lambda_2^2 - 32 = 0.$$

The caustic equation of function (3.2) is as follows:

$$\lambda_2(108\lambda_1^2 - 108\lambda_2\lambda_1 + 27\lambda_2^2 - 32)(\lambda_2 - \lambda_1^3) = 0.$$

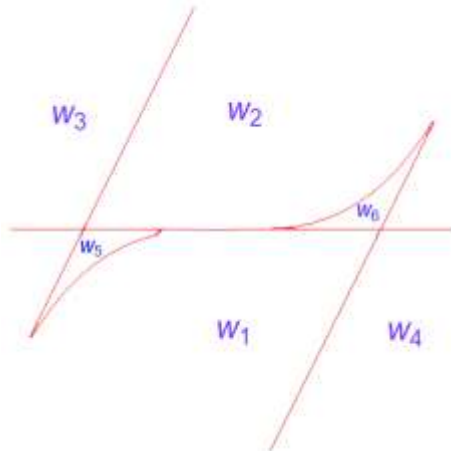


Figure 1: Describes the caustic of function (3.2) in $\lambda_1 \lambda_2$ -plane.

The caustic of function (3.2) as in figure 1 decomposes the plane of parameters into six regions $W_i, i = 1,2,3,4,5,6$; every region contains a fixed number of critical points such that the spreading of the critical points is as follows: if the parameters belong to

1. $W_1 \cup W_6$, then we have three critical points (one saddle boundary critical point, one saddle interior critical point and one minimum interior critical point), or
2. $W_2 \cup W_5$, then we have two critical points (one saddle boundary critical point and one minimum interior critical point), or
3. W_3 , then we have two critical points (one minimum boundary critical point and one minimum interior critical point), or
4. W_4 , then we have one only minimum boundary critical point.

The matrices of bif-spreadings are as follows:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In figure 2, the parts (a), (b), (c) and (d) show the locations of contour lines with respect to the boundary of the domain of function (3.2), number and type of critical points corresponding for all region in caustic of function (3.2), where showing of (a) corresponds region $W_1 \cup W_6$, (b) corresponds region $W_2 \cup W_5$, (c) corresponds region W_3 and (d) corresponds region W_4 .

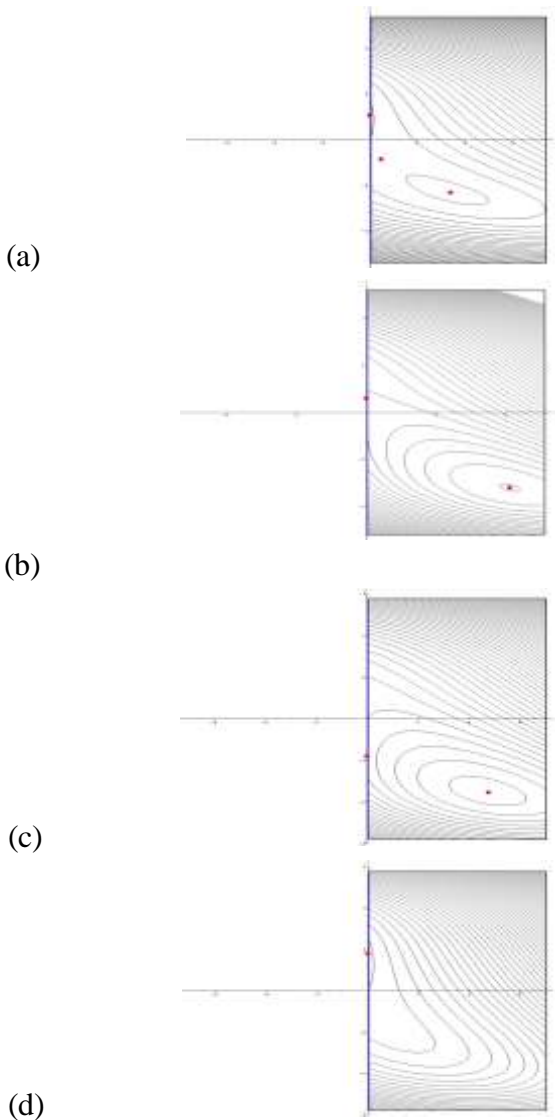


Figure 2:

4. SINGULARITIES OF THE FUNCTION OF CODIMENSION TWENTY FOUR

In this section, we consider the function that has codimension twenty four at the origin [2] defined by

$$W(z, \rho) = \frac{x_1^6}{6} + \frac{x_2^6}{6} + x_1^2 x_2^4 + x_1^4 x_2^2 + \frac{x_1^4}{4} + \frac{x_2^4}{4} + x_1^2 x_2^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2, \tag{4.1}$$

where $z = (x_1, x_2), \rho = (\lambda_1, \lambda_2)$.

