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The Moment for Some quotient Stochastic Differential Equation with Application

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ABSTRACT

In this article we use the product stochastic differential equations in order to study the solution for some quotient stochastic differential equation by using itô's formula, Then we find their moments (mean, variance and the k.th moments). Also we gave some examples to explain the method.

1. Introduction

The main definition of the Stochastic differential equations (simply SDE's) is that differential equations in which one or more of its terms are stochastic (random) processes, for which their solutions may be stochastic process, [Arnold, 1974. The Stochastic differential equations (SDEs) used in many field of science such as biology, chemistry, climatology, mechanics, physics, economics and finance. Many researcheres have given their contribution in these field (Akinbo B.J., et, al. (2015)), Guangqiang LAN et, al. (2014) derive the new sufficient conditions of existence, moment of the solution of stochastic differential equation, J.C. Jimenez (2015) uses the explicit formulas for the mean and variance of the solutions of linear stochastic differential equation, Platens [5] study the strong and weak approximation methods for the numerical methods to get the solution of stochastic differential equations, Nayak and Chakraverty [6] worked on numerical solution of fuzzy stochastic differential equation. Christios H.skiadas, [7] Study the exact solution of stochastic differential equation (Gomertz, Generalized logistic and revised exponential. Akinbo B.J. et al [2] study numerical solution of stochastic differential equation, and so on.

In this paper we study some form of stochastic differential equation as a quotient stochastic differential equation, then we explain how to apply itô's integral formula to find the solution of those equations and we find the moments of their solutions.

2. Preliminaries and method

Definition 1 :(random variable) [5]

A random variable is a mapping or a function from the sample space Ω onto the real line R, (i.e. X: Ω →R)

Definition 2 :(Expectation) or (mean) of a random variable:[5]

Let X is a random variable defined on the probability space (Ω, F, P), then the expected values or the mean of X is:

$E(X) = \mu = \sum_i x_i p(x_i)$.

That is the average of X over the entire probability space.

For a random variable continuous over R:

$E(X) = \int_{-\infty}^{\infty} x f(x) dx$

Definition 3: (Variance):[5]

The Variance is a measure of the spread of data about the mean μ $Var(X) = E((X-\mu)^2) = E(X^2) - (E(X))^2$

Definition 4: The kth -order moment:[5]

The kth -order moment of a continuous random variable is defined by:

$E(X^k) = \int_{-\infty}^{\infty} x^k f(x) dx$

Where f(x) is the probability density function

Or $\mathbf{E}(\mathbf{x}^{\mathbf{k}}) = \sum_{\mathbf{i}} \mathbf{x}_{\mathbf{i}}^{\mathbf{k}} \mathbf{p}(\mathbf{x}_{\mathbf{i}})$ **;** (For discrete time and $\mathbf{p}(\mathbf{x}_{\mathbf{i}})$ is probability mass function)

Definition 5: (stochastic process) [1]

A stochastic process is a family of random variables denoted by $\{x(t), t \in T\}$ where t is time parameter and $T \in R$.

Definition 6: (Wiener process) [1]

A wiener process (Brownian motion) over [0, T] denoted by {w (t)} is a continuous-time stochastic process satisfying:

1: W (0) =0

2: For all t, s ≥ 0, W (t)-W(s) is normally distributed with mean zero and variance$|t-s|$.

3: The increment's W (t)-W(s) and W (v)-W (u) are independent.

Definition 7 :(Itô –formula)[1]

Let X (.) be a real-valued stochastic process which satisfying

$$x(a) = x(b) + \int_a^b F dt + \int_a^b G dw \quad \dots (1)$$

For some G ϵ $L^2(0, T)$, F ϵ $L^1(0,T)$ and 0<a<b<T. Then we say that X (.) has a stochastic differential equation

dX =Fdt+Gdw; for 0<t<T …(2)

Remark. [3]

L^1 [0, T], L^2[0, T] denotes the space of all real-valued, adaptive processes $\{x_t\}$, $\{y_t\}$ respectively , such that

$$E\left(\int_0^T |x_t| dt\right) < \infty$$

$$E\left(\int_0^T |y_t| dt\right) < \infty$$

If $u: R \times [0,T] \to R$ is continuous and their first and second partial derivative for t exist and are continuous.

