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Prime–Extending Module and S-Prime Module.

Tamadher A. Ibrahiem
Department of Mathematics, College of Science for Women, University of Baghdad, Iraq.

Abstract

In this paper, the notation of prime-extending module, is introduced and studied where an R-module M is Prime-extending if every nonzero proper submodule of M is essential in prime direct summand .Beside these, the notation of S-prime is given, where an R module is called S-prime if every proper direct summand is prime for this notation we proved: 1.Under the condition of S-prime module, M is Prime-extending if and only if M is extending .2. If M is semisimple module, then M is Prime-extending if and only if M is S-prime.

Keywords: Extending module, Prime –Extending module, Prime direct summand, Prime submodule, uniform submodule

1.Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary left R-modules. Following [1], an R-module M is extending if every submodule of M is essential in a direct summand of M. This class of modules have been extensively studied in recent years in particular [1].In section one of this article we turn our attention to study prime-extending module. A proper submodule P of M is called prime if for any $r\in R$ and $x\in M$ Such that $r\,x\in P$, then either $x\in P$ or $r\in [P: M]$. A nonzero submodule N of an R-module M is called essential in M ($N \leq^{e} M$), if $N \cap L \neq (0)$ for every nonzero submodule L of M [1].

In section two of this paper, as a link between extending modules and prime – extending modules, we introduce and study the concept S-prime module. several properties of S-prime modules are given.

2.Prime- Extending modules

In this section we introduce the definition of prime-extending modules and give examples, some basic properties and characterizations of this concept .An R-module module M is called prime if <0> is prime submodule of M.A submodule N of an R-module M is called essential if $N\cap W\neq$<0>, for each nonzero submodule W of M

Definition (2.1):

An R-module M is called Prime-extending module if for every nonzero proper submodule N of M is essential in a Prime direct summand of M.

Remarks and Examples (2.2)

1. It is clear that every Prime-extending module is extending module but not conversely in general .Where each of Q, Z and $Z_{P^{\infty}}$ are extending module which is not Prime-extending module where the only prime submodule of the Z-module Q is <0>[2]. Hence, all submodules $0\neq N\leq Q$, N is not essential in prime direct summand. Also for the Z module Z we can show that the submodule $<\bar{3}>$ is essential in Z but Z is not prime submodule of Z. Finally $Z_{P^{\infty}}$ has no prime submodule[2].
2. Z_{12} as Z-module is not Prime-extending since $<\bar{2}>\leq^{e}<\bar{2}>$ and $<\bar{2}>$is prime submodule[3], but $<\bar{2}>$is not direct summand in Z_{12} .Also $<\bar{2}>\leq^{e} Z_{12}$ and Z_{12} is not prime submodule of Z_{12} since Z_{12} is not proper submodule of Z_{12}.
3. Z_6 as Z-module is Prime-extending module .Since $<\bar{2}>$, $<\bar{3}>$ are the only nonzero proper submodules .Also $<\bar{2}>\leq^{e}<\bar{2}>\leq^{\oplus} Z_6$ and $<\bar{3}>\leq^{e}<\bar{3}>\leq^{\oplus} Z_6$ with $<\bar{2}>$ and $<\bar{3}>$ is prime.
4. A ring R is fully prime if and only if every R-module is prime [4].
5. Every uniform module is not Prime-extending module. To show this:
   Let $0\neq N<M$ .Since M is uniform then the only direct summand are <0> and M [1] Hence there is no prime submodule P of M such that$0\neq N\leq^{e} P\leq^{\oplus} M$. So M is not Prime-extending module.

6. The uniform and Prime-extending concepts are independent. (See example (3)).

The following result gives a relationship between semisimple module and Prime-Extending modules.

Lemma(2.3):

Let M be an R-module and be a R is fully prime ring .Then every submodule N of M, is a prime submodule.

Proof:

Let N be a submodule of M .Since R is fully prime ring, so $\frac{M}{N}$ is a prime R-module. Let $r\in R$, $x\in M$, such that $rx\in N$. Then $rx + N=N$. So $r(x +N) =N$. It follows that either $x +N=N$ or $r \in [N:\frac{M}{N}]$.Hence either $x \in N$ or $r\frac{M}{N} \leq N$. That is $rM \leq N$ .That is $r \in [N: M]$.Thus N is a prime submodule of M.□

Proposition (2.4):

Let R be a fully prime ring. Then each semi-simple R-module M is Prime-extending module.