Function (4.1) has multiplicity twenty five and then it has codimension twenty four. The main purpose is to find geometrical description (bifurcation diagram) of the caustic of function (4.1) and then to determine the spreading of the critical points of this function. To avoid some difficulties in the study of function (4.1), we assume the following $x_1^2 = x$ and $x_2^2 = y$. So the study of function (4.1) is equivalent to the study of the following function,

$$W(z, \rho) = \frac{x^3}{6} + \frac{y^3}{6} + xy^2 + x^2y + \frac{x^2}{4} + \frac{y^2}{4} + xy + \lambda_1 x + \lambda_2 y, \tag{4.2}$$

where, $z = (x, y), x \geq 0, y \geq 0$ and $\rho = (\lambda_1, \lambda_2)$. Since the germ of function (4.2) is $W_0 = \frac{x^3}{6} + \frac{y^3}{6}$. So, from section 2 we have,

$$I = (x \frac{\partial W_0}{\partial x}, y \frac{\partial W_0}{\partial y}) = (\frac{x^3}{2}, \frac{y^3}{2}) = (x^3, y^3).$$

Accordingly, the multiplicity of function (4.2) is $\hat{\mu} = 9$ where $\mu = 3$ and $\mu_0 = 6$. Since multiplicity is equal to the number of critical points [3], hence the number of critical points of function (4.2) is nine, three points lie on the boundary $x = 0$, three points lie on the boundary $y = 0$ and three points lie in the interior. So the caustic of function (4.2) is the union of six sets,

$$\Sigma = \Sigma_{0,0} \cup \Sigma_{0,1}^{int} \cup \Sigma_{0,1}^{ext} \cup \Sigma_{1,0}^{int} \cup \Sigma_{1,0}^{ext} \cup \Sigma_{1,1}$$

where $\Sigma_{0,0}$ is the subset (component) of the caustic corresponding to degeneration at the vertex $(0, 0)$, $\Sigma_{0,1}^{int}$ and $\Sigma_{0,1}^{ext}$ are the subsets (components) of the caustic corresponding to the degeneration of boundary singularities along the boundary $x = 0$ and along the normal, respectively, $\Sigma_{1,0}^{int}$ and $\Sigma_{1,0}^{ext}$ are the subsets (components) of the caustic corresponding to the degeneration of boundary singularities along the boundary $y = 0$ and along the normal, respectively, while $\Sigma_{1,1}$ is the component corresponding to the degeneration of interior (non-boundary) critical points.

4.1 Degeneration At The Vertex (0, 0)

To determine the set $\Sigma_{0,0}$, we must find the following union

$$\{ (\lambda_1, \lambda_2) : \frac{\partial W(0,0,\lambda_1,\lambda_2)}{\partial x} = 0 \} \cup \{ (\lambda_1, \lambda_2) : \frac{\partial W(0,0,\lambda_1,\lambda_2)}{\partial y} = 0 \},$$

this implies the union consists of all pairs (λ_1, λ_2) which satisfy the conditions,

$$\lambda_1 = 0 \text{ or } \lambda_2 = 0.$$

Hence, from these relations, we have that the set $\Sigma_{0,0}$ consists of the union of the lines $\lambda_1 = 0$ and $\lambda_2 = 0$.

4.2 Degeneration Along The Boundary $x = 0$ (Internal Degeneration)

To determine the set $\Sigma_{0,1}^{int}$, we consider boundary critical points of function (4.2) such that the second-order partial derivative of this function with respect to y vanishes at these points, i.e, the following relations are valid:

$$\frac{\partial W(0,y,\lambda_1,\lambda_2)}{\partial y} = \frac{\partial^2 W(0,y,\lambda_1,\lambda_2)}{\partial y^2} = 0, \quad y > 0$$

or

$$\lambda_2 + \frac{y^2}{2} + \frac{y}{2} = y + 1/2 = 0.$$

From these relations, we can clearly observe that $y = -\frac{1}{2} < 0$. But $y > 0$ by the above assumption, hence the set $\Sigma_{0,1}^{int}$ is empty.