If we take $Y(t) = u(x(t), t)$,then we have the following Itô formula:

$$dY = \left(\frac{\partial u}{\partial t} + F\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial u^2}{\partial x^2}G^2\right)dt + \frac{\partial u}{\partial x}G dw$$

Theorem :(1) [1]

Let $u(x)=x^m$, m=0,1,2,… then $d(x^m)=mx^{m-1}dx+\frac{1}{2}m(m-1)\,x^{m-2}G^2dt$ …(3)

see [1]

Lemma (1): [2]

Let w_t is a Brownian motion then, by Itô's Formula, we have:

$(dw_t)^2$= dt, $(dt)(dw_t) = 0$ and $(dt)^2 = 0$. …(4)

Lemma(2): [4]

Suppose $\{w_t\}$ is a Brownian motion then, by using Ito's Formula, we get:

dw_t^2= $2w_t dw_t$ + dt, $(dw_t)^2$=dt, $dtdw_t = 0$ and

$dw_t^3 = 3w_t^2 dw_t + 3w_t dt$ …(5)

Theorem (2): (Itô product rule) [1]

Let $dx_i = \alpha_i(t)dt + \beta_i(t)dW(t)$; $(i = 1,2)$

(0≤ t ≤T): $\alpha_i(t)\epsilon L^1(0,T)$, $\beta_i(t)\epsilon L^2(0,T)$,. Then

$d(x_1(t)x_2(t)) = X_1(t)dX_2(t) + X_2(t)dX_1(t) + \beta_1(t)\beta_2(t)dt$.

Let $\alpha_i(t) = \alpha_i$; $\beta_i(t) = \beta_i$ independent of t, where $i = 1,2$

Therefore $d(x_1(t)x_2(t)) = X_1(t)dX_2(t) + X_2(t)dX_1(t) + \beta_1\beta_2 dt$ … (6)

3: Propositions:

a:The quotient stochastic differential equation of the first order:

let x_1 and $x_2$are two stochastic processes and time independent .Then by using Ito Formula we have:

$$d\left(\frac{x_1}{x_2}\right) = \left(x_1F_2 + \frac{F_1}{x_2} + G_1G_2\right)dt + \left(x_1G_2 + \frac{G_1}{x_2}\right)dw \quad \dots(7)$$

Proof: from equation (6) then we have

$$d\left(\frac{x_1}{x_2}\right) = x_1 d\left(\frac{1}{x_2}\right) + \frac{1}{x_2}dx_1 + G_1G_2dt \quad \dots(8)$$

let z=$\frac{1}{x_2}$ and dz=d($\frac{1}{x_2}$) from equation (2) then

$dx_1 = F_1dt + G_1dw$; $dz = F_2dt + G_2dw$

$d\left(\frac{x_1}{x_2}\right) = d(x_1z) = x_1dz + zdx_1 + G_1G_2dt =$

$x_1(F_2dt + G_2dw) + z(F_1dt + G_1dw) + G_1G_2dt =$

$x_1F_2dt + x_1G_2dw + zF_1dt + zG_1dw + G_1G_2dt =$

$(x_1F_2 + zF_1 + G_1G_2)dt + (x_1G_2 + zG_1)dw$

Since z=$\frac{1}{x_2}$ then we have

$$d(x_1z) = d\left(\frac{x_1}{x_2}\right) = \left(x_1F_2 + \frac{F_1}{x_2} + G_1G_2\right)dt + \left(x_1G_2 + \frac{G_1}{x_2}\right)dw$$

