Composition Series of the Modules $S_2^3(m,n)$ for the Weyl Groups of Type B_n

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الخلاصة

الغرض من هذا البحث هو دراسة زمرة فايل وبنية موديولات سبخت $S_2^3(m,n)$. ايضا ، في هذه الدراسة ، تمكنا من اثبات المتسلسلات التركيبية التالية للموديول $S_2^3(m,n)$:

$$0 \subset \mathrm{Im}\gamma_1^2 \subset \mathrm{Im}\beta_1^3 \subset \mathrm{Ker}\gamma_2^3 \subset S_2^3(m,n)$$

$$0 \subset \mathrm{Im}\gamma_1^2 \subset \mathrm{Im}\alpha_2^2 \subset \mathrm{Ker}\gamma_2^3 \subset S_2^3(m,n)$$

ABSTRACT

The purpose of this paper is to study the Weyl groups and structure of the specht modules $S_2^3(m,n)$. Also, in this study, we were able to prove the following composition series of the modules $S_2^3(m,n)$:

$$0 \subset \mathrm{Im}\gamma_1^2 \subset \mathrm{Im}\beta_1^3 \subset \mathrm{Ker}\gamma_2^3 \subset S_2^3(m,n)$$

$$0 \subset \mathrm{Im}\gamma_1^2 \subset \mathrm{Im}\alpha_2^2 \subset \mathrm{Ker}\gamma_2^3 \subset S_2^3(m,n)$$

1. INTRODUCTION

Many outstanding problems in representation theory can be solved with a proper understanding of the specht modules for the Weyl groups W_n of type B_n [1, 2].

Peel [3] gives the analysis of the specht modules corresponding to the partitions of the form $(n-r, 1^r)$, $0 \leq r \leq n-1$. The diagram of such a partition is a hook and accordingly, these specht modules are referred to as hook representation modules. Halicioglu [4] show how to construct submodules of Specht modules for Weyl groups. Geck and Jacon [5] present the irreducible representations of H_n and consider the canonical basic sets for certain values of the parameters in type B_n. Brandt [6] present the analogues of more results in the symmetric groups for the Weyl groups of type B_n. The intention of this research is to study the structure of the modules $S_2^3(m,n)$.

Now we give an indication of the layout of this paper. In section 2, we present definition and some properties of the Weyl group. In section 3, we introduce a classification of the group algebra KW_n. In section 4, we introduce the fundamental definitions and examples of compositions, partitions and diagram. In section 5, we introduce λ – tableaux and (λ, μ) – tableaux and related definitions. In section 6, we give basic definitions and analysis of specht modules $S_2^3(m,n)$. Finally in section 7, we review the background theory that we shall need; we also combine our results to complete the proof of the main theorem.

2. THE WEYL GROUP W_n

We begin by defining a family of integral representation modules of the Weyl groups W_n. Let $x_1, \dots, x_n$ be independent indeterminate over the field K characteristic P, which may be zero or a prime number and P not equal to 2, let W_n be the set of all one to one mapping w from the set $\{\pm x_1, \dots, \pm x_n\}$ onto $\{\pm x_1, \dots, \pm x_n\}$ such that $w(-xi) = -w(xi)$, $i = 1,\dots,n$.

The pair (W_n, composition) forms a group known as the Weyl groups [1].

Properties of $\mathbf{W_n}$**:**

1) Each element w belongs to W_n is called a permutation.
2) w can be represented as

$$W = \begin{pmatrix} x_1 & \dots & x_n & -x_1 & \dots & -x_n \\ w(x_1) & \dots & w(x_n) & -w(x_1) & \dots & -w(x_n) \end{pmatrix}$$

3) The order of the Weyl groups W_n is denoted by $| W_n |$ which is equal to 2 .n! .

***Example(2.1)*:-**

$$W_2 = \left[\begin{pmatrix} x_1 & x_2 & -x_1 & -x_2 \\ x_1 & x_2 & -x_1 & -x_2 \end{pmatrix}, \begin{pmatrix} x_1 & x_2 & -x_1 & -x_2 \\ -x_1 & x_2 & x_1 & -x_2 \end{pmatrix}, \begin{pmatrix} x_1 & x_2 & -x_1 & -x_2 \\ x_1 & -x_2 & -x_1 & x_2 \end{pmatrix}, \begin{pmatrix} x_1 & x_2 & -x_1 & -x_2 \\ -x_1 & -x_2 & x_1 & x_2 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} x_1 & x_2 & -x_1 & -x_2 \\ x_2 & x_1 & -x_2 & -x_1 \end{pmatrix}, \begin{pmatrix} x_1 & x_2 & -x_1 & -x_2 \\ -x_2 & x_1 & x_2 & -x_1 \end{pmatrix}, \begin{pmatrix} x_1 & x_2 & -x_1 & -x_2 \\ x_2 & -x_1 & -x_2 & x_1 \end{pmatrix}, \begin{pmatrix} x_1 & x_2 & -x_1 & -x_2 \\ -x_2 & -x_1 & x_2 & x_1 \end{pmatrix}\right]$$

Notice that $| W_2 | = 2^2 . 2! = 8$.

3. THE GROUP ALGEBRA KW_n

Definition(3.1) :

Let $K[x_1, \dots , x_n]$ be the set of all polynomials in $x_1, \dots , x_n$ with coefficients in the field K .

Then any permutation $w \in W_n$ can be regarded as a one mapping from $K[x_1, \dots , x_n]$ onto $K[x_1, \dots , x_n]$ by defining $wf(x_1, \dots , x_n) = f(wx_1, \dots ,wx_n)$ for each Polynomial $f(x_1, \dots , x_n) \in K[x_1, \dots , x_n]$.

That is w changes each variable x_i by the variable $w(x_i)$ in each polynomial $f(x_1, \dots , x_n) \in K[x_1, \dots , x_n]$.

Now since $kf(x_1, \dots , x_n) \in K[x_1, \dots , x_n]$ for each polynomial $f(x_1, \dots , x_n) \in K[x_1, \dots , x_n]$ and for each $k \in K$, Then the multiplication of a permutation $w \in W_n$ by a scalar $k \in K$ is a function $kwf(x_1, \dots , x_n) = kf(wx_1, \dots , wx_n)$ for each $f(x_1, \dots , x_n) \in K[x_1, \dots , x_n]$.

Let KW_n be the set of all k-linear combination of the permutations $w_i \in W_n$, i.e. $KW_n=\{\Sigma k_i w_i / k_i \in K\}$

4. COMPOSITIONS, PARTITIONS AND DIAGRAM

In this section we introduce the fundamental definitions of compositions, partitions and diagram. Thereby we follow [1, 2 and 6].

***Definition (4.1)*:**

1) $\lambda = (\lambda_1, \lambda_2, \dots)$ is a composition of n, if $\lambda_1, \lambda_2, \dots$ are non–negative integers with $|\lambda| = \sum_{i=1}^{\infty} \lambda_i = n$. The non–zero λ_i are called the parts of λ.

2) A partition of n is a composition λ of n for which

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$$

***Definition (4.2)*:**

A pair of partitions (λ, μ) of a positive integer n, is a pair of sequences of positive integers $\lambda = (\lambda_1, \dots ,\lambda_s)$ and $\mu = (\mu_1, \dots , \mu_t)$ such that

1) $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0$, $\mu_1 \geq \mu_2 \geq \dots \geq \mu_t > 0$.
2) $\lambda_1 + \lambda_2 + \dots + \lambda_s + \mu_1 + \mu_2 + \dots + \mu_t = n$.

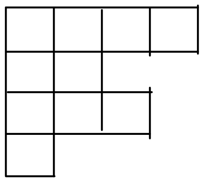
***Definition (4.3)*:**

If λ is a composition of n, then the diagram $[\lambda]$ is the set $\{(i , j) \mid i , j \in Z , 1 \leq i , 1 \leq j \leq \lambda_i \}$. If $(i , j) \in [\lambda]$, then (i , j) is called a node of $[\lambda]$. The k^{th} row (respectively, column) of a diagram consists of those nodes whose first (respectively, second) coordinate is k .

We shall draw diagrams as in the following example.