4.3 Degeneration Along The Boundary $x = 0$ (External Degeneration)

To determine the set $\Sigma_{0,1}^{ext}$ we consider boundary critical points of function (4.2) such that the first-order partial derivative of this function with respect to x vanishes at these points, i.e, the following relations are valid:

$$\frac{\partial W(0,y,\lambda_1,\lambda_2)}{\partial y} = \frac{\partial W(0,y,\lambda_1,\lambda_2)}{\partial x} = 0, \quad y > 0$$

or

$$\lambda_2 + \frac{y^2}{2} + \frac{y}{2} = \lambda_1 + y^2 + y = 0,$$

this implies $\lambda_2 + \frac{y^2}{2} + \frac{y}{2} = \frac{1}{2}\lambda_1 + \frac{1}{2}y^2 + \frac{1}{2}y = 0$ which implies $\lambda_1 = 2\lambda_2$ which represents the set $\Sigma_{0,1}^{ext}$.

4.4 Degeneration Along The Boundary $y = 0$ (Internal Degeneration)

To determine the set $\Sigma_{1,0}^{int}$, we consider boundary critical points of function (4.2) such that the second-order partial derivative of this function with respect to x vanishes at these points, i.e, the following relations are valid:

$$\frac{\partial W(x,0,\lambda_1,\lambda_2)}{\partial x} = \frac{\partial^2 W(x,0,\lambda_1,\lambda_2)}{\partial x^2} = 0, \quad x > 0$$

or

$$\lambda_1 + \frac{x^2}{2} + \frac{x}{2} = x + 1/2 = 0.$$

These relations give a contradiction with the assumption $x > 0$, so the set $\Sigma_{1,0}^{int}$ is empty.

4.5 Degeneration Along The Boundary $y = 0$ (External Degeneration)

To determine the set $\Sigma_{1,0}^{ext}$ we consider boundary critical points of function (4.2) such that the first-order partial derivative of this function with respect to y vanishes at these points, i.e., the following relations are valid:

$$\frac{\partial W(x,0,\lambda_1,\lambda_2)}{\partial x} = \frac{\partial W(x,0,\lambda_1,\lambda_2)}{\partial y} = 0, x > 0$$

or

$$\lambda_1 + \frac{x^2}{2} + \frac{x}{2} = \lambda_2 + x^2 + x = 0.$$

this implies $\lambda_1 + \frac{x^2}{2} + \frac{x}{2} = \frac{1}{2}\lambda_2 + \frac{1}{2}x^2 + \frac{1}{2}x = 0$ which implies $\lambda_2 = 2\lambda_1$ of the elements of the set $\Sigma_{1,0}^{ext}$.

4.6 Degeneration Of Interior (Non-boundary)

To determine the set $\Sigma_{1,1}$, we consider the critical points of function (4.2) defined by the system,

$$\frac{\partial W(x,y,\lambda_1,\lambda_2)}{\partial x} = \frac{\partial W(x,y,\lambda_1,\lambda_2)}{\partial y} = 0, x > 0, y > 0$$

or

$$\lambda_1 + \frac{x^2}{2} + 2xy + \frac{x}{2} + y^2 + y = \lambda_2 + x^2 + 2xy + \frac{y^2}{2} + \frac{y}{2} + x = 0. \tag{4.3}$$

Then, make the determinate of the Hessian matrix of function (4.2) equal to zero to get the equation,

$$-3xy - 2x^2 - \frac{5}{2}x - 2y^2 - \frac{5}{2}y - \frac{3}{4} = 0. \tag{4.4}$$

Theoretically, it is difficult to solve the relations (4.3) with equation(4.4). So, we use Mathematica 11.3 program in solving these relations to get the parameter equation of the set $\Sigma_{1,1}$,

$$114688\lambda_1^4 + \lambda_1^3(-51200 - 344064\lambda_2) + \lambda_1^2(12992 + 54272\lambda_2 + 487424\lambda_2^2) + \lambda_1(-80 - 26368\lambda_2 + 54272\lambda_2^2 - 344064\lambda_2^3) = 9 + 80\lambda_2 - 12992\lambda_2^2 + 51200\lambda_2^3 - 114688\lambda_2^4.$$

If we write every equation of the caustic components the equations by making its left-hand side=its right-hand side=0, then the parametric equation of caustic of function (4.2) will consist of the product of multiplying of the left-hand sides of all the equations of caustic components with making it equal to zero.