Then from equation (7) we have:

$$\frac{x_1(t)}{x_2(t)} = \frac{x_1(0)}{x_2(0)} + \int_0^t \left(x_1F_2 + \frac{F_1}{x_2} + G_1G_2\right)ds + \int_0^t \left(x_1G_2 + \frac{G_1}{x_2}\right)dw_s \quad \dots(9)$$

b: The quotient stochastic differential equation of degree two (i.e. $d(\frac{x_1}{x_2})^2$) :

$$d\left(\frac{x_1}{x_2}\right)^2 = \left(\frac{2x_1F_1}{x_2^2} + \frac{2x_1^2F_2}{x_2} + \frac{4x_1G_1G_2}{x_2}\right)dt + \left(\frac{2x_1G_1}{x_2^2} + \frac{2x_1^2G_2}{x_2}\right)dw \quad \dots(10)$$

proof:

let z=$\frac{1}{x_2}$, $z^2 = \frac{1}{x_2^2}$ and $dz^2 = d\frac{1}{x_2^2}$. From (2) then we have $dx_1 = F_1dt + G_1dw$; $dz = F_2dt + G_2dw$

$$d\left(\frac{x_1}{x_2}\right)^2 = d(x_1^2z^2) = 2x_1z^2dx_1 + 2x_1^2zdz + dx_1^2dz^2$$

By using theorem(1), we have

$dx_1^2 = 2x_1dx_1 + G_1^2dt$; $dz^2 = 2zdz + G_2^2dt$

Then,

$d\left(\frac{x_1}{x_2}\right)^2 = d(x_1^2z^2) = 2x_1z^2dx_1 + 2x_1^2zdz +$
$dx_1^2dz^2 = 2x_1z^2(F_1dt + G_1dw) + 2x_1^2z(F_2dt +$
$G_2dw) + (2x_1dx_1 + G_1^2dt)(2zdz + G_2^2dt) =$
$2x_1z^2F_1dt + 2x_1z^2G_1dw + 2x_1^2zF_2dt +$
$2x_1^2zG_2dw +$
$(2x_1(F_1dt + G_1dw) + G_1^2dt)(2z(F_2dt + G_2dw) +$
$G_2^2dt) = (2x_1z^2F_1 + 2x_1^2zF_2)dt + (2x_1z^2G_1 +$
$2x_1^2zG_2)dw + (2x_1F_1dt + 2x_1G_1dwG_1^2dt)(2zF_2dt +$
$2zG_2dw + G_2^2dt) = (2x_1z^2F_1 + 2x_1^2zF_2)dt +$
$(2x_1z^2G_1 + 2x_1^2zG_2)dw + 4x_1zF_1F_2(dt)^2 +$
$4x_1zF_1G_2dwdt + 2x_1zF_1G_2^2(dt)^2 +$
$4x_1zF_2G_1dwdt + 4x_1zG_1G_2(dw)^2 +$
$2x_1G_1G_2^2dwdt + 2zF_2G_1^2(dt)^2 + 2zG_2G_1^2dwdt +$
$G_1^2G_2^2(dt)^2$

From equation (4) and (5) then we have

$d(x_1^2z^2)=(2x_1z^2F_1+2x_1^2zF_2)dt+(2x_1z^2G_1+2x_1^2zG_2)dw+4x_1zG_1G_2dt=(2x_1z^2F_1+2x_1^2zF_2+4x_1zG_1G_2)dt+(2x_1z^2G_1+2x_1^2zG_2)dw$

since $z=\frac{1}{x_2}$ then . $d(\frac{x_1}{x_2})^2=\left(\frac{2x_1F_1}{x_2^2}+\frac{2x_1^2F_2}{x_2}+\frac{4x_1G_1G_2}{x_2}\right)dt+\left(\frac{2x_1G_1}{x_2^2}+\frac{2x_1^2G_2}{x_2}\right)dw$

The integral of (10) is.

$$\frac{x_1^2(t)}{x_2^2(t)}=\frac{x_1^2(0)}{x_2^2(0)}+\int_0^t\left(\frac{2x_1F_1}{x_2^2}+\frac{2x_1^2F_2}{x_2}+\frac{4x_1G_1G_2}{x_2}\right)ds+\int_0^t\left(\frac{2x_1G_1}{x_2^2}+\frac{2x_1^2G_2}{x_2}\right)dw_s \quad ...(11)$$

C:The quotient stochastic differential equation in the general form :

$$d(\frac{x_1}{x_2})^m=\left(\frac{mx_1^{m-1}F_1}{x_2^m}+\frac{mx_1^mF_2}{x_2^{m-1}}+\frac{m^2x_1^{m-1}G_1G_2}{x_2^{m-1}}\right)dt+\left(\frac{mx_1^{m-1}G_1}{x_2^m}+\frac{mx_1^mG_2}{x_2^{m-1}}\right)dw \ ...(12)$$

Proof:

let $z=\frac{1}{x_2}$, $z^m=\frac{1}{x_2^m}$ and $dz^m=d\frac{1}{x_2^m}$. from (2) then we have $dx_1=F_1dt+G_1dw\,;dz=F_2dt+G_2dw$

$d(\frac{x_1}{x_2})^m=d(x_1^mz^m)=$

$mx_1^{m-1}z^mdx_1+mx_1^mz_1^{m-1}dz+dx_1^mdz^m=mx_1^{m-1}z^m(F_1dt+G_1dw)+mx_1^mz_1^{m-1}(F_2dt+G_2dw)+dx_1^mdz^m$

From Theorem (1), we have $dx_1^m=mx_1^{m-1}dx_1+\frac{1}{2}m(m-1)x_1^{m-2}G_1^2dt$

and $dz^m=mz^{m-1}dz+m(m-1)z^{m-2}dt$

Then,

$d(\frac{x_1}{x_2})^m=mx_1^{m-1}z^mF_1dt+mx_1^{m-1}z^mG_1dw+mx_1^mz_1^{m-1}F_2dt+mx_1^mz_1^{m-1}G_2dw+\left(mx_1^{m-1}dx_1+\frac{1}{2}m(m-1)x_1^{m-2}G_1^2dt\right)(mz^{m-1}dz+m(m-1)z^{m-2}dt)=mx_1^{m-1}z^mF_1dt+mx_1^{m-1}z^mG_1dw+mx_1^mz_1^{m-1}F_2dt+mx_1^mz_1^{m-1}G_2dw+\left(mx_1^{m-1}(F_1dt+G_1dw)+\frac{1}{2}m(m-1)x_1^{m-2}G_1^2dt\right)(mz^{m-1}(F_2dt+G_2dw)+m(m-1)z^{m-2}dt)$

From equation (4) and equation (5), we have

$d(\frac{x_1}{x_2})^m=mx_1^{m-1}z^mF_1dt+mx_1^{m-1}z^mG_1dw+mx_1^mz_1^{m-1}F_2dt+mx_1^mz_1^{m-1}G_2dw+m^2x_1^{m-1}z^{m-1}G_1G_2dt$

$=(mx_1^{m-1}z^mF_1+mx_1^mz_1^{m-1}F_2+m^2x_1^{m-1}z^{m-1}G_1G_2)dt+(mx_1^{m-1}z^mG_1+mx_1^mz_1^{m-1}G_2)dw$

Since $z=\frac{1}{x_2}$ then:

$$d(\frac{x_1}{x_2})^m=\left(\frac{mx_1^{m-1}F_1}{x_2^m}+\frac{mx_1^mF_2}{x_2^{m-1}}+\frac{m^2x_1^{m-1}G_1G_2}{x_2^{m-1}}\right)dt+\left(\frac{mx_1^{m-1}G_1}{x_2^m}+\frac{mx_1^mG_2}{x_2^{m-1}}\right)dw$$

Or equivalently by integrated $d(\frac{x_1}{x_2})^m$, we have

$$\frac{x_1^m(t)}{x_2^m(t)}=\frac{x_1^m(0)}{x_2^m(0)}+\int_0^t\left(\frac{mx_1^{m-1}F_1}{x_2^m}+\frac{mx_1^mF_2}{x_2^{m-1}}+\frac{m^2x_1^{m-1}G_1G_2}{x_2^{m-1}}\right)ds+\int_0^t\left(\frac{mx_1^{m-1}F_1}{x_2^m}+\frac{mx_1^mF_2}{x_2^{m-1}}+\frac{m^2x_1^{m-1}G_1G_2}{x_2^{m-1}}\right)dw_s \quad ...(13)$$

4. The moment

In this paragraph we find the moments to the solution of the Quotient stochastic differential equation(Mean, Variance and the k-moment) by using the above proposition:

Let we have

$$d\left(\frac{x_1}{x_2}\right)=\left(x_1F_2+\frac{F_1}{x_2}+G_1G_2\right)dt+\left(x_1G_2+\frac{G_1}{x_2}\right)dw$$

Then the **mean** of ($\frac{x_1}{x_2}$) is

$$E(\frac{x_1(t)}{x_2(t)})=E(\frac{x_1(0)}{x_2(0)})+E(\int_0^t\left(x_1F_2+\frac{F_1}{x_2}+G_1G_2\right)ds)+E(\int_0^t\left(x_1G_2+\frac{G_1}{x_2}\right))dw_s$$

$$=\frac{x_1(0)}{x_2(0)}+E(\int_0^t\left(x_1F_2+\frac{F_1}{x_2}+G_1G_2\right)ds) \quad ...(14)$$

And from equation (11) we can find the expected value to $(\frac{x_1}{x_2})^2$ as:

$$E(\frac{x_1^2(t)}{x_2^2(t)})=E(\frac{x_1^2(0)}{x_2^2(0)})+E(\int_0^t\left(\frac{2x_1F_1}{x_2^2}+\frac{2x_1^2F_2}{x_2}+\frac{4x_1G_1G_2}{x_2}\right)ds)+E(\int_0^t\left(\frac{2x_1G_1}{x_2^2}+\frac{2x_1^2G_2}{x_2}\right)dw_s$$

$$=\frac{x_1^2(0)}{x_2^2(0)}+E(\int_0^t\left(\frac{2x_1F_1}{x_2^2}+\frac{2x_1^2F_2}{x_2}+\frac{4x_1G_1G_2}{x_2}\right)ds) \ ...(15)$$

Then , $\mathbf{Var(X)}=\mathbf{E}(X^2)-(E(X))^2$, where $X=\frac{x_1}{x_2}$

The moment for general form (equation (13)) or the **k'th moment** is:

$$E(\frac{x_1^m(t)}{x_2^m(t)})=E(\frac{x_1^m(0)}{x_2^m(0)})+E(\int_0^t\left(\frac{mx_1^{m-1}F_1}{x_2^m}+\frac{mx_1^mF_2}{x_2^{m-1}}+\frac{m^2x_1^{m-1}G_1G_2}{x_2^{m-1}}\right)ds)+E\left(\int_0^t\left(\frac{mx_1^{m-1}G_1}{x_2^m}+\frac{mx_1^mG_2}{x_2^{m-1}}\right)dw_s\right)$$

$$=\frac{x_1^m(0)}{x_2^m(0)}+E(\int_0^t\left(\frac{mx_1^{m-1}F_1}{x_2^m}+\frac{mx_1^mF_2}{x_2^{m-1}}+\frac{m^2x_1^{m-1}G_1G_2}{x_2^{m-1}}\right)ds) \ ...(16)$$

Example: (1)

Suppose $d\left(\frac{x_1}{x_2}\right)=\left(\frac{s}{r}\right)dw$ or we can write it as $dx_1=sdw$ and $dx_2=rdw$

s and r are constants, also let $x_1=t^2+1$, $x_2=t$, where t is a scalar($t\neq$ time). By using Itô's formula find E $(\frac{x_1}{x_2})$ and var $(\frac{x_1}{x_2})$.

Solution: To find the mean of $(\frac{x_1(t)}{x_2(t)})$:

Let $z=\frac{1}{x_2}$ then $dz=d(\frac{1}{x_2})$, by equation (2) we get

$dx_1=F_1dt+G_1dw\,;dz=F_2dt+G_2dw$

Then

$d(x_1z)=(x_1F_2+zF_1+G_1G_2)dt+(x_1G_2+zG_1)dw$
$=(0+0+sr)dt+(sz+rx_1)dw=srdt+(sz+rx_1)dw$

So

$\int_0^t d(x_1z)=\int_0^t srds+\int_0^t(sz+rx_1)dw$

$\frac{x_1(t)}{x_2(t)}=\frac{x_1(0)}{x_2(0)}+\int_0^t srds+\int_0^t\left(\frac{s}{x_2}+rx_1\right)dw$

Then the expected value (mean) of $(\frac{x_1(t)}{x_2(t)})$ is:

$$E\left(\frac{x_1(t)}{x_2(t)}\right)=\frac{x_1(0)}{x_2(0)}+E\left(\int_0^t srds\right)+E\left(\int_0^t\left(\frac{s}{x_2}+rx_1\right)dw\right)$$

$=\frac{x_1(0)}{x_2(0)}+\int_0^t E(sr)ds=\frac{x_1(0)}{x_2(0)}+srt$. Where $x_2(0)\neq 0$

The variance: First we need to find $E\left((\frac{x_1}{x_2})^2\right)$.

Let $z=\frac{1}{x_2}$, $z^2=\frac{1}{x_2^2}$ and $dz^2=d\frac{1}{x_2^2}$

$d(x_1^2z^2)=(2x_1z^2F_1+2x_1^2zF_2+4x_1zG_1G_2)dt+(2x_1z^2G_1+2x_1^2zG_2)dw$

$\int_0^t d(x_1^2z^2)=\int_0^t(2x_1z^2F_1+2x_1^2zF_2+4x_1zG_1G_2)ds+\int_0^t(2x_1z^2G_1+2x_1^2zG_2)dw_s$

$x_1^2(t)z^2(t)=x_1^2(0)z^2(0)+\int_0^t 4srx_1zds+\int_0^t(2sx_1z^2+rx_1^2z)dw_s$

Where $z=\frac{1}{x_2}$ and $z^2=\frac{1}{x_2^2}$ then

$E\left(\frac{x_1^2(t)}{x_2^2(t)}\right)=\frac{x_1^2(0)}{x_2^2(0)}+E\int_0^t 4sr\frac{x_1}{x_1}ds=\frac{x_1^2(0)}{x_2^2(0)}+4sr\int_0^t E\left(\frac{x_1}{x_2}\right)ds$

$=\frac{x_1^2(0)}{x_2^2(0)}+4sr\int_0^t(\frac{x_1(0)}{x_2(0)}+srt)ds=\frac{x_1^2(0)}{x_2^2(0)}+4sr\left(\frac{x_1(0)}{x_2(0)}t+\frac{1}{2}srt^2\right)=\frac{x_1^2(0)}{x_2^2(0)}+4sr\frac{x_1(0)}{x_2(0)}t+2s^2r^2t^2$

$var\left(\frac{x_1}{x_1}\right)=E\left(\frac{x_1^2}{x_2^2}\right)-(E\left(\frac{x_1}{x_2}\right))^2$

$var\left(\frac{x_1}{x_1}\right)=\left\{\frac{x_1^2(0)}{x_2^2(0)}+4sr\frac{x_1(0)}{x_2(0)}t+2s^2r^2t^2\right\}-\left\{\frac{x_1(0)}{x_2(0)}+srt\right\}^2=\frac{2srx_1(0)}{x_2(0)}+s^2r^2t^2$

In the same way we can find the higher moment.

Example: (2)

Suppose $dX_i=F_idt+G_idw$, i=1,2 , $X_i=\frac{x_1}{x_2}$ and let $dx_1=t^3dt+2tdw$ and $dx_2=t^2dt+4tdw$,where $x_1=t^2+1$and $x_2=t$ where t is a scalar(t≠ time). Then by using Itô's formula find E $(\frac{x_1}{x_2})$, var$(\frac{x_1}{x_2})$, where $x_1(0)=0$, $x_2(0)\neq 0$.

Solution: we have

$d\left(\frac{x_1}{x_2}\right)=\left(x_1F_2+\frac{F_1}{x_2}+G_1G_2\right)dt+\left(x_1G_2+\frac{G_1}{x_2}\right)dw=\left((t^2+1)t^2+\frac{t^3}{t}+8t^2\right)dt+\left((t^2+1)(4t)+\frac{2t}{t}\right)dw==(t^4+t^2+t^2+8t^2)dt+(4t^3+4t+2)dw=(t^4+10t^2)dt+(4t^3+4t+2)dw$

Then from equation (9), we have

$\frac{x_1(t)}{x_2(t)}=\frac{x_1(0)}{x_2(0)}+\int_0^t(s^4+10s^2)ds+\int_0^t(4s^3+4s+2)dw_s$

Since $x_1(0)=0$, then by taking the expectation for both sides, we get

$E\left(\frac{x_1(t)}{x_2(t)}\right)=E\left(\int_0^t(s^4+10s^2)ds\right)+E\left(\int_0^t(4s^3+4s+2)dw_s\right)$

$=E\left(\int_0^t(s^4+10s^2)ds\right)=\frac{1}{5}t^5+\frac{10}{3}t^3$

To find the variance of $(\frac{x_1}{x_2})$ we need $E((\frac{x_1}{x_2})^2)$

From equation (10) , we have:

$d(\frac{x_1}{x_2})^2=\left(\frac{2x_1F_1}{x_2^2}+\frac{2x_1^2F_2}{x_2}+\frac{4x_1G_1G_2}{x_2}\right)dt+\left(\frac{2x_1G_1}{x_2^2}+\frac{2x_1^2G_2}{x_2}\right)dw$

$=\left(\frac{2(t^2+1)t^3}{t^2}+\frac{4(t^2+1)^2t^2}{t}+\frac{4(t^2+1)(2t)(4t)}{t}\right)dt+\left(\frac{2(t^2+1)2t}{t^2}+\frac{4(t^2+1)(4t)}{t}\right)$

$=(2t^3+2t+4t^5+8t^3+4t+32t^2+32t)dt+\left(4t+\frac{4}{t}+16t^2+16\right)dw$

$=(4t^5+10t^3+38t)dt+\left(16t^2+4t+\frac{4}{t}+16\right)dw$

So that:

$(\frac{x_1(t)}{x_2(t)})^2=(\frac{x_1(0)}{x_2(0)})^2+\int_0^t(4s^5+10s^3+38s)ds+\int_0^t\left(16s^2+4s+\frac{4}{s}+16\right)dw_s$

By taking the expectation for both side (since $x_1(0)=0$) , then

$E((\frac{x_1(t)}{x_2(t)})^2)=E(\int_0^t(4s^5+10s^3+38s)ds)+E(\int_0^t\left(16s^2+4s+\frac{4}{s}+16\right)dw_s)$

$E((\frac{x_1(t)}{x_2(t)})^2)=E(\int_0^t(4s^5+10s^3+38s)ds)=E(\frac{3}{2}t^6+\frac{5}{2}t^4+19t^2)=\frac{3}{2}t^6+\frac{5}{2}t^4+19t^2$

$Var\left(\frac{x_1}{x_2}\right)=E((\frac{x_1}{x_2})^2)-(E\left(\frac{x_1}{x_2}\right))^2$

Then

$Var\left(\frac{x_1}{x_2}\right)=\left(\frac{3}{2}t^6+\frac{5}{2}t^4+19t^2\right)-\left(\frac{1}{5}t^5+\frac{10}{3}t^3\right)^2=\frac{5}{2}t^4+19t^2-\frac{1}{25}t^{10}-\frac{5}{3}t^8-\frac{891}{18}t^6$

Example: (3)

Suppose $dx_1=dt$ and $dx_2=2tdw$, where $x_1=t^2$and $x_2=2t$, where t is a scalar(t≠ time). Then by using Itô's formula find E $(\frac{x_1}{x_2})$ and var$(\frac{x_1}{x_2})$,Where $x_1(0)=0$, $x_2(0)\neq 0$.

Solution: from equation (7) then

$d\left(\frac{x_1}{x_2}\right)=\left(x_1F_2+\frac{F_1}{x_2}+G_1G_2\right)dt+\left(x_1G_2+\frac{G_1}{x_2}\right)dw=\left(t^2(0)+\frac{1}{2t}+0(2t)\right)dt+\left(t^2(2t)+\frac{1}{2t}(0)\right)dw=\frac{1}{2t}dt+2t^3dw$

$\frac{x_1(t)}{x_2(t)}=\frac{x_1(0)}{x_2(0)}+\int_0^t\frac{1}{2s}ds+\int_0^t 2s^3dw_s=\int_0^t\frac{1}{2s}ds+\int_0^t 2s^3dw_s$

$E\left(\frac{x_1(t)}{x_2(t)}\right)=E\left(\int_0^t\frac{1}{2s}ds\right)+E\left(\int_0^t 2s^3dw_s\right)=E\left(\int_0^t\frac{1}{2s}ds\right)=\frac{1}{2}\ln(t)$

To find the variance of $(\frac{x_1}{x_2})$, we need $E((\frac{x_1}{x_2})^2)$

From equation (10) we have:

$d(\frac{x_1}{x_2})^2=\left(\frac{2x_1F_1}{x_2^2}+\frac{2x_1^2F_2}{x_2}+\frac{4x_1G_1G_2}{x_2}\right)dt+\left(\frac{2x_1G_1}{x_2^2}+\frac{2x_1^2G_2}{x_2}\right)dw=\left(\frac{2t^2}{4t^2}+\frac{2t^4(0)}{2t}+\frac{4t^2(0)(2t)}{2t}\right)dt+\left(\frac{2t^2(0)}{4t^2}+\frac{2t^4(2t)}{2t}\right)dw=\frac{1}{2}dt+2t^4dw$

Then,

$(\frac{x_1(t)}{x_2(t)})^2=(\frac{x_1(0)}{x_2(0)})^2+\int_0^t\frac{1}{2}ds+\int_0^t 2s^4dw_s=\int_0^t\frac{1}{2}ds+\int_0^t 2s^4dw_s$

And then,

$E((\frac{x_1(t)}{x_2(t)})^2)=E(\int_0^t\frac{1}{2}ds)=\frac{1}{2}t$

So that

$$Var\left(\frac{x_1}{x_2}\right) = E\left(\left(\frac{x_1}{x_2}\right)^2\right) - \left(E\left(\frac{x_1}{x_2}\right)\right)^2 = \frac{1}{2}t - \left(\frac{1}{2}\ln(t)\right)^2 = \frac{1}{2}t - \frac{1}{4}(\ln(t))^2 = \frac{1}{2}t - \frac{1}{2}\ln(t)$$

For the higher moment we can use the same way.

5. Conclusion

In this paper, we showed using Itˆo's formula that the quotient stochastic differential equations can be found by the same method for product stochastic differential equations with some attention when we using their theorem's .(That is, Itˆo's formula is valid for rational form of the functions $u(x(t), t)$ of the variables $x_1$and x_2). Also we find the moments to the solution of the Quotient stochastic differential equation by using the above proposition with some examples.

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العزوم لبعض المعادلات التفاضلية التصادفية الكسرية مع التطبيق

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الملخص

في هذا البحث سوف نستخدم صيغة المعادلات التفاضلية التصادفية في حالة الضرب لدراسة وايجاد حل بعض المعادلات التفاضلية التصادفية الكسرية باستخدام صيغة ito التكاملية ،ثم نحاول ايجاد العزوم لها المتمثلة (المتوسط mean) ، التباين variance والعزم-k'th moment k) ، و قدمنا بعض الامثلة لتوضيح الطريقة.