Example (4.1):

For $\lambda = (4,2,3,1)$ we have $[\lambda]$ =



Definition (4.4):

Let $(\lambda, \mu) = ((\lambda_1, \ldots, \lambda s),(\mu_1, \ldots , \mu_t))$ be a pair of partitions of a positive integer n . Then the set

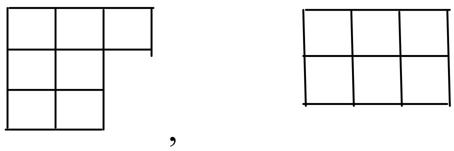
$D_{(\lambda, \mu)} = \{(i, j, k) \mid 1 \le i \le s , 1 \le j \le \lambda_i$ if $k=1$ and $1 \le i \le t , 1 \le j \le \mu_i$ if $k=2\}$ is called the (λ, μ) – diagram.

Example (4.2):

Let $(\lambda, \mu) = ((3,2,2), (3,3))$ which is a pair of partitions of 13 , then the (λ, μ) – diagram is the set

$D_{(\lambda, \mu)} = \{ (1,1,1) , (1,2,1) , (1,3,1) , (2,1,1) , (2,2,1) , (3,1,1) , (3,2,1) , (1,1,2) , (1,2,2) , (1,3,2) , (2,1,2) , (2,2,2) , (2,3,2) \}$.

The following pair of nodes is also a (λ, μ) – diagram.



5. λ – TABLEAUX AND (λ, μ) – TABLEAUX

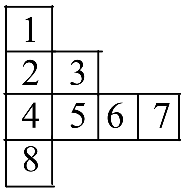
We continue with the introduction of λ – tableaux and (λ, μ) – tableaux and related definitions. Our main references are [1, 6].

Definition (5.1):

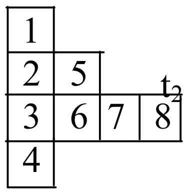
Let λ be a composition of n. A λ – tableau is one of the n! Arrays of integers obtained by replacing each node in $[\lambda]$ by one of the integers 1, 2, …, n, allowing no repeats.

Example (5.1):

Let $\lambda = (1,2,4,1)$. Then t_1 =



t_2 =



are λ – tableaux.

Definition (5.2):

Let $(\lambda, \mu) = ((\lambda_1, \ldots , \lambda_s) , (\mu_1, \ldots , \mu_t))$ be a pair partitions of a positive integer n. Then any one to one mapping Z from $D_{(\lambda, \mu)}$ into the set $\{\pm x_1 , \ldots , \pm x_n\}$ such that $Z(i_1, j_1, k_1) \neq \pm Z(i_2, j_2, k_2)$ if $(i_1, j_1, k_1) \neq (i_2, j_2, k_2)$ is called a (λ, μ) – tableau.

***Example (5.2)*:**

Let $(\lambda , \mu) = ((3,2) , (1,1))$ be a pair of partitions of 7, then a ((3,2), (1,1)) – tableaux is the mapping
$Z : D_{((3,2), (1,1))} \longrightarrow \{\pm x_1 , . . . , \pm x_7\}$ defined by $Z(1,1,1) = x_1$, $Z(1,2,1) = x_3$, $Z(1,3,1) = -x_5$, $Z(2,1,1) = x_2$, $Z(2,2,1) = -x_4$, $Z(1,1,2) = x_6$, $Z(2,1,2) = x_7$.

Z can also be considered as the following pair of arrays

x_1	x_3	$-x_5$
x_2	$-x_4$	

,

x_6
x_7

6. SPECHT MODULE $S_2^3(m , n)$

***Definition (6.1)*:**

Let $(\lambda, \mu) = ((\lambda_1, \dots ,\lambda_s),(\mu_1, \dots , \mu_t))$ be a pair of partitions of a positive integer n, and let Z be any (λ, μ) – tableau, then the cyclic KW_n – module $S_K(\lambda, \mu)$ generated over KW_n by f(Z) (i.e. $S_K(\lambda, \mu) = KW_nf(Z)$) is called the Specht module over K corresponding to the pair of partitions (λ, μ) of n.

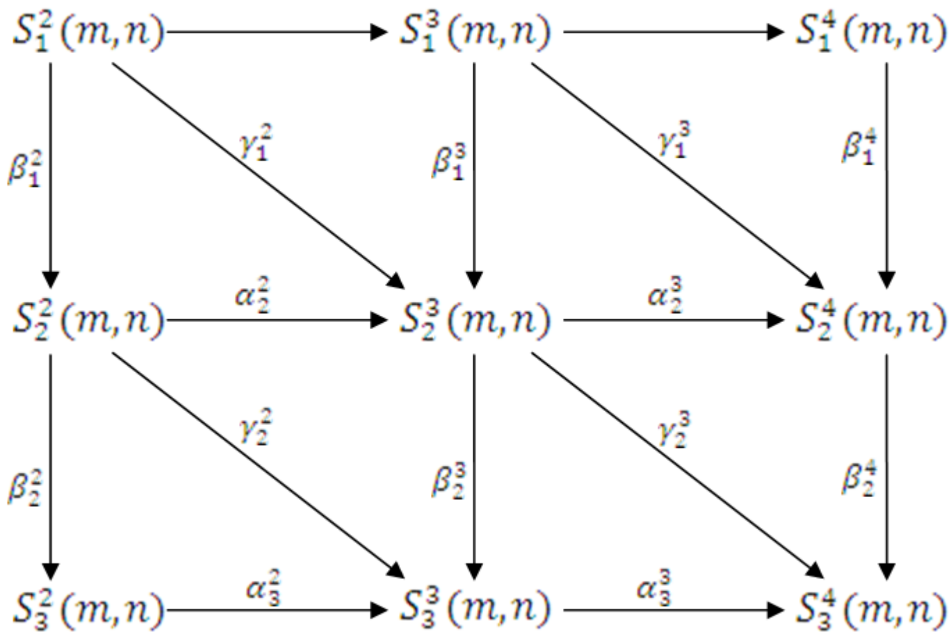
The Specht modules $S_K((m-2,1^{t-1}), (n-m-1,1^{r-1}))$ will be denoted by $S_r^t(m,n)$, where t , r , m and n are positive integers such that $t \leq m \leq n$ and $r \leq n-m$.

***Definition (6.2)*:**

Let $\alpha_r^t : S_r^t(m,n) \longrightarrow S_r^{t+1}(m,n)$, $\beta_r^t : S_r^t(m,n) \longrightarrow S_{r+1}^t(m,n)$ and $\gamma_r^t : S_r^t(m,n) \longrightarrow S_{r+1}^{t+1}(m,n)$ are the linear transformation. Then α_r^t , β_r^t , γ_r^t are KW_n – homomorphism if P divides m , n – m and both m and
n – m respectively .

ANALYSIS OF $S_2^3(m , n)$

Let K be a field of characteristic P not equal to 2 and P divides both m and n-m. Depending on the results founds by AL-Aamily [2], the analysis of the module $S_2^3(m,n)$ will be done.



7. THE MAIN RESULT

In this section, we give our main theorem. First we review the background concerning of the irreducible KW_n– module required to enable us to state our results.

7.1 USEFUL BACKGROUND RESULTS

Here, we summarize some theorems without prove (see proof in [7]) which we shall need in order to prove our Theorem 7.1.

Theorem7.1.1: $\operatorname{Im}\gamma_1^2$ is an irreducible KW_n– module.

Theorem 7.1.2: $\dfrac{\operatorname{Im}\beta_1^3}{\operatorname{Im}\gamma_1^2}$ is an irreducible KW_n – module.

Theorem 7.1.3: $\dfrac{\operatorname{Im}\alpha_2^2}{\operatorname{Im}\gamma_1^2}$ is an irreducible KW_n – module.

Theorem 7.1.4: $\dfrac{\ker\gamma_2^3}{\operatorname{Im}\alpha_2^2}$ is an irreducible KW_n – module.

Theorem 7.1.5: $\dfrac{\ker\gamma_2^3}{\operatorname{Im}\beta_1^3}$ is an irreducible KW_n – module.

To conclude, we can formulate and prove the main theorem motivating of this paper.

***Theorem7.1*:**

If K is a field of characteristic P not equal to 2 and P divides both m and $n-m$ then $S_2^3(m,n)$ has the following composition series:

1. $0 \subset \operatorname{Im}\gamma_1^2 \subset \operatorname{Im}\beta_1^3 \subset \operatorname{Ker}\gamma_2^3 \subset S_2^3(m,n)$
2. $0 \subset \operatorname{Im}\gamma_1^2 \subset \operatorname{Im}\alpha_2^2 \subset \operatorname{Ker}\gamma_2^3 \subset S_2^3(m,n)$

Proof:

From the above theorems, we have $\operatorname{Im}\gamma_1^2$, $\frac{\operatorname{Im}\beta_1^3}{\operatorname{Im}\gamma_1^2}$, $\frac{\operatorname{Im}\alpha_2^2}{\operatorname{Im}\gamma_1^2}$, $\frac{\ker\gamma_2^3}{\operatorname{Im}\alpha_2^2}$, $\frac{\ker\gamma_2^3}{\operatorname{Im}\beta_1^3}$ are irreducible, and from [2] we have $\operatorname{Im}\gamma_2^3$ is irreducible.

Hence $\frac{S_2^3(m,n)}{\ker\gamma_2^3}$ is irreducible. Therefore, $S_2^3(m,n)$ has (1) and (2) as composition series.

To prove that $S_2^3(m,n)$ has no non-zero proper submodule other than $\operatorname{Im}\gamma_1^2$, $\operatorname{Im}\beta_1^3$, $\operatorname{Im}\alpha_2^2$ and $\operatorname{Ker}\gamma_2^3$, let M be any non-zero proper KW_n– submodule, then either M is not a submodule of $\operatorname{Ker}\gamma_2^3$ or M is a submodule of $\operatorname{Ker}\gamma_2^3$.

If M is not a submodule of $\operatorname{Ker}\gamma_2^3$, then $S_2^3(m,n) = M + \operatorname{Ker}\gamma_2^3$, because $\operatorname{Ker}\gamma_2^3$ is maximal in $S_2^3(m,n)$.

But $S_2^3(m,n)$ is indecomposable. Therefore, M is a submodule of $\operatorname{Ker}\gamma_2^3$ and hence $M =$ $\operatorname{Ker}\gamma_2^3$ or M is a proper submodule of $\operatorname{Ker}\gamma_2^3$.

If M is a proper submodule of $\operatorname{Ker}\gamma_2^3$ then $M = \operatorname{Im}\beta_1^3$ or $M \cap \operatorname{Im}\beta_1^3 = 0$

1) If $M \cap \operatorname{Im}\beta_1^3 = \operatorname{Im}\gamma_1^2$, then either $M \subseteq \operatorname{Im}\alpha_2^2$ or $M + \operatorname{Im}\alpha_2^2 = \operatorname{Ker}\gamma_2^3$

   If $M + \operatorname{Im}\alpha_2^2 = \operatorname{Ker}\gamma_2^3$, then

   $\operatorname{Im}\beta_1^3 \cap \operatorname{Ker}\gamma_2^3 = \operatorname{Im}\beta_1^3 \cap (M + \operatorname{Im}\alpha_2^1) = \operatorname{Im}\gamma_1^2$

   This is not true. Therefore, $M \leq \operatorname{Im}\alpha_2^2$ and hence either $M = \operatorname{Im}\alpha_2^2$ or $M = \operatorname{Im}\gamma_1^2$.

2) If $M \cap \operatorname{Im}\beta_1^3 = 0$, let $M \cap \operatorname{Im}\alpha_2^2 \neq 0$, then $\operatorname{Im}\gamma_1^2 \leq M \cap \operatorname{Im}\alpha_2^2$, which contradicts the assumption since $\operatorname{Im}\gamma_1^2 \leq \operatorname{Im}\beta_1^3$. Therefore,

$M \cap \mathrm{Im}\alpha_2^2 = 0$. But it is clear that $\mathrm{Ker}\gamma_2^3 = \mathrm{Im}\alpha_2^2 + \mathrm{Im}\beta_1^3$ therefore, $M = 0$.

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