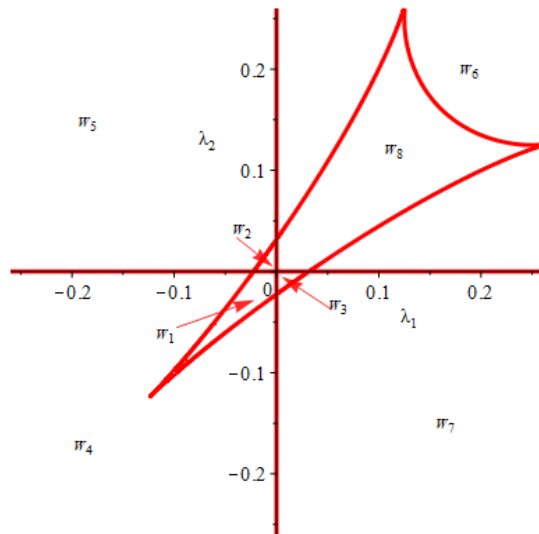


Figure 7: Describes the caustic of function (4.2) in $\lambda_1 \lambda_2$ -plane.

The caustic of function (4.2) as in figure 7 decomposes the plane of parameters into eight regions $W_i, i = 1,2,3,4,5,6,7,8$; every region contains a fixed number of critical points such that the spreading of the critical points is as follows: if the parameters belong to

1. W_1 , then we have three critical points (one minimum point on boundary $x = 0$, one minimum point on boundary $y = 0$ and one saddle point in the interior), or

2. W_4 , then we have two critical points (one minimum point on boundary $x = 0$ and one saddle point in the interior), or
3. $W_3 \cup W_7$, then we have one minimum critical point on boundary $x = 0$, or
4. $W_2 \cup W_5$, then we have one minimum critical point on boundary $y = 0$, or
5. $W_6 \cup W_8$, then we have not any real critical point lie in domain $\{(x, y): x \geq 0, y \geq 0\}$.

The matrices of bif-spreadings are as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In addition, the values of the Morse index at a given vertex corresponds to one of the previous eight regions are defined as follows: $index = 0 \Leftrightarrow vertex \in W_6 \cup W_8$; $index = 2 \Leftrightarrow vertex \in W_1 \cup W_4$; and $index = 1 \Leftrightarrow vertex \in W_2 \cup W_3 \cup W_5 \cup W_7$.

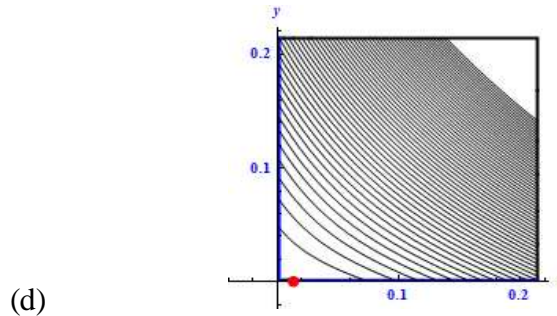
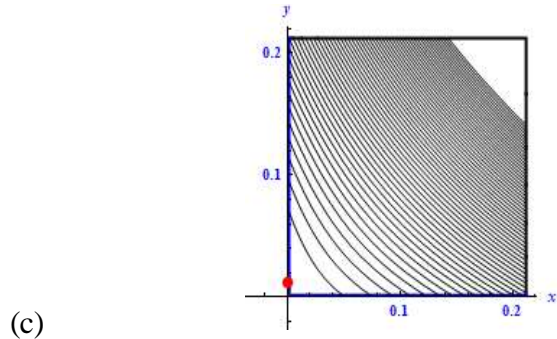
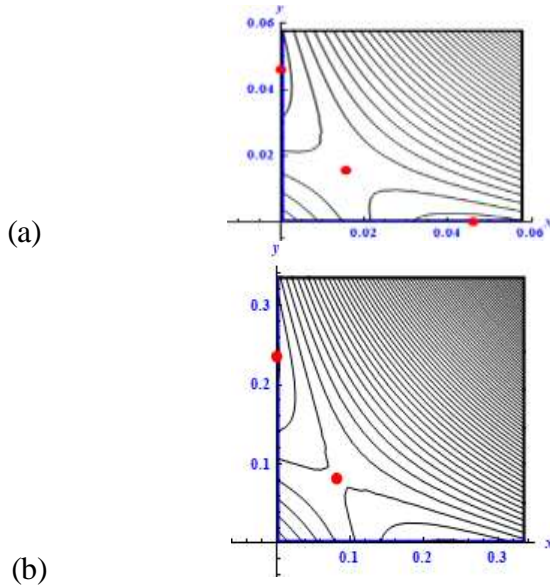


Figure 4:

In figure 4, the parts (a), (b), (c), and (d) show the locations of contour lines with respect to the boundaries of the domain of function (4.2), number and type of critical points corresponding for all region in caustic of function (4.2), where showing of (a) corresponds region W_1 , (b) corresponds region W_4 , (c) corresponds region $W_3 \cup W_7$, and (d) corresponds region $W_2 \cup W_5$.

