Bochner curvature tensor of Almost Kahler manifold

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Abstract

In this paper we study the Bochner curvature tensor of almost Kahler manifold. We found the components of Bochner tensor of almost Kahler manifold in the adjoint G-structure space by using Kirichenko's tensors. It has been proved that an almost Kahler manifold is a manifold of class $\beta_1$ if and only if it is Kahler.

1. Introduction

The Bochner tensor was given by S. Bochner (1949)[4]. He found this tensor in the Kahler manifold as Weyl's tensor(conformal curvature of Riemannian manifold). S. Tachibana (1967)[13] gave it the real form and proved that the Bochner tensor has a meaning on any almost Hermitian manifold. M. Mastumoto(1969)[10] proved that Kahler manifold of constant scalar curvature tensor with zero Bochner tensor is local symmetric. S. Tachibana (1970)[14] proved that Kahler manifold of constant scalar curvature tensor with zero Bochner tensor is local holomorphic-isometric of product complex spaces. Z. Olsgak(1984)[12] gave classification of 4-dimensionsal compact flat Bochner of Kahler manifold with non positive scalar curvature tensor. M. Petrovic and L. Verstraelen(1987)[11] are classified flat Bochner of Kahler manifold that the Weyl's tensor satisfies some conditions. A. Al-Othman(1993)[2] studied the Bochner tensor of Nearly Kahler manifold, he found the classification of flat Bochner tensor of NK-manifold and studied the Bochner tensor of B-constant type of almost Hermitian manifold and he defined the holomorphic Bochner curvature of almost Hermitian manifold. A. Al-Othman(2008)[1] studied the Bochner-recurrent Nearly Kahler manifold, he proved that Bochner-Recurrent Nearly Kahler manifold is either Bochner-symmetrical or Bochner-recurrent Kahler manifold.

In this present work we give the components of Bochner tensor of Almost Kahler manifold and study the almost Kahler manifold of class $\beta_1$.

2. Preliminaries

Definition 1.2 [5]

A tensor field $J$ of type (1,1) is called an almost complex structure, such that, at each point $p \in M$ can be defined an endomorphism of the tangent space $T_p(M)$ with the property $J^2 = -id$, where $id: T_p(M) \to T_p(M)$ is the identity transformation.
Definition 2.2 [5]
A manifold provided by the almost complex structure is called an almost complex manifold. It is well-known, that every complex manifold has even dimension and it is orientable. In general the converse is not true [8].

In the module $X^c(M)$ can be defined two projections $\sigma = \frac{1}{2}(id - J\sqrt{-1})$ and $\bar{\sigma} = \frac{1}{2}(id + J\sqrt{-1})$, where $X^c(M)$ is the complexification of the module $X^e(M)$.

The setting of projections $\sigma$ and $\bar{\sigma}$ is equivalent to the decomposition of the module $X^c(M)$ in the direct sum of these projections, i.e. $\forall X \in X^c(M), \ X = \sigma(X) + \bar{\sigma}(X)$.

Definition 3.2 [8]
The pair $\{J, g = <\cdot, \cdot>\}$ is called an almost Hermitian structure (AH-structure) on the manifold $M$, where $J$ is the almost complex structure on $M$, $g = <\cdot, \cdot>$ is a Riemannian metric on $M$, such that $<X, Y> = <JX, JY>$, $X, Y \in X(M)$.

Definition 4.2 [8]
A manifold provided by AH-structure is called an almost Hermitian manifold. It is known [6] that the setting of an almost Hermitian structure on $M$ is equivalent to the setting of a $G$-structure on $M$ with structure group is a unitary group $U(n)$. This $G$-structure is called an adjoint $G$-structure space [7].

Assume that the value of indices $a, b, c, d, e, g, h, \ldots$ is in the range 1 to $n$, and the indices $i, j, k, l, \ldots$ is in the range 1 to $2n$. Denote $\hat{a} = a + n$, then the indices are $a, b, c, d, e, f, g, \ldots, \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}, \hat{f}, \hat{g}, \ldots$.

In the space of the adjoint $G$-structure, the components of the tensor fields $J$ and $g$ are given by the matrices [9]

$$
\begin{pmatrix}
0 & \mathbb{I}_n \\
\mathbb{I}_n & 0
\end{pmatrix}
$$

(2.1)

Where $\mathbb{I}_n$ is the unit matrix of order $n$.

Definition 5.2 [9]
An AH-structure is called an almost Kahler structure (AK structure) if the fundamental form $\Omega(X, Y) = <X, JY>$ is closed i.e. $d\Omega = 0$.

A manifold $M$ with AK-structure is called an almost Kahler manifold (AK-manifold).