Proof:

Let M be a semisimple R-module. Let N be a submodule of M .Then N is a direct summand of M [1] .But R is fully prime ring. Then N is prime submodule (Lemma2.3).Then N is essential in a prime direct summand of M Therefore M is Prime –Extending module.

Proposition (2.5):

Let M be a Prime-extending module and N be a direct summand of M .Then N is a uniform submodule.

Proof:

Let U be a nonzero proper submodule of a direct summand N of M. So there exists a prime direct summand P of M such that U essential in P. Since N is direct summand of M and P is direct summand of M, then $N\oplus W=M$ and $P\oplus L=M$ for some $W\leq M$ and $L\leq M$. But $M\cap M=M$. So $(N\oplus W) \cap (P\oplus L) =M$. Hence $M= (P\cap N) \oplus(L\cap W)$.Now let $x\in N$ so $x\in M$ .Thus $x = a + b$ where $a\in P\cap N$, $b\in L\cap W$. Since $x-a =b$ .Thus $x-a\in N\cap L\cap W=0$. Hence $x=a \in P\cap N$ .Therefore $N\leq P\cap N$. Thus $N=N\cap P$ .that gives $N\leq P$. Now to prove $U<^e N$ .Let $0\neq K<N$, so $K<P$. Hence $U\cap K\neq 0$ (where $U<^e P$). .Thus $U<^e N$ .Thus N is uniform.□

Remark (2.6):

From proposition (2.5) and remark (2.2-5) we can note that a direct summand of a Prime-extending is not necessarily Prime-extending.

Recall that a submodule N of M is closed if it has no proper essential extension in M(i.e. whenever N is a submodule of M such that L is essential extension of N then N=L)[1][5]. In [1], we have an R-module M is extending if and only if every closed submodule is a direct summand .We have the following proposition for Prime-extending module.

Proposition (2.5):

An R-module M is Prime-extending if and only if every closed submodule of M is prime direct summand.

Proof:

→) Let N be a closed submodule of Prime-extending module M .Then N is essential in a prime direct summand L of M. But N is closed submodule of M .Hence N=L .i.e. N is prime direct summand.

←) Since every closed submodule is prime direct summand, then M is an extending module[1]. Let A be a nonzero proper submodule of M .Then A is essential in U, where U is a direct summand of M. Hence U is closed, and so that U is prime direct summand of M. Thus A is essential in prime direct summand of M .Therefore M is Prime-extending R-module. □

The following theorem gives us many characterizations of Prime–Extending module.

Theorem (2.6):

Let M be an R-module .Then the following statements are equivalent:

1. M is a Prime-extending R-module.
2. Every closed submodule of M is a prime direct summand.
3. If N is a direct summand of the injective envelope of M(denote by E (M)), then $N\cap M$ is a prime direct summand.

***Proof*:**

(1)→ (2**)** By proposition (2.5).

(2)→ (3) Let N be a direct summand of E(M), E(M)=N⊕ L , for submodule L of E(M) .We claim that N∩M is closed in M .Suppose that N∩M is an essential in K, where K is a submodule of M, and let $k\in K$ .Thus k=n +ℓ,where $n \in N$ and $\ell\in L$ .Now consider that $k\notin N$,so $\ell\neq 0$. But M is essential in E (M) and $0\neq\ell\in L\leq$ E (M).Therefore, there exist $r\in R$ such that $0\neq r\ell \in M$ .r k= r n+ r ℓ and r n = r k –r ℓ $\in N\cap M \leq K$ .Thus r ℓ= r k-r n $\in L\cap K$. But N∩M is essential in K, so 0= (N∩M) ∩L is essential in K∩L and hence K∩L=0 .Then rℓ= 0 which is a contradiction. So N∩M is closed in M and hence by (2) N∩M is prime direct summand of M.

(3)→ (1) Let N be a submodule of M Then N⊕L is essential in M, where L is a relative complement of N [6] .Since M is essential in E(M),thus N⊕L is essential in E(M) [6,p.209] and so E(M) =E(N⊕L)=E(N) ⊕E(L).Since E(N) is a summand of E(M), then by (3) E(N)∩M is prime direct summand of M .Now, N=N∩M is essential in E(N)∩M, where N is essential in E(N).Thus N is essential in a prime direct summand of M. So, M is a prime-extending module. □

The following result gives a decomposition theorem by Prime-extending modules.

Theorem (2.7):

An R-module M is Prime-extending module if and only if for each submodule N of M, there is a direct decomposition $M=M_1 \oplus M_2$ such that $N \leq M_1$ where M_1 is prime submodule of M and $N + M_2$ is essential submodule of M.

***Proof*:**

→) Suppose that M is Prime-extending module .Let N be a submodule of M. Then N is essential in prime direct summand K of M. Hence $M = K \oplus K_1$ where K_1 is a submodule of M. Now, since N is essential in K and K_1 is essential in K_1, then $N+K_1$ is essential in $K+K_1=M$ [6].That is $N+K_1$ is essential submodule of M .

←) Let N be a submodule of M .By hypothesis, there is a direct decomposition $M =M_1\oplus M_2$ such that $N \leq M_1$, where M_1 is prime submodule of M and $N + M_2$ is essential in M. We claim that N is essential in M_1 .Let K be a nonzero submodule of M_1, hence K is a submodule of M, and since $N + M_2$ is essential in M. Then $(N + M_2) \cap K\neq 0$. Let $0\neq k =n + m_2$, where $k \in K$, $n\in N$ and $m_2\in M_2$, thus $m_2 = k-n \in M_1 \cap M_2 = (0)$ .Therefore $k = n \in K \cap N \neq (0)$. So N is essential in M_1 .That means M is a Prime-extending module. □

Proposition (2.9):

Let M be faithful multiplication and finitely generated module. Then M is Prime - Extending if and only if R is Prime-extending R-module.

***Proof*:**

→) Let I be an ideal in R .Since M is multiplication module ,then N=IM , where N is a submodule of M .But M is Prime-extending module, so there exists a prime direct summand W of M such that N is essential in W .But W= PM for some prime ideal P of R [2, proposition 4.6].That is IM is essential in PM and PM is prime summand of M .Then I is a submodule of P .Now ,there exists a submodule Z of M such that M= PM ⊕ Z =PM⊕TM where T is an ideal in R .But M=PM⊕TM =(P⊕T)M=RM. Hence P⊕T=R[7] .i.e P is prime summand of R. Now, it is enough to show that I is essential in P .Let $A \neq 0$ be a subideal of P .Suppose (I∩A) M=0.But (I∩A) M=IM∩AM =0 [7] .But AM is submodule of PM, and IM is essential in PM, so AM=0 .That gives $A \leq \operatorname{ann} M = 0$ (where M is faithful) and hence A=0 which is a contradiction .Thus I is essential in a prime direct summand P of R .Therefore R is Prime-extending R-module.

←) Suppose that R is Prime-extending R-module. Let N submodule of M .since M is multiplication, then N=IM, where I is an ideal of R .But R is Prime-extending R-module .So, there exists a prime ideal which is direct summand of R(say P) such that I is essential in P. Hence P⊕T=R, where T is an ideal of R. Thus IM is essential in PM[5]. Now P⊕T=R, then M = RM = (P⊕T) M = PM⊕TM. But M is multiplication and P is prime in R. Therefore PM is prime in M [4, proposition4.6] .Also, PM is a summand of M .Hence N=IM is essential in prime direct summand PM of M. □

By remarks and examples (1.2-1) we observed that every prime-extending module is extending module, but not conversely in general .The following proposition gives the sufficient condition under which the converse is true. Also, this condition will be study extensively in the next section.

Proposition (2.10):

Let M be an R module such that every direct summand of M is prime .Then M is Prime-extending if and only if M is extending.

Proof:

It is straightforward.

3. S-prime module

In this section we observed that the concepts of Prime-extending modules and extending modules are equivalent under the condition (every direct summand is prime). This leads as to introduce and study this condition under the class of semisimple uniform and others types of modules.

Definition (3.1):

An R- module M is called S-prime module if every proper direct summand of M is a prime. Also, we call a ring R is S- prime ring if R is S- prime module as R- module.

Examples (3.2).

1. The Z_3 as Z-module is S-prime, where Z_3 has no nonzero proper direct summand N.
2. The Z_{12} as Z-module is not S-prime module .To show that :

Let N= $<\bar{4}>$.So N+$<\bar{3}>=<\bar{4}>\oplus<\bar{3}>$ =Z_{12} .Hence N=$<\bar{4}>$is a direct summand which is not prime, 2. $\bar{2}\in<\bar{4}>$but $\bar{2}\notin<\bar{4}>$ and $2\notin[<\bar{4}>:Z_{12}] = 4Z$.Thus Z_{12} is not S-prime

Proposition (3.3):

Let M be an R- module where R is fully prime ring, then every module is S-prime module.

Proof:

Since R is fully prime ring, and then by lemma (2.3) every submodule of M is a prime. Thus M is S-prime.□

Proposition(3.4).:

Let M be S-prime module .Then M is Prime-extending module if and only if M is extending.

Proof:

Let M be extending module .and N is a nonzero proper submodule of M. So N is essential in a direct summand W of M .But M is S-prime .Hence W is prime direct summand .Therefore M is Prime-extending module.
The converse is obvious. □

Proposition(3.5):

An R-module M is Prime-extending if and only if M is extending and S-prime module.

Proof:
Obvious.□

The following proposition gives a characterization between the Prime-extending and S-prime concepts.

Proposition (3.6):

Let M be a semisimple R-module. Then M is a Prime-extending module if and only if M is S-prime module.

***Proof*:**

→) suppose that M is Prime-extending module .Let N be a proper direct summand of M .Then N is closed submodule of M. Now, since M is a Prime-extending, then N is prime (proposition, 2.6).That is M is S-prime module.

←) Let N be a proper submodule of M .Then N is a direct summand of M (where M is a semisimple module). But M is S-prime module, so N is prime in M .Thus N is prime direct summand of M, and N is essential in N. Therefore M is a Prime-extending module.□

The following Proposition shows that the direct summands of S-prime module inherit Property. Before that we give the following lemma:

Lemma (3.7):

Let M be an R-module with submodules K and N where $K \leq N \leq M$. If K is prime in M and N is prime in M then K is prime in N.

Proof:

Let $x \in N$, $r \in R$ such that $r\,x \in K$. To show that K is prime in N, we must prove that either $x \in K$ or $r \in [K: N]$ .since K is prime in M ,then either $x \in K$ or $r \in [K:M]$.Suppose that $r \in [K: M]$.i.e $rM \leq K$. But $N \leq M$, so $rN \leq K$. Hence $r \in [K: N]$.Thus K is prime in N .If $x \in K$ then nothing to prove. □

<u>Preposition (3.8):</u>

Every direct summand of S-prime Module is S-prime module.

Proof:

Let M be S-prime module and N be a summand of M. Then N is prime direct summand .Now let K be summand of N ,then K is a summand of M [2].Hence, K is Prime in M, so K is prime direct summand of N (lemma 3.7). Therefore N is S-prime module.□

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الخلاصة

ليكن M مقاس ايسر على الحلقة R. في بحثنا هذا نقدم دراسة عن مقاسات التوسع الاولية حيث ان المقاس M يسمى مقاس توسع اولي اذا كان كل مقاس جزئي فعلي غير صفري جوهري في جداء جمع داخلي اولي في M بالاضافة الى ذلك درسنا مفهوم المقاس الاولي من النمط –S .حيث ان المقاس M يسمى اولي من النمط -S الذا كان كل جمع داخلي في M اولي.وفيما يخص هذا المفهوم حصلنا على:1. تحت شرط المقاس الاولي من النمط – S فانه M مقاس توسع اولي اذا وفقط اذا كان M مقاس توسع.

2. اذا كان M مقاس شبه بسيط فانه M مقاس توسع اولي اذا وفقط اذا كان M مقاس اولي من النمط -S.