5. APPLICATIONS

To illustrate the result obtained in section 4, we give the following example which is a nonlinear fourth order differential equation. This equation describes the oscillations and motion of wave of the elastic beams on elastic foundations that can be described by means of the following ODE,

$$u'''' + \alpha u'' + \beta u + u^3 + u^5 = 0, \quad (5.1)$$

$$u(0) = u(1) = u''(0) = u''(1) = 0$$

where α, β are the parameters of the problem, $u = u(x), x \in [0,1], ' = \frac{d}{dx}$.

Suppose that $f: E \rightarrow M$ is a nonlinear Fredholm operator of index zero from Banach space E to Banach space M , where $E = C^4([0,1], \mathbb{R})$ is the space of all continuous functions that have derivative of order at most four, $M = C^0([0,1], \mathbb{R})$ is the space of all continuous function and f are defined by the operator equation,

$$f(u, \lambda) = u'''' + \alpha u'' + \beta u + u^3 + u^5 = 0, \quad (5.2)$$

where $\lambda = (\alpha, \beta)$. Every solution of the equation (5.1) is a solution of the operator equation (5.2). Since, the operator f has variational property, then there exist functional V such that,

$$f(u, \lambda) = \text{grad}_H V(u, \lambda)$$

where,

$$V(u, \lambda) = \int_0^1 \left(\frac{(u'')^2}{2} - \alpha \frac{(u')^2}{2} + \beta \frac{u^2}{2} + \frac{u^4}{4} + \frac{u^6}{6} \right) dx.$$

In this case every solution of the equation (5.2) is a critical point of the functional V .

In the following theorem, we show that the study of bifurcation of extremals of the functional V is reduced to the study of bifurcation of extremals of the function (4.1).

Theorem 5.1: The normal form of the key function W_1 corresponding to the functional V is given by,

$$W_1(y, \rho) = \frac{x_1^6}{6} + \frac{x_2^6}{6} + x_1^2 x_2^4 + x_1^4 x_2^2 + \frac{x_1^4}{4} + \frac{x_2^4}{4} + x_1^2 x_2^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2,$$

$$y = (x_1, x_2), \rho = (\lambda_1, \lambda_2).$$

Proof. By using the scheme of Lyapunov-Schmidt, the linearized equation corresponding to the equation (5.2) at the point $(0, \lambda)$ has the form,

$$A h = 0, h \in E$$

$$h(0) = h(1) = h''(0) = h''(1) = 0$$

$$\text{where } A = \frac{d^4}{dx^4} + \alpha \frac{d^2}{dx^2} + \beta.$$

The solution of the linearized equation which satisfies the initial conditions is given by $e_p(x) = c_p \sin(p\pi x)$, $p = 1, 2, \dots$ and the characteristic equation corresponding to this solution is

$$(p\pi)^4 - \alpha(p\pi)^2 + \beta = 0.$$

This equation gives in $\alpha\beta$ -plane characteristic lines ℓ_p . The characteristic lines ℓ_p consist of the points (α, β) for which the linearized equation has non-zero solutions [8]. The point of intersection of the characteristic lines in $\alpha\beta$ -plane is bifurcation point, so the bifurcation point for the equation (5.2) is $(\alpha, \beta) = (5\pi^2, 4\pi^4)$. Localized parameters α, β as following, $\alpha = 5\pi^2 + \delta_1$, $\beta = 4\pi^4 + \delta_2$, δ_1, δ_2 are small parameters, lead to bifurcation along the modes, $e_1(x) = c_1 \sin(\pi x)$, $e_2(x) = c_2 \sin(2\pi x)$. Since, $\|e_1\| = \|e_2\| = 1$ then we have $c_1 = c_2 = \sqrt{2}$.