Definition 6.2 [9]
The components of the fundamental form in the adjoint $G$-structure space are given by the matrix:

$$
\Omega_{ij} = \begin{pmatrix}
0 & \mathbb{I}_n \sqrt{-1} \\
-\mathbb{I}_n \sqrt{-1} & 0
\end{pmatrix}
$$

(2.2)

Definition 7.2 [9]
The Riemannian curvature tensor $R$ for $M$ is 4-covariant tensor:

$R: T_p(M) \times T_p(M) \times T_p(M) \times T_p(M) \rightarrow \mathbb{R}$ which is defined by:

$R(X_1, X_2, X_3, X_4) = g(R(X_3, X_4)X_2, X_1)$ where $X_i \in T_p(M)$ \ \ $\forall i = 1, \ldots, 4$ and satisfied the following properties:

1. $R(X_1, X_2, X_3, X_4) = -R(X_2, X_1, X_3, X_4)$
2. \( R(X_1, X_2, X_3, X_4) = -R(X_1, X_2, X_4, X_3) \)
3. \( R(X_1, X_2, X_3, X_4) = R(X_2, X_1, X_4, X_3) \)
4. \( R(X_1, X_2, X_3, X_4) + R(X_1, X_3, X_4, X_2) + R(X_1, X_4, X_2, X_3) = 0 \)

Proposition 8.2 [9]

The components of Riemannian curvature tensor of \( AK \)-manifold in the adjoint G-structure space are:

1. \( R^\alpha_{\beta \gamma \delta} = A^{\beta \gamma}_{\alpha \delta} + 2B^{\beta \gamma \delta}_{\alpha \delta} - 4B_{\bar{\alpha} \bar{\beta} \gamma \delta} \)
2. \( R^\alpha_{\beta \gamma \delta} = 4B^{\gamma \delta \alpha}_{\beta \gamma} - A^{\alpha \beta}_{\gamma \delta} - 2B^{\alpha \gamma \beta}_{\delta \gamma} \)
3. \( R^\alpha_{\beta \gamma \delta} = A^{\alpha \beta}_{\gamma \delta} + 2B^{\alpha \beta \gamma}_{\delta \gamma} - 4B_{\bar{\alpha} \bar{\beta} \gamma \delta} \)
4. \( R^\alpha_{\beta \gamma \delta} = 4B_{\gamma \delta \alpha \beta} - A^{\beta \gamma}_{\alpha \delta} - 2B^{\gamma \delta \beta \alpha}_{\alpha \delta} \)
5. \( R^\alpha_{\beta \gamma \delta} = -2B^{\beta \gamma \delta \alpha} \)
6. \( R^\alpha_{\beta \gamma \delta} = 2B^{\beta \gamma \delta \alpha} \)
7. \( R^\alpha_{\beta \gamma \delta} = 2B^{\beta \gamma \delta \alpha} \)
8. \( R^\alpha_{\beta \gamma \delta} = 2B^{\beta \gamma \delta \alpha} \)
9. \( R^\alpha_{\beta \gamma \delta} = 4B^{\beta \gamma \delta \alpha} \)
10. \( R^\alpha_{\beta \gamma \delta} = -2B^{\beta \gamma \delta \alpha} \)
11. \( R^\alpha_{\beta \gamma \delta} = 2B^{\beta \gamma \delta \alpha} \)
12. \( R^\alpha_{\beta \gamma \delta} = -4B^{\beta \gamma \delta \alpha} \)
13. \( R^\alpha_{\beta \gamma \delta} = 2B^{\beta \gamma \delta \alpha} \)
14. \( R^\alpha_{\beta \gamma \delta} = -2B^{\beta \gamma \delta \alpha} \)
15. \( R^\alpha_{\beta \gamma \delta} = 2B^{\beta \gamma \delta \alpha} \)
16. \( R^\alpha_{\beta \gamma \delta} = 4B^{\beta \gamma \delta \alpha} \)

3. Bochner curvature tensor

Definition 1.3 [2]

Bochner curvature tensor on \( AH \)-manifold defined as the following form:

\[
\beta(X, Y, Z, W) = R(X, Y, Z, W) + L(X, W)g(Y, Z) - L(X, Z)g(Y, W) + L(Y, Z)g(X, W)
- L(Y, W)g(X, Z) + L(JX, W)g(JY, Z) - L(JX, Z)g(JY, W)
+ L(JY, Z)g(JX, W) - L(JY, W)g(JX, Z) - 2L(JX, Y)g(JZ, W)
- 2L(JZ, W)g(JX, Y)
\]

Where \( L(X, Y) = -\frac{1}{2n+4} g(rX, Y) + \frac{K}{2(2n+2)(2n+4)} g(X, Y) \), \( r \) is a Ricci tensor and \( K \) is a scalar curvature tensor and \( X, Y, Z, W \in X(M) \).

