Fourth Order Exponential Compact Alternating Direction Implicit (ADI) Scheme for Solving Three-Dimensional Convection-Diffusion Equations

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Abstract

In this paper, fourth order exponential compact alternating direction implicit (ADI) finite difference scheme for solving unsteady 3D convection-diffusion equation with constant coefficients is derived. This scheme is second order in time and fourth order in space. The formulation is solved with an efficient numerical method corresponds the solution of tridiagonal systems. The accuracy and efficiency of this scheme are discussed. We proved a higher order discretization scheme is unconditionally stable with respect to the initial values by using Fourier analysis. Numerical results are presented and compared with the fourth order compact ADI scheme by (Karaa , 2006).

Key words: Three-dimensional unsteady convection-diffusion, high order compact exponential scheme, ADI method, stability.

1. Introduction

This article is concerned with the development of accurate solution of the Three-dimensional unsteady convection-diffusion equation with $u(x,y,z,t)$ transport variable.

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} - b\frac{\partial^2 u}{\partial y^2} - d\frac{\partial^2 u}{\partial z^2} + c\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} = 0, \quad (x,y,z,t)\in\Omega\times J \quad (1)$$

with initial condition

$$u(x,y,z,0) = u_0(x,y,z), \ \ u(x,y,z)\in\Omega \ (2)$$

and boundary conditions

$$u(x,y,z,t) = g(x,y,z,t), \ \ u(x,y,z,t)\in\Gamma\times J \ (3)$$

where Ω is the rectangular domain in R^3 with boundary $\Gamma = \partial\Omega$, $(0,T]$ is the time interval, and c, p and q are constants convective velocities, a,b and d are positive constant diffusion coefficients in x,y and z directions, respectively.

The convection-diffusion equation (1) is an important class of partial differential equation and arises in physics and engineering sciences involving the applications in fluid dynamics(Roache, 1976)., the modelling of transport phenomenon which may present temperature, concentration of a contaminant sea water (Noye,1986a)., Groundwater pollutants problems and energy and chemical separation processes (Parlarge 1980, Patankar 1980), the spread of pollutants in rivers and streams and transport of pollutants in the atmosphere (Chatwin et al 1985, Chaudhry et al 1983), flow in porous media (Fattah et al 1985,Kumar 1988).

Class finite difference methods (FDM), such as second-order central difference (CD) scheme and the first-order upwind (UD) difference scheme, they are working well for solving the convection-diffusion equation, but it is not always very accurate one to be need a large number of mesh point. However, their solutions suffer from excessive time truncation errors, moreover they put a strong limitation on the courant number and hence require very small time steps to generate stable solutions.

Various numerical finite difference schemes have been proposed to solve the convection-diffusion equations approximately. Noye and Tan (1988a) drive several high order implicit schemes for unsteady (1D) convection-diffusion equations and in (Noye et al 1988b) they proposed a compact nine-point high order compact (HOC) implicit scheme for unsteady (2D) convection-diffusion equations. These schemes have large interval of stability and its third order accurate in space and second order accurate in time. In Rigal, (1988, 1999)derived two classes of compact difference schemes of order 2 in time and 4 in space with different choices of weighting parameters. Spotz and Carye (2001) extended the 2D HOC scheme in (Gupta, et al 1984) for solving steady state equations to solve unsteady state 1D convection-diffusion equations with variable coefficients and 2D diffusion equations. Also, in (Kalita, et al 2002)and (Karaa, submitted for publication). derived a HOC schemes with weight time discretization to solve the unsteady 2D and 3D convection-diffusion equations, respectively.

To obtain satisfactory higher order numerical results with reasonable computational cost, there have been attempts to develop higher order compact ADI methods. Michell and Faireweather(1964) obtained

a high order split formula and later Dai and Nassar (2002)are using it for 2D diffusion problems. A higher order compact ADI for solving 2D convection-diffusion equations of order 2 in time and 4 in space is deriving by Karaa and Zhang(2004) and extended this scheme for 3D convection-diffusion equations in (Karaa,2006).A class of HOC compact exponential finite difference methods that is proposed for solving 1D and 2D steady and unsteady state convection-diffusion equations with variable coefficients in (Tain et al 2007a) extended by Tian and Ge (2007b) for 2D unsteady state convectiondiffusion equations with ADI scheme.

In this paper, we extension the work of Tian and Ge (2007b) by derived a fourth order compact exponential ADI finite difference scheme for 3D unsteady state convection-diffusion equations. Numerical results that are obtained more accurate than those in (Karaa,2006) . In section 2, we establish an efficiency exponential fourth order ADI method for solving 3D unsteady convection diffusion problems with constant coefficients, and the Crank–Nicholson method is used for the time discretization.

Stability analyzed by using Von-Neumann method verifying the proposed present exponential fourth order ADI method is unconditionally stable described in section 3. The tridiagonal system of equation produced by present exponential fourth order ADI scheme is strictly diagonally dominant is given in section 4. In section 5, numerical results for two test problems produced by present exponential fourth order ADI method are compared with Karra (2006) results. Section 5,introduced the concludes of this paper .

2. Exponential compact Fourth-order ADI finite difference method

In order to develop the HOC exponential finite difference schemes for solving the convection diffusion equation

$$-au_{xx}+cu_x=f(x), \qquad (4)$$

where a is the positive constant conductivity, c is the constant convective velocity, f is a sufficiently smooth function. Consider the finite difference scheme for equation (4) with constant convection coefficient at a grid point. Let the interval $[x_0, x_N]$ be discretized into N grid steps of size Δx, where $x_i=ih_x$, $h_x=x_{i+1}-x_i$, $u_i=u(x_i)$, $f_i=f(x_i)$, i is an index of any grid point in x direction. Derivatives in (4) at interior grid points x_i can be defined using Taylor's expansion as

$$u_{x_i}=D_{h_x}u_i-\sum_{n=1}^{\infty}\frac{h_x^{2n}}{(2n+1)!}D_x^{2n+1}u_i, \quad (5)$$

$$u_{xx_i}=D_{h_x}^2u_i-\sum_{n=1}^{\infty}\frac{2h_x^{2n}}{(2n+1)!}D_x^{2n+2}u_i, \quad (6)$$

where $D_{h_x}u_i=(u_{i+1}-u_{i-1})/(2h_x)$ and $D_{h_x}^2u_i=(u_{i+1}-2u_i+u_{i-1})/h_x^2$ are the central difference approximations for the first and second derivatives and D_x^n is the nth-order exact derivative operator at any interior x. Rewrite equation (4) as

$$-e^{\frac{cx}{a}}(ae^{-\frac{cx}{a}}u_x)_x=f_i,$$

$$-(ae^{-\frac{cx}{a}}u_x)_x=f_i\,e^{-\frac{cx}{a}}. \qquad (7)$$

Integration of equation (7) over a spaced interval from $x_{i-\frac{1}{2}}$ to $x_{i+\frac{1}{2}}$

$$c(e^{-\frac{ch_x}{2a}}(u_x)_{i+1/2}-e^{\frac{ch_x}{2a}}(u_x)_{i-1/2}) = (e^{-\frac{ch_x}{2a}}-e^{\frac{ch_x}{2a}})f_i. \qquad (8)$$

Assume $K=\frac{ch_x}{2a}$ and using the central difference approximate u_{x_i} in equation (8), we get

$$c\left[e^{-K}\frac{u_{i+1}-u_i}{h_x}-e^{K}\frac{u_i-u_{i-1}}{h_x}\right]=(e^K-e^{-K})f_i.$$

By adding and replace u_{i+1} in the second term we get

$$\frac{ch_x}{2}e^{-K}\frac{2u_{i+1}-2u_i+u_{i-1}-u_{i-1}}{h_x^2}$$

$$-ce^K\frac{2u_i-2u_{i-1}+u_{i+1}-u_{i+1}}{2h_x}=(e^K-e^{-K})f_i.$$

Rearrange this equation, give

$$\frac{c}{2h_x}\left[\begin{array}{l}u_{i+1}\left(e^K+e^{-K}\right)-2u_i\left(e^K+e^{-K}\right)\\+u_{i+1}\left(e^K+e^{-K}\right)\end{array}\right]$$

$$+\frac{c}{2h_x}\left[u_{i+1}\left(e^{-K}-e^{-K}\right)-u_{i-1}\left(e^{-K}-e^{K}\right)\right]$$

$$=(e^{-K}-e^{K})f_i,$$

and then

$$-\frac{ch_x}{2}\frac{e^K+e^{-K}}{e^K-e^{-K}}\left[\frac{u_{i+1}-2u_i+u_{i-1}}{h_x^2}\right]$$

$$+c\frac{e^{-K}+e^{K}}{e^{-K}-e^{K}}\left[\frac{u_{i+1}-u_{i-1}}{2h_x}\right]=f_i.$$

Hence, the equation may be written as;

$$-\frac{ch_x}{2}\coth(K)D_{h_x}^2u_i+cD_{h_x}u_i=f_i.$$

Thus, we get

$$-\alpha D_{h_x}^2u_i+cD_{h_x}u_i=f_i, \qquad (9)$$

$$\alpha = \begin{cases} -\frac{ch_x}{2}\coth\left(\frac{ch_x}{2a}\right), & c \neq 0 \\ a & , c = 0 \end{cases} \qquad (10)$$

(12)

Equation (9) represents the second order exponential FD schemes for the convection-diffusion equation (4) and it's equivalent to the standard second order central FD formula applied to the following equation:

$$-\frac{ch_x}{2}\coth\left(\frac{ch_x}{2a}\right)u_{xx} + cu_x = f. \quad (11)$$

Equation (11) show that when equation (9) is used, an artificial diffusion coefficient $a\left[(ch_x/2a)\coth(ch_x/2a)-1\right]$ is perturbed to equation (4).

To drive Exponential compact high-order ADI finite difference method, we consider the FD scheme for equation (4) with constant convection coefficient at grid point x_i as;

$$-\alpha D_{h_x}^2 u_i + cD_{h_x}u_i = \alpha_0 f_i + \alpha_1 f_{xi} + \alpha_2 f_{xxi}, \quad (12)$$

In order to determine the parameters $\alpha_0, \alpha_1, \alpha_2$ rewrite equation (12) as

$$\begin{aligned} -\alpha D_{h_x}^2 u_i + cD_{h_x}u_i &= \alpha_0(-au_{xx}+cu_x)_i \\ &+\alpha_1(-au_{xx}+cu_x)_{xi} + \alpha_2(-au_{xx}+cu_x)_{xxi}. \end{aligned} \quad (13)$$

From equations (5) and (6), we have

$$D_{h_x}u_i = u_{x_i} + \frac{h_x^2}{6}D_x^3 u_i + o(h_x^4)$$

$$D_{h_x}^2 u_i = u_{xx_i} + \frac{h_x^2}{12}D_x^4 u_i + o(h_x^4)$$

Substituting these equations into equation (13) , we obtain

$$\begin{aligned} &-\alpha\left(u_{xx} + \frac{h_x^2}{12}D_x^4 u_i\right) + c\left(u_x + \frac{h_x^2}{6}D_x^3 u_i\right) = \\ &-a\alpha_0 u_{xx} + \alpha_0 c u_{xi} - a\alpha_1 D_x^3 u_i + \alpha_1 c u_{xxi} \\ &-a\alpha_2 D_x^4 u_i + c\alpha_2 D_x^3 u_i. \end{aligned} \quad (14)$$

Rearrange equation (14),we obtain

$$\begin{aligned} &(\alpha_0 - 1)cu_{xi} + (-a\alpha_0 + \alpha_1 c + \alpha)u_{xxi} \\ &+ (-a\alpha_1 + \alpha_2 c - \frac{ch_x^2}{6})D_x^3 u_i \\ &+\left(\frac{\alpha h_x^2}{12} - a\alpha_2\right)D_x^4 u_i + o(h_x^4) = 0. \end{aligned} \quad (15)$$

Add f_i to both sides, and using equation (4), we have

$$\begin{aligned} &-au_{xx} + cu_x = f_i + (\alpha_0 - 1)cu_{xi} \\ &+(-a\alpha_0 + \alpha_1 c + \alpha)u_{xxi} + (-a\alpha_1 + \alpha_2 c - \frac{ch_x^2}{6})D_x^3 u_i \\ &+\left(\frac{\alpha h_x^2}{12} - a\alpha_2\right)D_x^4 u_i + o(h_x^4) = 0. \end{aligned} \quad (16)$$

From (15), we have

$$\begin{aligned} &(\alpha_0 - 1)c = 0, \\ &-a\alpha_0 + \alpha_1 c + \alpha = 0, \\ &-a\alpha_1 + \alpha_2 c - \frac{ch_x^2}{6} = 0, \\ &\frac{\alpha h_x^2}{12} - a\alpha_2. \end{aligned} \quad (17)$$

Solution equation (17), gives the parameters

$$\alpha_2 = \begin{cases} \frac{a(a-\alpha)}{c^2} + \frac{h_x^2}{6}, & c \neq 0 \\ \frac{h_x^2}{12}, & c = 0 \end{cases} \quad (18)$$

Equation (12) with equation (18) is an $o(h^4)$ compact FD scheme and it's a diagonally dominant tri-diagonal system of equations. The 4OC exponential FD scheme (12) for the equation (4) is equivalent to the standard second-order central FD scheme applied to the following equation:

$$\left(-\alpha\delta_x^2 + c\delta_x\right)u_i = \left(1+\alpha_1 D_x + \alpha_2 D_x^2\right) f_i, \quad (19)$$

which an artificial diffusion coefficient $a\left[\left(ch_x/2a\right)\coth\left(ch_x/2a\right)-1\right]$ and an artificial source term $\alpha_1 f_x + \alpha_2 f_{xx}$ have been added.

The modified differential equation corresponding to the equation (19) can be obtained by expressing Taylor series. From (16) with (18), we have

$$-au_{xx} + cu_x = f_i + \left(\frac{\alpha h_x^2}{12} - a\alpha_2\right) D_x^4 u_i \quad (20)$$

$$-\frac{ch_x^4}{120} D_x^5 u_i + \frac{\alpha h_x^4}{360} D_x^6 u_i + o(h_x^6).$$

Equation (19) can be formulated symbolically as

$$\left(1+\alpha_1\delta_x + \alpha_2\delta_x^2\right)^{-1}\left(-\alpha\delta_x^2 + c\delta_x\right)u_i = f_i. \quad (21)$$

Here the operator $\left(1+\alpha_1\delta_x + \alpha_2\delta_x^2\right)^{-1}$ has symbolic meaning only.

This symbolic construction can be used to derive higher order compact schemes for 3D convection-diffusion equation (Li et al (1995),Tian and Ge(2003), Zhang(2002), Hirsh (1975), Karra and Zhang (2004).

An analogous symbolic fourth compact approximation operator can also be obtained for y and z variables. For convenience, we define several finite difference operators

$$L_x = 1+\alpha_1\delta_x + \alpha_2\delta_x^2, \quad A_x = -\alpha\delta_x^2 + c\delta_x,$$
$$L_y = 1+\beta_1\delta_y + \beta_2\delta_y^2, \quad A_y = -\beta\delta_y^2 + p\delta_y,$$
$$L_z = 1+\gamma_1\delta_z + \gamma_2\delta_z^2, \quad A_z = -\gamma\delta_z^2 + q\delta_z,$$

and

$$\beta = \begin{cases} \dfrac{ph_y}{2}\coth\left(\dfrac{ph_y}{2b}\right), & p \neq 0 \\ b, & p = 0 \end{cases},$$

$$\beta_1 = \begin{cases} \dfrac{b-\beta}{p}, & p \neq 0 \\ 0 & p = 0 \end{cases},$$

$$\beta_2 = \begin{cases} \dfrac{b(b-\beta)}{p^2} + \dfrac{h_y^2}{6}, & p \neq 0 \\ \dfrac{h_y^2}{12}, & p = 0 \end{cases}, \quad (21)$$

$$\gamma = \begin{cases} \dfrac{qh_z}{2}\coth\left(\dfrac{qh_z}{2d}\right), & q \neq 0 \\ d, & q = 0 \end{cases},$$

$$\gamma_1 = \begin{cases} \dfrac{d-\gamma}{q}, & q \neq 0 \\ 0 & q = 0 \end{cases},$$

$$\gamma_2 = \begin{cases} \dfrac{d(d-\gamma)}{q^2} + \dfrac{h_z^2}{6}, & q \neq 0 \\ \dfrac{h_z^2}{12}, & q = 0 \end{cases}, \quad (22)$$

where δ_y, δ_z and δ_y^2, δ_z^2 are the first and the second central difference operator with mesh sizes Δy and Δz in the y and z directions, respectively.
Applying the forth order compact difference operators L_x^{-1}, L_y^{-1} and L_z^{-1} to the equation stady-state 3D convection diffusion equation

(4), we obtain the following exponential fourth-order compact approximation:

$$\left(L_x^{-1}A_x + L_y^{-1}A_y + L_z^{-1}A_z\right)u_{ijk}^n = f_{ijk} + o(h^4), \quad (24)$$

where

$o(h^4)$ denotes the $o(h_x^4)+o(h_y^4)+o(h_z^4)$ term and (i,j,k) denote (x_i, y_j, z_k).

An exponential fourth-order ADI approximation on to the unsteady 3D convection-diffusion equation (1) can be obtain by replacing f by $-\dfrac{\partial u}{\partial t}$ in equation (24).

$$\left(\frac{\partial u^n}{\partial t}\right)_{ijk} = \left(L_x^{-1}A_x + L_y^{-1}A_y + L_z^{-1}A_z\right)u_{ijk}^n + o(h^4), \quad (25)$$

where u^n is the approximate solution at time $t_n = n\Delta t, n$ represents the time increment, and $\Delta t = t^{n+1} - t^n$ is the time step size. A fourth-order semi –discrete approximation (25) to the unsteady 3D convection-diffusion equation (1) was also used in (Karra, 2005).

From Taylor series expansions, we have

$$u_{ijk}^{n+1} = \left(\begin{array}{l} 1 + \Delta t\dfrac{\partial}{\partial t} + \dfrac{\Delta t^2}{2!}\dfrac{\partial^2}{\partial t^2} + \dfrac{\Delta t^3}{3!}\dfrac{\partial^3}{\partial t^3} \\ + \dfrac{\Delta t^4}{4!}\dfrac{\partial^4}{\partial t^4} + \cdots \end{array}\right)u_{ijk}^n, \quad (26)$$

$$u_{ijk}^{n+1} = \exp\left(\Delta t\frac{\partial}{\partial t}\right)u_{ijk}^n.$$

Substituting (25) into equation (26), we have

$$u_{ijk}^{n+1} = \exp\left(-\Delta t\left(L_x^{-1}A_x + L_y^{-1}A_y + L_z^{-1}A_z\right)\right)u_{ijk}^n, \quad (27)$$

$$\exp\left(\frac{\Delta t}{2}\left(L_x^{-1}A_x + L_y^{-1}A_y + L_z^{-1}A_z\right)\right)u_{ijk}^{n+1} = \exp\left(\frac{-\Delta t}{2}\left(L_x^{-1}A_x + L_y^{-1}A_y + L_z^{-1}A_z\right)\right)u_{ijk}^n, \quad (28)$$

since, the operators $A_x, A_y, A_z, L_x, L_y, L_z$ have commutative property commutative , then yields

$$\exp\left(\frac{\Delta t}{2}L_x^{-1}A_x\right)\exp\left(\frac{\Delta t}{2}L_y^{-1}A_y\right)\exp\left(\frac{\Delta t}{2}L_z^{-1}A_z\right)u_{ijk}^{n+1} = \exp\left(\frac{-\Delta t}{2}L_x^{-1}A_x\right)\exp\left(\frac{-\Delta t}{2}L_y^{-1}A_y\right)\exp\left(\frac{-\Delta t}{2}L_z^{-1}A_z\right)u_{ijk}^n. \quad (29)$$

Using the Taylor expansions, equation (29) becomes

$$\left(1+\frac{\Delta t}{2}L_x^{-1}A_x\right)\left(1+\frac{\Delta t}{2}L_y^{-1}A_y\right)\left(1+\frac{\Delta t}{2}L_z^{-1}A_z\right)u_{ijk}^{n+1} = \left(1-\frac{\Delta t}{2}L_x^{-1}A_x\right)\left(1-\frac{\Delta t}{2}L_y^{-1}A_y\right)\left(1-\frac{\Delta t}{2}L_z^{-1}A_z\right)u_{ijk}^n + o(\Delta t^3) + o(\Delta t h^4), \quad (30)$$

which is the Crank-Niclson time discretization if $o(\Delta t^3) + o(\Delta t h^4)$ is neglect.

Rearrange equation (30), we obtain

$$\left(L_x+\frac{\Delta t}{2}A_x\right)\left(L_y+\frac{\Delta t}{2}A_y\right)\left(L_z+\frac{\Delta t}{2}A_z\right)u_{ijk}^{n+1}=\left(L_x-\frac{\Delta t}{2}A_x\right)\left(L_y-\frac{\Delta t}{2}A_y\right)\left(L_z-\frac{\Delta t}{2}A_z\right)u_{ijk}^{n}+o(\Delta t^3)+o(\Delta th^4). \quad (31)$$

This approximation is second-order accurate in time and fourth-order accurate in space. Crank- Niclson method for time discretization is unconditionally stabile, but to solving system of algebraic equations, we need large amounts of computations. Hence, we used the ADI method (Thomas, 1995), which is applicable to the iterative solution of unsteady 3D convection-diffusion equation and it's requires to solving one-dimensional implicit problems for each time step. In order to applying an exponential Foruth-order ADI scheme with the boundary conditions which will be use in our numerical solutions for the unsteady 3D convection-diffusion problem, we introduce an intermediate variable u^*, equation (31) leads to

$$\left(L_x+\frac{\Delta t}{2}A_x\right)u^*=\left(L_x-\frac{\Delta t}{2}A_x\right)\left(L_y-\frac{\Delta t}{2}A_y\right)\left(L_z-\frac{\Delta t}{2}A_z\right)u^n, \quad (32a)$$

$$\left(L_y+\frac{\Delta t}{2}A_y\right)u^{**}=u^*, \quad (32b)$$

$$\left(L_z+\frac{\Delta t}{2}A_z\right)u^{n+1}=u^{**}. \quad (32c)$$

The intermediate variable u^* satisfying the initial and boundary conditions (2) and (3),

(i) $u^0=u_0,$ at all mesh points,

(ii) $u^n=g^n,\ldots,N,$ on the boundary $\partial\Omega$.

Equation (32) clear that this scheme has the same of accuracy as formula (31) in time and space. In section 4 , the resulting (EHOC ADI) scheme (32) in each ADI solution step give a rise to strictly diagonally dominant tridiagonal matrix equation which can inverted by simple tridiagonal Gaussian decomposition .

The intermediate variable u^* introduce in each ADI scheme above is not necessarily approximation to the solution at any time levels, then from equation (3)

$$u^{**}=\left(L_z+\frac{\Delta t}{2}A_z\right)g^{n+1}. \quad (33)$$

This formula give u^{**} explicitly in terms of the central difference of g^{n+1} with respect to the z. If the boundary conditions are independent of the time, the formulae giving u^{**} on the boundary $\partial\Omega$ reduce to

$$u^{**}=\left(L_z+\frac{\Delta t}{2}A_z\right)g \quad (34)$$

3-Stability analysis

We use Von Neumann stability analysis to define the stability limit of (EHO ADI) scheme to 3D convection-diffusion equation.

Let the numerical solution $u\left(x_i,y_j,z_k,t_n\right)$ be represented by a finite Fourier series, and for linear stability, we can examine the behaviour of single term of the series, as follows

$$u_{ijk}^n=\eta^n\exp\{I\vartheta_x i\}\exp\{I\vartheta_y j\}\exp\{I\vartheta_z k\}, \quad (35)$$

where $I=\sqrt{-1}$, η^n is amplitude at time level n, and $\vartheta_x=v_x h_x$, $\vartheta_y=v_y h_y$, and $\vartheta_z=v_z h_z$ are phase angles with the wave numbers $v_x, v_y,$ and v_z in the x, y and z

directions, respectively. Substituting (35) into (31), we obtain the amplification factor $G\left(\vartheta_x,\vartheta_y,\vartheta_z\right)=\dfrac{\eta^{n+1}}{\eta^n}$ and we can be written as

$$G\left(\vartheta_x,\vartheta_y,\vartheta_z\right)\le\left|l(\vartheta_x)\right|\left|l(\vartheta_y)\right|\left|l(\vartheta_z)\right|,$$

Where

$$l(\vartheta_x)=\frac{Q+R\exp\{I\vartheta_x\}+S\exp\{-I\vartheta_x\}}{K+M\exp\{I\vartheta_x\}+N\exp\{-I\vartheta_x\}},\ \text{(38a)}$$

where

$$K=1-\frac{1}{h_x^2}\left(2\alpha_2-\alpha\Delta t\right),$$

$$M=\frac{1}{2h_x}\left(\alpha_1+\frac{c\Delta t}{2}\right)+\frac{1}{h_x^2}\left(\alpha_2-\frac{\alpha\Delta t}{2}\right),$$

$$N=\frac{1}{2h_x}\left(-\alpha_1-\frac{c\Delta t}{2}\right)+\frac{1}{h_x^2}\left(\alpha_2-\frac{\alpha\Delta t}{2}\right),$$

$$Q=1-\frac{1}{h_x^2}\left(2\alpha_2+\alpha\Delta t\right),$$

$$R=\frac{1}{2h_x}\left(\alpha_1-\frac{c\Delta t}{2}\right)+\frac{1}{h_x^2}\left(\alpha_2+\frac{\alpha\Delta t}{2}\right),$$

$$S=\frac{1}{2h_x}\left(-\alpha_1+\frac{c\Delta t}{2}\right)+\frac{1}{h_x^2}\left(\alpha_2+\frac{\alpha\Delta t}{2}\right).\text{(38b)}$$

Rearrange equation (38) and using the formula,

$$S\text{in}\,\vartheta_x=\frac{1}{2I}\left(\exp\{I\vartheta_x\}-\exp\{-I\vartheta_x\}\right)$$

and

$$Cos\vartheta_x=\frac{1}{2}\left(\exp\{I\vartheta_x\}+\exp\{-I\vartheta_x\}\right),$$

we give

$$Q+R\exp\{I\vartheta_x\}+S\exp\{-I\vartheta_x\}$$
$$=1-\frac{4\alpha_2}{h_x^2}Sin^2(\frac{\vartheta_x}{2})-\frac{2\alpha\Delta t}{h_x^2}Sin^2(\frac{\vartheta_x}{2})$$
$$+\frac{\alpha_1 I}{h_x}Sin\vartheta_x-\frac{Ic\Delta t}{2h_x}Sin\vartheta_x.$$

Similarly, from equation (38), we obtain

$$K+M\exp\{I\vartheta_x\}+N\exp\{-I\vartheta_x\}$$
$$=1-\frac{4\alpha_2}{h_x^2}Sin^2(\frac{\vartheta_x}{2})+\frac{2\alpha\Delta t}{h_x^2}Sin^2(\frac{\vartheta_x}{2})$$
$$+\frac{\alpha_1 I}{h_x}Sin\vartheta_x+\frac{Ic\Delta t}{2h_x}Sin\vartheta_x.$$

Thus, we give

$$l(\vartheta_x)=\frac{\left(\lambda_1-\lambda_2\right)+I\left(\lambda_3-\lambda_4\right)}{\left(\lambda_1+\lambda_2\right)+I\left(\lambda_3+\lambda_4\right)},\ (39)$$

with

$$\lambda_1=1-\frac{4\alpha_2}{h_x^2}Sin^2(\frac{\vartheta_x}{2}),\ \lambda_2=\frac{2\alpha\Delta t}{h_x^2}Sin^2(\frac{\vartheta_x}{2}),$$

$$\lambda_3=\frac{\alpha_1}{h_x}Sin\vartheta_x,\ \lambda_4=\frac{c\Delta t}{2h_x}Sin\vartheta_x.$$

The other terms $l(\vartheta_y)$ and $l(\vartheta_z)$ are defined in a similar way by replacing x by y,z in the above expressions respectively.

To study the stability condition is $G\left(\vartheta_x,\vartheta_y,\vartheta_z\right)\le 1$ for all ϑ_x,ϑ_y, and $\vartheta_z\in\left[-\pi,\pi\right]$. To verified this condition directly equation (39) yields to

$$\left(\lambda_1-\lambda_2\right)^2+\left(\lambda_3-\lambda_4\right)^2$$
$$\le\left(\lambda_1+\lambda_2\right)^2+\left(\lambda_3+\lambda_4\right),$$

and then $\lambda_1\lambda_2+\lambda_3\lambda_4\geq 0$. Furthermore, calculate $\lambda_1\lambda_2+\lambda_3\lambda_4\geq 0$, we have

$$\lambda_1\lambda_2+\lambda_3\lambda_4=$$
$$\frac{2\alpha\Delta t}{h_x^2}\left(1-\frac{4\alpha_2}{h_x^2}Sin^2(\frac{\vartheta_x}{2})\right)Sin^2(\frac{\vartheta_x}{2}) \quad (40)$$
$$+\frac{c\alpha_1\Delta t}{2h_x^2}\left(1-Sin^2(\frac{\vartheta_x}{2})\right)Sin^2(\frac{\vartheta_x}{2}).$$

We must prove that $\lambda_1\lambda_2+\lambda_3\lambda_4\geq 0$.

1- when $c=0$, then

$$\alpha=a,\ \alpha_1=0,\ \alpha_2=\frac{h_x^2}{12}. \quad (41)$$

Substituting (41) into (40), gives

$$\lambda_1\lambda_2+\lambda_3\lambda_4=$$
$$\frac{2c\Delta t}{h_x^2}\left(1-\frac{1}{3}Sin^2(\frac{\vartheta_x}{2})\right)Sin^2(\frac{\vartheta_x}{2}) \quad (42)$$

We know that $c>0$ and $0\leq Sin^2(\frac{\vartheta_x}{2})\leq 1$, this deduce to $\lambda_1\lambda_2+\lambda_3\lambda\geq 0$ for all $\vartheta_x\in[-\pi,\pi]$

2- when $c\neq 0$, then

$$\alpha_1=\frac{a-\alpha}{c},\ \ \alpha_2=\frac{a(a-\alpha)}{c^2}+\frac{h_x^2}{6}. \quad (43)$$

Substituting (43) into (40), gives

$$\lambda_1\lambda_2+\lambda_3\lambda_4=\frac{2\Delta t\alpha}{h_x^2}\left(\frac{1}{3}-\frac{4a(a-\alpha)}{c^2h_x^2}\right)Sin^4(\frac{\vartheta_x}{2})$$
$$+\frac{2\Delta ta}{h_x^2}\left(1-Sin^2(\frac{\vartheta_x}{2})\right)Sin^2(\frac{\vartheta_x}{2})\geq 0$$

Similarly, we can prove $\left|l(\vartheta_y)\right|^2\leq 1$ and $\left|l(\vartheta_z)\right|^2\leq 1$.

We conclude that the (EFOCADI) scheme when applied to the 3D convection-diffusion equation, is unconditionally stabile.

4-The Diagonal Dominance

We prove the tridiagonal systems of equations formed by schemes (32), consider the equation (32c) and rewrite it as follows

$$\left\{1+\gamma_1\delta_z+\gamma_2\delta_z^2+\frac{\Delta t}{2}\left(-\gamma\delta_z^2+q\delta_z\right)\right\}u^{n+1}=u^{**}$$

$$u_{i,j,k}^{n+1}+\gamma_1\frac{u_{i,j,k+1}^{n+1}-u_{i,j,k-1}^{n+1}}{2h_z}+\gamma_2\frac{u_{i,j,k+1}^{n+1}-2u_{i,j,k}^{n+1}+u_{i,j,k-1}^{n+1}}{h_z^2}$$
$$-\frac{\Delta t\gamma}{2}\frac{u_{i,j,k+1}^{n+1}-2u_{i,j,k}^{n+1}+u_{i,j,k-1}^{n+1}}{h_z^2}+\frac{\Delta tq}{2}\frac{u_{i,j,k+1}^{n+1}-u_{i,j,k-1}^{n+1}}{2h_z}=u_{i,j,k}^{**}$$

Hence,

$$A_{i,j,k}u_{i,j,k-1}^{n+1}+B_{i,j,k}u_{i,j,k}^{n+1}$$
$$+C_{i,j,k}u_{i,j,k+1}^{n+1}=u_{i,j,k}^{**}. \quad (44)$$

This execute to the coefficient matrix of the linear system

$$Q=tri\left\lfloor A_{i,j,k}+B_{i,j,k}+C_{i,j,k}\right\rfloor,$$

where

$$A_{i,j,k}=\left[\frac{\gamma_2}{h_z^2}-\frac{\gamma_1}{2h_z}-\frac{\Delta t}{2}\left(\frac{\gamma_2}{h_z^2}+\frac{q}{2h_z}\right)\right],$$

$$B_{i,j,k}=\left[1-\frac{2\gamma_2}{h_z^2}+\frac{\Delta t\gamma}{2h_z^2}\right],$$

$$C_{i,j,k}=\left[\frac{\gamma_2}{h_z^2}-\frac{\gamma_1}{2h_z}-\frac{\Delta t}{2}\left(\frac{\gamma}{h_z^2}-\frac{q}{2h_z}\right)\right].$$

The matrix Q is a diagonal dominance if the conditions satisfied

$$\left|B_{i,j,k}\right|>\left|A_{i,j,k}\right|+\left|C_{i,j,k}\right|.$$

It easily found that

$$C_{i,j,k}=\left[\frac{\gamma_2}{h_z^2}-\frac{\gamma_1}{2h_z}-\frac{\Delta t}{2}\left(\frac{\gamma}{h_z^2}-\frac{q}{2h_z}\right)\right].$$

$$\left|1-\frac{\gamma_2}{h_z^2}\right|+\left|\frac{\Delta t\gamma}{h_z^2}\right|>\left|\frac{\gamma_2}{h_z^2}-\frac{\gamma_1}{2h_z}\right|+\left|\frac{\gamma_2}{h_z^2}+\frac{\gamma_1}{2h_z}\right|$$

$$+\left|\frac{\Delta t}{2}\left(\frac{\gamma}{h_z^2}+\frac{q}{2h_z}\right)\right|+\left|\frac{\Delta t}{2}\left(\frac{\gamma}{h_z^2}-\frac{q}{2h_z}\right)\right|.$$

Is a sufficient condition for $\left|B_{i,j,k}\right|>\left|A_{i,j,k}\right|+\left|C_{i,j,k}\right|$.

$$\left|-\frac{\gamma}{h_z^2}-\frac{q}{2h_z}\right|=\left|-\frac{\gamma h_z\coth x}{2h_z^2}-\frac{q}{2h_z}\right|$$

$$=\left|-\frac{2qe^x}{2h_z\left(e^x-e^{-x}\right)}\right|=\frac{qe^x}{h_z\left(e^x-e^{-x}\right)} \quad (45)$$

$$=\frac{\gamma}{h_z^2}+\frac{q}{2h_z}$$

$$\left|-\frac{\gamma}{h_z^2}+\frac{q}{2h_z}\right|=\left|-\frac{\gamma h_z\coth x}{2h_z^2}+\frac{q}{2h_z}\right|$$

$$=\frac{qe^{-x}}{h_z\left(e^x-e^{-x}\right)}=\frac{\gamma}{h_z^2}-\frac{q}{2h_z}. \quad (46)$$

$$\left|\frac{2\gamma}{h_z^2}\right|=\frac{q\left(e^x+e^{-x}\right)}{h_z\left(e^x-e^{-x}\right)}=\frac{2\gamma}{h_z^2}, \quad (47)$$

where $x=\frac{qh_z}{2d}$, from (45), (46) and (47) produce that

$$\left|\frac{\Delta t\gamma}{h_z^2}\right|=\left|\frac{-\Delta t}{2}\left(\frac{\gamma}{h_z^2}+\frac{q}{2h_z}\right)\right|+\left|\frac{-\Delta t}{2}\left(\frac{\gamma}{h_z^2}-\frac{q}{2h_z}\right)\right|. \quad (48)$$

We must prove that

$$\left|1-\frac{2\gamma_2}{h_z^2}\right|>\left|\frac{\gamma_2}{h_z^2}-\frac{\gamma_1}{2h_z}\right|+\left|\frac{\gamma_2}{h_z^2}+\frac{\gamma_1}{2h_z}\right|. \quad (49)$$

1-When $q=0,$ then

$$\gamma=d,\ \gamma_1=0,\ \gamma_2=\frac{h_z^2}{12}. \quad (50)$$

Substituting (50) into (49), gives

$$1-\frac{2\gamma_2}{h_z^2}=\frac{5}{6},\quad \frac{\gamma_2}{h_z^2}-\frac{\gamma_1}{2h_z}=\frac{1}{12},$$

$$\frac{\gamma_2}{h_z^2}+\frac{\gamma_1}{2h_z}=\frac{1}{12}.$$

The above equation satisfying equation (49).

2- When $q\neq 0,$ then

$$1-\frac{2\gamma_2}{h_z^2}=\frac{2}{3}-\frac{1-\frac{qh_z}{2d}\coth x}{2\left(\frac{qh_z}{2d}\right)^2} \quad (51)$$

$$=\frac{2}{3}-\frac{1-x\coth x}{2x^2}.$$

Similarly, we obtain

$$\frac{\gamma_2}{h_z^2}+\frac{\gamma_1}{2h_z}= \frac{1}{6}+\frac{1-x\coth x}{4x^2}+\frac{1-x\coth x}{4x}, \qquad (52)$$

and

$$\frac{\gamma_2}{h_z^2}-\frac{\gamma_1}{2h_z}= \frac{1}{6}+\frac{1-x\coth x}{4x^2}-\frac{1-x\coth x}{4x}. \qquad (53)$$

Since $1-x\coth x<0$,for all non zero real x, we have

$$\frac{1-x\coth x}{x^2}<0. \qquad (54)$$

From reference (Tian and Ge 2007),

$-1<\frac{1-x\coth x}{x}\le 0,$ if $x>0$ and

$0<\frac{1-x\coth x}{x}<1,$ if $x<0$ when $x>0$,

yield to

$$\left|\frac{1}{6}+\frac{1-x\coth x}{4x}\right|= \begin{cases}\frac{1}{6}+\frac{1-x\coth x}{4x}, & \frac{1}{6}+\frac{1-x\coth x}{x}>0\\ \frac{x\coth x-1}{4x}-\frac{1}{6}, & \frac{1-x\coth x}{x}\le 0\end{cases} \qquad (55)$$

$$\left|\frac{1}{6}+\frac{1-x\coth x}{4x}\right|= \begin{cases}\frac{1}{6}-\frac{1-x\coth x}{4x}, & -1<\frac{1-x\coth x}{x}<\frac{4}{6}\\ \frac{x\coth x-1}{4x}-\frac{1}{6}. & \frac{4}{6}\le\frac{1-x\coth x}{x}\end{cases} \qquad (56)$$

Adding (56) and (55), we obtain

$$\left|\frac{1}{6}+\frac{1-x\coth x}{4x}\right|+\left|\frac{1}{6}-\frac{1-x\coth x}{4x}\right|= \begin{cases}\frac{1}{3} & -\frac{4}{6}<\frac{1-x\coth x}{x}<0\\ \frac{x\coth x-1}{2x}. & -1\le\frac{1-x\coth x}{x}\le\frac{-4}{6}\end{cases} \qquad (57)$$

Since $-1<\frac{1-x\coth x}{x}\le 0$ for $x>0$ and from equation (57), we see that

$$\left|\frac{1}{6}+\frac{1-x\coth x}{4x}\right|+\left|\frac{1}{6}-\frac{1-x\coth x}{4x}\right|<\frac{1}{2}. \qquad (58)$$

So, we find by using triangle inequality and equations (51), (54) and (58) the following

$$\left|\frac{\gamma_2}{h_z^2}+\frac{\gamma_1}{2h_z}\right|+\left|\frac{\gamma_2}{h_z^2}-\frac{\gamma_1}{2h_z}\right|\le\left|1-\frac{2\gamma_2}{h_z^2}\right|. \qquad (59)$$

The discussion for $x<0$ is equivalent, therefore, we obtain

$$\left|1-\frac{2\gamma_2}{h_z^2}\right|>\left|\frac{\gamma_2}{h_z^2}-\frac{\gamma_1}{2h_z}\right|++\left|\frac{\gamma_2}{h_z^2}+\frac{\gamma_1}{2h_z}\right|. \qquad (60)$$

Then for all real x,

$$\left|1-\frac{2\gamma_2}{h_z^2}\right|+\left|\frac{\Delta t\gamma}{h_z^2}\right|>\left|\frac{\gamma_2}{h_z^2}-\frac{\gamma_1}{2h_z}\right|+\left|\frac{\gamma_2}{h_z^2}+\frac{\gamma_1}{2h_z}\right| +\left|\frac{\Delta t}{2}\left(\frac{\gamma}{h_z^2}+\frac{q}{2h_z}\right)\right|+\left|\frac{\Delta t}{2}\left(\frac{\gamma}{h_z^2}-\frac{q}{2h_z}\right)\right|.$$

The proof complete.

5-Numerical Results

Test 1:

We first examine a diffusion problem in the cubic region with diffusion coefficients

$a = b = d = 1$ and $c = p = q = 0.$. The exact solution of this test problem given by

$$u(x, y, z) = e^{-3\pi^2 t} \sin(\pi x)\sin(\pi y)\sin(\pi z).$$

The initial and boundary condition are taken from this solution. We consider uniform grids with different mesh sizes and different regions and compare the accuracy of the computed solutions from the present exponential fourth order ADI scheme and fourth order ADI scheme of samir karaa . The quantity that we compare is the *average absolute* errors of the computed solution with respect to the exact solution. In Table 1, we show that, the new scheme has the same of accuracy compare with samir karaa scheme. This is because if $c = p = q = 0$, then the exponential forth order compact scheme has the same formula with forth order compact scheme

Test 2:

Also, we use another test problem which has the analytic solution given by

$$u(x, y, z) = e^{x+y+z+t},$$

with coefficient $a = 2, b = d = 1$ and $c = p = q = 1.$ The boundary and the initial conditions are directly taken from this solution. In Fig. 1 we plot the *average absolute* errors at each point $x = y = z$ for $t = 1$ and (a) $h = 0.1$ and region $[1,1]^3$, (b) $h = 0.2$ and region $[2,2]^3$,(c) $h = 0.3$ and region $[3,3]^3$, (d) $h = 0.4$ and region $[4,4]^3$, (e) $h = 0.5$ and region $[5,5]^3$, at $dt = 0.001$.

The figures show the superiority of the present exponential fourth order ADI scheme.

In Table 2 we show that, the *average absolute* of the new scheme is less than from the Samir Karaa scheme

5. Conclusions

We have introduced a fourth order exponential compact ADI scheme with Crank-Nicholson technique for solving three-dimensional unsteady convection-diffusion equation. The unconditionally stability of new scheme is proved with respect to initial values. Our Numerical results showed that present scheme is computationally more efficient and more accurate than the fourth order scheme of samir karaa.

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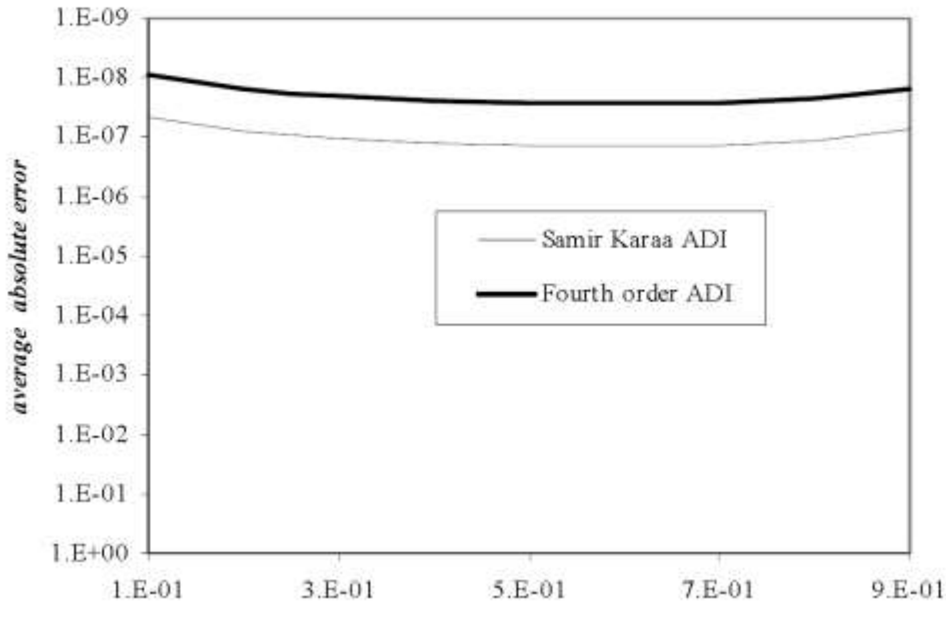
Table 1. *Average absolute* error at t = 1 computed by two different schemes. For test1.

h	0.1	0.2	0.3	0.4	0.5
region	$[1,1]^3$	$[2,2]^3$	$[3,3]^3$	$[4,4]^3$	$[5,5]^3$
Δt=0.1					
Samir Karaa ADI method	3.89E-14	4.21E-14	4.47E-14	4.03E-14	1.97E-14
Present ADI method	3.89E-14	4.21E-14	4.47E-14	4.03E-14	1.97E-14
Δt =0.01					
Samir Karaa ADI method	1.09E-15	1.94E-16	3.83E-15	1.57E-14	2.85E-14
Present ADI method	1.09E-15	1.94E-16	3.83E-15	1.57E-14	2.85E-14
Δt =0.002					
Samir Karaa ADI method	1.82E-16	1.41E-15	5.23E-15	1.67E-14	2.97E-14
Present ADI method	1.82E-16	1.41E-15	5.23E-15	1.67E-14	2.97E-14
Δt =0.001					
Samir Karaa ADI method	6.55E-16	1.90E-15	5.36E-15	1.63E-14	2.93E-14
Present ADI method	6.55E-16	1.90E-15	5.36E-15	1.63E-14	2.93E-14

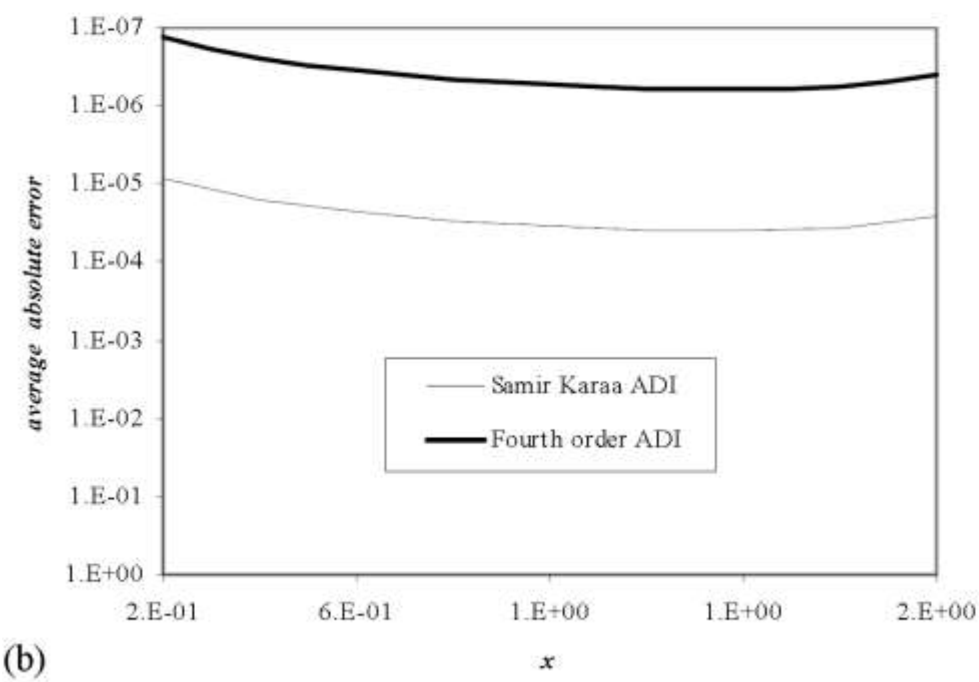
Table 2. *Average absolute* error at t = 1 computed by two different schemes. For test 2 .

h	0.1	0.2	0.3	0.4	0.5
region	$[1,1]^3$	$[2,2]^3$	$[3,3]^3$	$[4,4]^3$	$[5,5]^3$
Δt =0.1					
Samir Karaa ADI method	1.77E-04	4.10E-03	5.28E-02	6.24E-01	7.8580891
Present ADI method	1.77E-04	4.07E-03	5.11E-02	5.64E-01	6.2207654
Δt =0.01					
Samir Karaa ADI method	2.18E-06	6.95E-05	2.27E-03	6.69E-02	1.7197148
Present ADI method	2.09E-06	4.17E-05	5.18E-04	5.85E-03	6.99E-02
Δt =0.002					
Samir Karaa ADI method	1.71E-07	2.95E-05	1.77E-03	6.15E-02	1.659899
Present ADI method	8.39E-08	1.69E-06	2.35E-05	4.05E-04	9.98E-03
Δt =0.001					
Samir Karaa ADI method	1.08E-07	2.83E-05	1.76E-03	6.13E-02	1.6580297
Present ADI method	2.10E-08	4.37E-07	8.04E-06	2.35E-04	8.11E-03

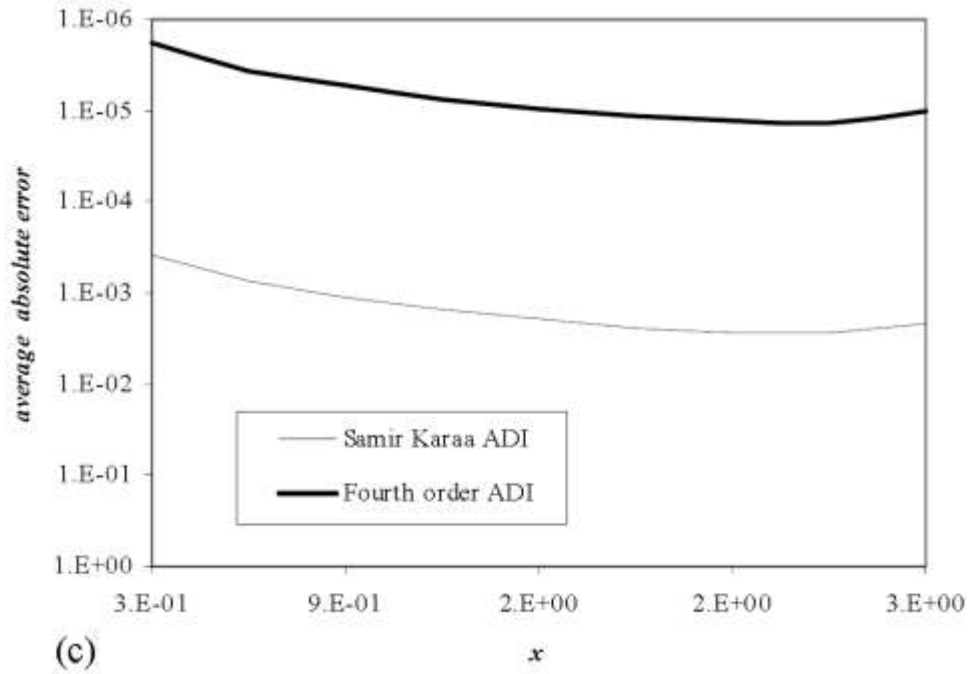
FIG.1. Comparison of *average absolute* error at t = 1 of the present scheme with Samir Karaa scheme at dt = 0.001. (a) h= 0.1, (b)h= 0.2, (c) h= 0.3, (d) h=0.4, (e) h= 0.5.



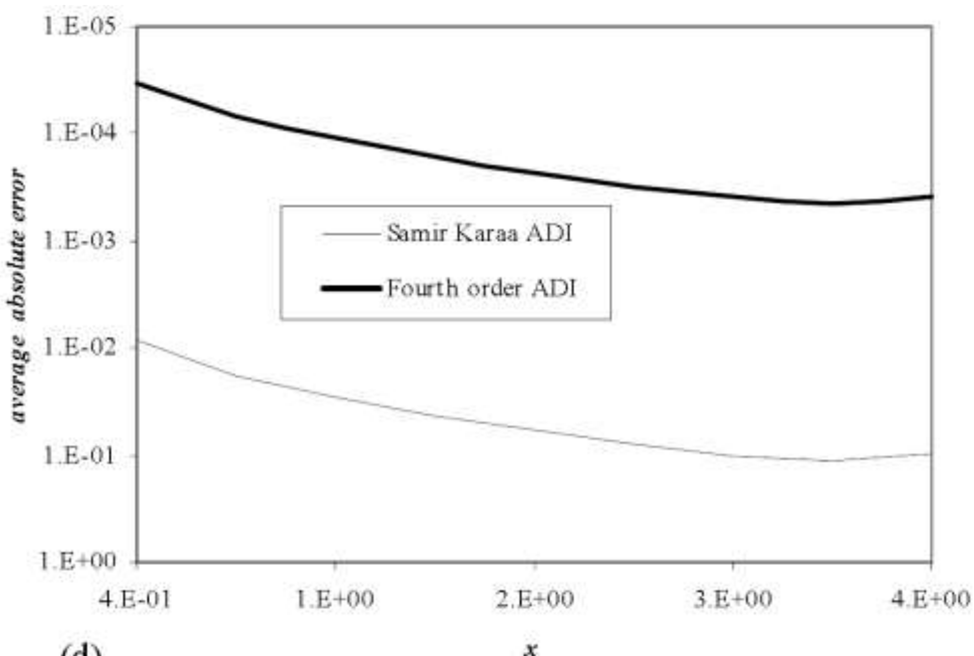
(a)



(b)


1.E-06
1.E-05
1.E-04
1.E-03
1.E-02
1.E-01
1.E+00
average absolute error
Samir Karaa ADI
Fourth order ADI
3.E-01
9.E-01
2.E+00
2.E+00
3.E+00
x

(c)


1.E-05
1.E-04
1.E-03
1.E-02
1.E-01
1.E+00
average absolute error
Samir Karaa ADI
Fourth order ADI
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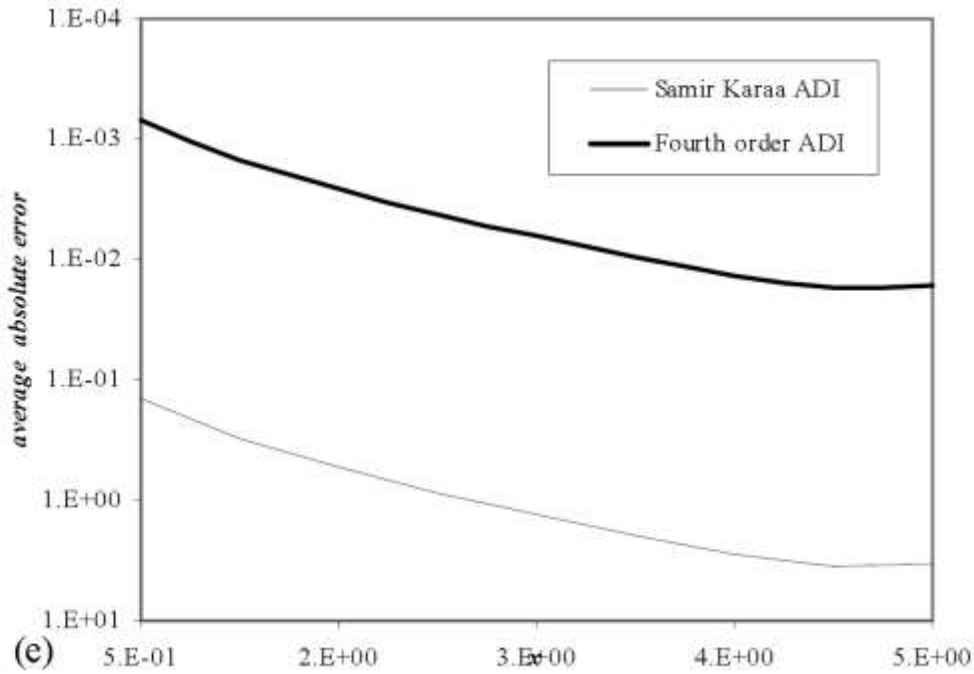
2.E+00
3.E+00
4.E+00
x

(d)



(e)

الأسلوب الأسي الضمني المضغوط (المتناوب الأتجاه) من الرتبة الرابعة لحل معادلات الأنتقال والأنتشار ثلاثية البعد

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الخلاصة

في هذا البحث، تم أشتقاق أسلوب الفروقات المحددة الأسية الضمنية المضغوطة (متناوب الأتجاه) من الرتبة الرابعة لحل معادلة الأنتقال والأنتشار ثلاثية البعد مع المعاملات الثابتة والمعتمدة على الزمن. هذا الأسلوب من الرتبة الثانية بالنسبة للزمن والرتبة الرابعة للحيز. وقد تم حل الصيغة بأستخدام طريقة عددية كفوءة تقابل حل نظام ثلاثي الاقطار. الدقة والكفاءة لهذا الأسلوب تم مناقشتها. أثبتنا أن هذا الأسلوب لرتب عليا تكون استقراريتها غير مقيدة وغير مشروطة بالنسبة للقيم الأبتدائية بأستخدام تحليل فورير. النتائج العددية تم عرضها ومقارنتها مع الأسلوب الأسي المضغوط (المتناوب الأتجاه) من الرتبة الرابعة والمقدم من قبل Karra.