Let $N = \text{Ker}(A) = \text{span} \{e_1, e_2\}$, then the space E can be decomposed in the direct sum of two subspaces, N and the orthogonal complement to N ,

$$E = N \oplus N^\perp, N^\perp = \{v \in$$

$$E: \int_0^1 v e_k dx = 0, k = 1, 2 \}.$$

Similarly, the space M can be decomposed in the direct sum of two

subspaces, N and the orthogonal complement to N ,

$$M = N \oplus \tilde{N}^\perp, \tilde{N}^\perp = \{ \omega \in \int_0^1 \omega e_k dx = 0, k = 1, 2 \}.$$

There exists two projections $P: E \rightarrow N$ and $I - P: E \rightarrow N^\perp$ such that $Pu = \omega$ and $(I - P)u = v$, (I is the identity operator). Hence every vector $u \in E$ can be written in the form,

$$u = \omega + v, \omega = x_1 e_1 + x_2 e_2 \in N, v \in N^\perp, x_i = \langle u, e_i \rangle.$$

Thus, by the implicit function theorem, there exists a smooth map $\Theta: N \rightarrow N^\perp$, such that

$$\tilde{W}(z, \gamma) = V(\Theta(z, \gamma), \gamma), z = (x_1, x_2), \gamma = (\delta_1, \delta_2).$$

and then the key function \tilde{W} can be written in the form,

$$\tilde{W}(z, \gamma) = V(x_1 e_1 + x_2 e_2 + \Theta(x_1 e_1 + x_2 e_2, \gamma), \gamma) = W_2(z, \gamma) + o(|z|^6) + O(|z|^6) O(\gamma),$$

where,

$$W_2(z, \gamma) = \frac{5}{12} x_1^6 + \frac{5}{12} x_2^6 + \frac{15}{4} x_1^2 x_2^4 + \frac{25}{8} x_1^4 x_2^2 + \frac{3}{8} x_1^4 + \frac{3}{8} x_2^4 + \frac{3}{2} x_1^2 x_2^2 + \left(\frac{1}{2} \pi^4 - \frac{1}{2} \pi^2 \alpha + \frac{1}{2} \beta \right) x_1^2 + \left(8 \pi^4 - 2 \pi^2 \alpha + \frac{1}{2} \beta \right) x_2^2.$$

The geometrical form of bifurcations of critical points and the first asymptotic of branches of bifurcating for the function \tilde{W} are completely determined by its principal part W_2 . If, we replace x_1 and x_2 by $\sqrt[6]{\frac{2}{5}} x_1$ and $\sqrt[6]{\frac{2}{5}} x_2$ in function W_2

respectively, then W_1 and W_2 are contact equivalence, since in this case, they have the same germ,

$$W_0(x_1, x_2) = \frac{1}{6} x_1^6 + \frac{1}{6} x_2^6$$

and deformation. Therefore the caustic of the function W_2 coincides with the caustic of the function W_1 .

Thus, the function W_1 has all the topological and analytical properties of functional V , so the study of bifurcation analysis of the equation (5.2) is equivalent to the study of bifurcation analysis of the function W_1 . This shows that the study of bifurcation of extremals of the functional V is reduced to the study of bifurcation of extremals of the function (4.1). ■

Conclusion

In this paper, the singularities of the functions (3.1) and (4.1) were studied. For the study of the function (3.1), it was a theoretical study, whereas the study of the function (4.1) was reinforced by an applied example. We found the geometric description of the branching diagram with the spread of the critical points in the areas connected by this diagram, such that each region contains a constant number in terms of the number and quality of the critical points. In the applied study we found the solution areas of the equation (5.1) such that each critical point in the area represents a solution to the equation (5.1), such that this critical point represents the critical point of functional V which in turn corresponds to a critical point of the key function of V . Studying the branching solutions for the equation (5.1) is an application for studying singularities of the function (4.1)

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تحليل حلول التفرع بواسطة المتفرقات الحدودية

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الخلاصة:

يقدم هذا البحث دراسة حول تفرع النهايات للدوال ذات البعد المرافق ثمانية و البعد المرافق أربعة وعشرين عند نقطة الاصل. لقد استخدمنا المتفرقات الحدودية للدوال الملساء لدراسة تحليل التفرع لهذه الدوال. قمنا بإيجاد معادلة المعلمات للكاوستك (مجموعة التفرع) مع الوصف الهندسي للكاوستك. فضلا عن، وجدنا أنتشار التفرع للنقاط الحرجة مع تقديم تطبيق على واحدة من نتائجنا.

الكلمات المفتاحية: حلول التفرع ، المتفرقات الحدودية ، الكاوستك (مجموعة التفرع).