Let \( C(X, Y) = L(JX, Y) \) and we have \( g(JX, Y) = -\Omega(X, Y) \), thus

\[
\beta_{ijkl} = R_{ijkl} + L_{ij}g_{jk} - L_{ik}g_{jl} + L_{jk}g_{il} - L_{jl}g_{ik} - C_{ijl}\Omega_{jk} + C_{ikl}\Omega_{jl} - C_{jkl}\Omega_{il} + C_{jkl}\Omega_{il} + 2C_{ijl}\Omega_{kl} + 2C_{ikl}\Omega_{ij}.
\]
where \( L_{ij} = -\frac{1}{2n+4} r_{ij} + \tilde{R} g_{ij} \) (1.3)

and \( \tilde{R} = \frac{1}{2(2n+2)(2n+4)} \)

and \( C_{ij} = L(e_i, e_j) = -\frac{1}{2n+4} J_i^k r_{kj} + \tilde{R} J_i^k g_{kj} \) (2.3)

Theorem 1.3

The components of Bochner curvature tensor of \( AK \)-manifold are:

1. \( \beta_{abcd} = R_{abcd} \)
2. \( \beta_{abcd} = R_{abcd} + \frac{1}{2n+4} (r_{bd} \delta^a_c - r_{bc} \delta^a_d) + \frac{1}{2n+4} (r_{cd} \delta^a_b - r_{ac} \delta^a_d - 2r_{ad} \delta^a_b) \)
3. \( \beta_{abcd} = R_{abcd} + \frac{1}{2n+4} (r_{ac} \delta^d_b - r_{ad} \delta^b_c) + \frac{1}{2n+4} (r_{cd} \delta^a_b - r_{ca} \delta^a_d - 2r_{cb} \delta^a_d) \)
4. \( \beta_{abcd} = R_{abcd} + \frac{1}{2n+4} (r_{bd} \delta^a_c - r_{bc} \delta^a_d) + \frac{1}{2n+4} (2r_{cb} \delta^a_d - r_{cd} \delta^a_b - r_{cb} \delta^a_d) \)
5. \( \beta_{abcd} = R_{abcd} + \frac{1}{2n+4} (r_{ac} \delta^b_d - r_{bc} \delta^d_a) + \frac{1}{2n+4} (2r_{cb} \delta^a_d - r_{bc} \delta^a_d - r_{ac} \delta^a_b) \)
6. \( \beta_{abcd} = R_{abcd} \)
7. \( \beta_{abcd} = R_{abcd} - \frac{1}{n+2} (r_{bc} \delta^a_d + r_{ac} \delta^a_b + r_{b} \delta^{a}_d + r_{a} \delta^b) + 4\tilde{R} \delta^a_b \)
8. \( \beta_{abcd} = R_{abcd} + \frac{1}{n+2} (r_{ac} \delta^b_d + r_{ad} \delta^b_c + r_{bd} \delta^a_c + r_{c} \delta^a_d) - 4\tilde{R} \delta^{cd} \)

Proof

suppose that \( M \) is \( AK \)-manifold, in the adjoint \( G \)-structure space by using (2.1), (2.2), (1.3) and (2.3) we get:

1. put \( i = a, j = b \) we obtained \( L_{ab} = -\frac{1}{2n+4} r_{ab} \) (3.3)

2. put \( i = \hat{a}, j = \hat{b} \) we get \( L_{\hat{a}\hat{b}} = -\frac{1}{2n+4} r_{\hat{a}\hat{b}} \) (4.3)

3. put \( i = \hat{a}, j = b \) we get \( L_{\hat{a}b} = -\frac{1}{2n+4} r_{\hat{a}b} + \tilde{R} \delta^a_b \) (5.3)

4. put \( i = a, j = \hat{b} \) we get \( L_{ab} = -\frac{1}{2n+4} r_{a\hat{b}} + \tilde{R} \delta^b_a \) (6.3)

We compute the components of \( C_{ij} \), in the same computes we obtained:

\( C_{ab} = \frac{\sqrt{-1}}{2n+4} r_{cb} \delta^c_a \) (7.3)

\( C_{\hat{a}\hat{b}} = \frac{\sqrt{-1}}{2n+4} r_{\hat{a}\hat{b}} \delta^a_c \) (8.3)

\( C_{\hat{a}b} = \frac{\sqrt{-1}}{2n+4} r_{\hat{a}b} \delta^c_a - \sqrt{-1} \tilde{R} \delta^a_b \) (9.3)

\( C_{a\hat{b}} = \frac{\sqrt{-1}}{2n+4} r^b_a \delta^b_a + \sqrt{-1} \tilde{R} \delta^a_b \) (10.3)

Now we compute the components of Bochner curvature tensor:

1. put \( i = a, j = b, k = c, l = d \) then:

\( \beta_{abcd} = R_{abcd} + L_{ad} g_{bc} - L_{ac} g_{bd} + L_{bd} g_{ac} - L_{bc} g_{ad} - C_{ad} \Omega_{bc} + C_{ac} \Omega_{bd} - C_{bc} \Omega_{ad} + C_{bd} \Omega_{ac} + 2C_{ab} \Omega_{cd} + 2C_{cd} \Omega_{ab} \)

From equations (2.1), (2.2), (1.3) – (10.3) we get:

\( \beta_{abcd} = R_{abcd} \) (11.3)
From equations (2.1, 2.2, 1.3-10.3) we get:
\[ \beta_{abcd} = \frac{1}{2n+4} (r_{bd} \delta^a_c - r_{bc} \delta^a_d) + \frac{1}{2n+4} (r_{cd} \delta^b_a - r_{ca} \delta^b_d - 2r_{ad} \delta^b_c) \] (12.3)

4. put \( i = a, j = b, k = c, l = d \) then:
\[ \beta_{ab} = \frac{1}{2n+4} (r_{ad} \delta^a_b + r_{bd} \delta^a_c - r_{cd} \delta^a_d) + \frac{1}{2n+4} (r_{ca} \delta^b_a - r_{cb} \delta^b_d - 2r_{cd} \delta^b_c) \] (13.3)

5. put \( i = a, j = b, k = c, l = d \) then:
\[ \beta_{abc} = \frac{1}{2n+4} (r_{ad} \delta^a_b - r_{bd} \delta^a_c + r_{cd} \delta^a_d) \] (14.3)

6. put \( i = a, j = b, k = c, l = d \) then:
\[ \beta_{a} = \frac{1}{2n+4} (r_{ad} \delta^a_b + r_{bd} \delta^a_c + r_{cd} \delta^a_d) \] (16.3)

8. put \( i = a, j = b, k = c, l = d \) then:
\[ \beta_{abc} = \frac{1}{2n+4} (r_{ad} \delta^a_b + r_{bd} \delta^a_c + r_{cd} \delta^a_d) - 4K \delta^a_{cd} \] (18.3)

Definition 2.3 [2]

The Bochner curvature tensor is of:
1. class \( \beta_1 \) if \( \beta(X,Y,Z,W) = \beta(X,Y,Z,W) \)
2. class \( \beta_2 \) if \( \beta(X,Y,Z,W) = \beta(X,Y,Z,W) + \beta(X,Y,Z,W) + \beta(X,Y,Z,W) \)
3. class \( \beta_3 \) if \( \beta(X,Y,Z,W) = \beta(X,Y,Z,W) \)

Definition 3.3 [3]

An \( AH \)-manifolds is called a Kahler manifold if \( B_{abc} = 0 \) and called an almost Kahler manifold if \( B_{abc} = 0 \) where \( B_{abc} = 0 \) is structure tensor (Kirichenko's tensor), and the bracket ( ) denote to symmetric.

Theorem 2.3

Almost Kahler manifold \( M \) is of class \( \beta_1 \) if and only if \( M \) is Kahler manifold.

Proof
According to class \( \beta_1 \) we get:
\[ \beta_{abcd} = \beta(e_a, e_b, e_c, e_d) = \beta(e_a, e_b, J e_c, J e_d) \]
\[ = \beta(e_a, e_b, \sqrt{-1} e_c, \sqrt{-1} e_d) \]
\[ = (\sqrt{-1})(\sqrt{-1})\beta(e_a, e_b, e_c, e_d) \]
\[ = -\beta(e_a, e_b, e_c, e_d) = -\beta_{abcd} \]

Thus \( \beta_{abcd} + \beta_{abcd} = 0 \Rightarrow \beta_{abcd} = 0 \)
Since \( \beta_{abcd} = 4B^{ab}_{bc}B_{cd} \Rightarrow B^{ab}_{bc}B_{cd} = 0 \)

By folding (a and c) and (b and d) we get

\[
B^{ab}_{bc}B_{cd} = 0 \Longleftrightarrow \sum |B^{ab}_{cd}|^2 = 0 \Longleftrightarrow B_{cd} = 0
\]

According to [3] this Kahler condition.

Therefore \( M \) is Kahler manifold.

References: