Some properties of $b\tau$ -denseness set in bitopological spaces

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Abstract

In this paper we introduced the notion " bτ- dense sets" in bitopological spaces and proved some of their properties and related theorems by using the concept of bτ -open set .

المستخلص :

في هذا البحث قدمنا مفهوم المجموعات بي تاو- الكثيفه وبي تاو-غير الكثيفه في الفضاءات ثنائية التوبولوجيا واثبتا بعض الخواص المتعلقه بها باستخدام مفهوم المجموعات بي تاو-المفتوحه

Introduction:

In 2007, M. Ganster and M. Steiner [6] introduce the concept of $b\tau$-closed set, which the complement of it is called $b\tau$-open set where they defined a subset A of a topological space X to be $b\tau$-closed if $cl_b(A)\subset U$ whenever $A\subset U$ and U is open, where $cl_b(A)$ denoted to the intersection of all b-closed sets containing a subset A.in this paper we used a corresponding concept, i.e. $\tau_1\tau_2$bτ-open in bitopological spaces, where the study of Bitopological space initiated by Kelly [3],[5] is defined as : A set equipped with two topologies is called a bitopological space, denoted by (X,τ_1,τ_2) where (X,τ_1) and (X,τ_2) are two topological spaces defined on topological spaces defined on X

Definition (1-1): [8] A subset S of X is called $\tau_1\tau_2$ open if $S \in \tau_1 \cup \tau_2$ and the complement of $\tau_1\tau_2$- open set is $\tau_1\tau_2$-closed.
Example (1-2): Let $X = \{a, b, c\}$, $\tau_1 = \{\phi,X, \{a\}\}$ and $\tau_2 = \{\phi,X, \{b\}\}$. The sets in $\{\phi,X, \{a\}, \{b\}\}$ are called $\tau_1\tau_2$ open and the sets in $\{\phi,X, \{b, c\}, \{a, c\}\}$ are called $\tau_1\tau_2$ closed.
Definition (1-3):[1] A subset A of a space (X,τ_1,τ_2) is said to be $\tau_1\tau_2$b-open set if $A\subset \tau_1\tau_2 cl(\tau_1\tau_2 \operatorname{int}(A))\cup \tau_1\tau_2 \operatorname{int}(\tau_1\tau_2 cl(A))$
Remark(1-4):
1) The complement of $\tau_1\tau_2$b-open set is called $\tau_1\tau_2$b-closed set.
2) The intersection of all $\tau_1\tau_2$b-closed sets of X containing a subset A of (X,τ_1,τ_2) is called $\tau_1\tau_2$b-closure of A and is denoted by $\tau_1\tau_2$b-cl(A). Analogously the $\tau_1\tau_2$b- interior of A is the union of all $\tau_1\tau_2$b -open sets contained in A denoted by $\tau_1\tau_2$b-int(A).

Definition (1-5): A subset A of a bitopological space (X,τ_1,τ_2) is called to be $\tau_1\tau_2$bτ -open if$\tau_1\tau_2$b-cl(A) $\tau_1\tau_2$b-cl
Definition (1-6): Let (X, τ_1 , τ_2) be a bitopological space and $A \subseteq X$.Then A is called a $\tau_1\tau_2$bτ-neighbourehood of a point x in X ,if there exists$\tau_1\tau_2$bτ-open set U in X such that $x \in U \subset A$.
Definition (1-7): The union of all bτ - open sets contained in a setA is called bτ- ***interior*** of A and denoted by bτ-**int(A)** .
Definition (1-8): The intersection of all bτ -closed sets containing a set A is called $b\tau$ ***-closures*** of A and denoted by bτ- **cl (A)**.
Definition (1-9): A point $x\in X$ is said to be a bτ ***-limit Point*** if and only if U is $\tau_1\tau_2$bτ-open set implies $U \cap (A-\{x\}) \neq \emptyset$ .where $\emptyset$ is the empty set.

Definition (1-10): The set of all bτ -limit points of A $\subseteq$ X, is called the $b\tau$ ***-drived set*** of A and is denoted by $b\tau$**-D(A)** .

Definition (1-11): The set bτ cl(A) – bτint(A) is called $b\tau$***-frontier*** of A is denoted by $b\tau$-**f(A)** .

Remark (1-12) : every $\tau_1\tau_2$-closed set is $\tau_1\tau_2 b$-closed set and every $\tau_1\tau_2 b$-closed set is $\tau_1\tau_2 b\tau$-closed set .

Proposition (1-13) : Every $\tau_1\tau_2$-closed subset of a bitopological space (X, τ_1 , τ_2) is $\tau_1\tau_2 b$-closed.

Proof: Let $A\subseteq X$ be $\tau_1\tau_2$-closed set, since $A^c\subset\tau_1\tau_2 cl(A^c)$, hence $\tau_1\tau_2\operatorname{int}(A^c)\subset\tau_1\tau_2\operatorname{int}(\tau_1\tau_2 cl(A^c))$, but $\tau_1\tau_2\operatorname{int}(A)\subset A$ for any subset A, hence $A^c\subset\tau_1\tau_2\operatorname{int}(\tau_1\tau_2 cl(A^c))$, and $A^c\subset\tau_1\tau_2\operatorname{int}(\tau_1\tau_2 cl(A^c))\bigcup\tau_1\tau_2 cl(\tau_1\tau_2\operatorname{int}(A^c))$, hence A^c is $\tau_1\tau_2 b$-open set, hence A is $\tau_1\tau_2 b$-open set .□

Remark(1-14) : The converse of Proposition (1-13) is not true as the following example

Example (1-15): $\tau_1\tau_2 b$-closed set $\not\rightarrow\tau_1\tau_2$-closed set.Let $X=\{a,b,c\}$, $\tau_1=\{X,\phi,\{a\},\{b\},\{a,b\}\}$, $\tau_2=\{X,\phi,\{b\}\}$, then the sets in $\{X,\phi,\{b,c\},\{a,c\},\{c\}\}$ are $\tau_1\tau_2$-closed and the set in $\{X,\phi,\{b,c\},\{a,c\},\{c\},\{a\},\{b\}\}$ are all $\tau_1\tau_2 b$-closed, so $\{a\},\{b\}$ are $\tau_1\tau_2 b$-closed but not 123-closed set.

Proposition (1-16) : Every $\tau_1\tau_2 b$-closed subset of a bitopological space (X,τ , τ_2) is $\tau_1\tau_2 b\tau$-closed.

Proof:

Let $A\subseteq X$ be $\tau_1\tau_2 b$-closed set, and let $A\subseteq U$, where U is $\tau_1\tau_2$-open, since A is $\tau_1\tau_2 b$-closed set, hence $\tau_1\tau_2\operatorname{int}(\tau_1\tau_2 cl(A))\bigcap\tau_1\tau_2 cl(\tau_1\tau_2\operatorname{int}(A))\subset A$, but $A\subset U$, hence $\tau_1\tau_2\operatorname{int}(\tau_1\tau_2 cl(A))\bigcap\tau_1\tau_2 cl(\tau_1\tau_2\operatorname{int}(A))\subset U$, since $\tau_1\tau_2 cl_b(A)$ is the smallest $\tau_1\tau_2 b$-closed set containing A, so,

$$\begin{aligned}\tau_1\tau_2 cl_b(A)&=A\bigcup(\tau_1\tau_2\operatorname{int}(\tau_1\tau_2 cl(A))\bigcap 123cl(123\operatorname{int}(A)))\\&\subset A\bigcup U\\&\subset U,\end{aligned}$$

i.e. A is $\tau_1\tau_2 b\tau$-closed.□

Remark (1-17):the converse of Proposition (1-16) is not true as the following example

Example (1-18) $\tau_1\tau_2 b\tau$-closed set $\not\rightarrow\tau_1\tau_2 b$-closed set.For the same example (1.15), the set $\{a,b\}$ is $\tau_1\tau_2 b\tau$-closed but it is not $\tau_1\tau_2 b$-closed set.

Theorem (1-19): Let A be a subset of abitopological space (X, τ_1 , τ_2) . then $\tau_1\tau_2 b\tau$-cl(A) isclosed and A $\subseteq\tau_1\tau_2 b\tau$-cl(A) further A is closed if and only if A = $\tau_1\tau_2 b\tau$-cl(A) .

Theorem (1-20): Let A be a subset ofabitopological space (X,τ_1,τ_2).then $\tau_1\tau_2 b\tau$-int(A)is open $\tau_1\tau_2 b\tau$-int(A) $\subseteq$ A further A is open if and only if A= $\tau_1\tau_2 b\tau$-int(A).

Theorem (1-21): Let A be a subset of abitopological space (X, τ_1 , τ_2) Then $\tau_1\tau_2\, b\tau$-int(A) = X - $\tau_1\tau_2\, b\tau$-cl(X-A) and $\tau_1\tau_2\, b\tau$-cl(A) = X - $\tau_1\tau_2\, b\tau$-int(X-A) .

***Proof*:** Since X-A $\subseteq\tau_1\tau_2\, b\tau$-cl(X-A) we have X - $\tau_1\tau_2\, b\tau$-cl(X-A) $\subseteq$ A .But X - $\tau_1\tau_2\, b\tau$-cl(X-A) is open (by theorem (1-19)),so X - $\tau_1\tau_2\, b\tau$-cl(X-A) $\subseteq\tau_1\tau_2\, b\tau$-int(A). On the other hand, X-$\tau_1\tau_2\, b\tau$ -int(A) is closed by theorem(1-20), and X-A $\subseteq$ X-$\tau_1\tau_2\, b\tau$-int(A) , so $\tau_1\tau_2\, b\tau$-cl(X-A) $\subseteq$ X-$\tau_1\tau_2\, b\tau$-intA .And hence $\tau_1\tau_2\, b\tau$-int(A) $\subseteq$ X-$\tau_1\tau_2\, b\tau$-cl(X-A). This shows that $\tau_1\tau_2\, b\tau$-int(A) = X-$\tau_1\tau_2\, b\tau$-cl(X-A) , and the other relation follows from replace A by X-A ■

2- $b\tau$ *-densnesse* set in bitopological spaces

Definition(2-1): Let A be a subsets of thebitopological space (X, τ_1 , τ_2).Then A is said to be $\tau_1\tau_2$ $b\tau$ ***-dense*** in X if $\tau_1\tau_2$ $b\tau$-cl(A) =X .

Definition(2-2): A subset A of abitopological space (X, τ_1 , τ_2) is ***non*** $\tau_1\tau_2$ $b\tau$ ***-dense*** set if $\tau_1\tau_2$ $b\tau$-int($b\tau$-cl(A))= $\emptyset$ that is the $\tau_1\tau_2$ $b\tau$ - interior of the $\tau_1\tau_2$ $b\tau$-closure of A is empty.

Theorem(2-3): Let A be a subset of abitopologicalspace (X, τ_1, τ_2) Then the following statements are equivalent:

i) A is non $\tau_1\tau_2$ $b\tau$-dense in X.

ii) $\tau_1\tau_2$ $b\tau$-cl(A) contains no $\tau_1\tau_2$ $b\tau$-nhd.

Proof : (i) $\leftrightarrow$ (ii) we have A is non $\tau_1\tau_2$ $b\tau$-dense

$\leftrightarrow$ $\tau_1\tau_2$ $b\tau$-int (spcl(A)) = $\emptyset$

$\leftrightarrow$ No point of X is a $\tau_1\tau_2$ $b\tau$-int point of $b\tau$-cl(A)

$\leftrightarrow$ $\tau_1\tau_2$ $b\tau$ cl (A) has not a $\tau_1\tau_2$ $b\tau$-nhd of any of its Points

$\leftrightarrow$ $\tau_1\tau_2$ $b\tau$ cl (A) contains no $\tau_1\tau_2$ $b\tau$-nhds■

Theorem(2-4): Let A be a subset ofa bitopological spaces (X, τ_1 , τ_2) if A is non $\tau_1\tau_2$ $b\tau$-dense, then $\tau_1\tau_2$ $b\tau$-cl(A) is not the entire space X.

Proof: Since X is $\tau_1\tau_2$ $b\tau$-closed then X= $\tau_1\tau_2$ $b\tau$-cl(X). Again since X is $\tau_1\tau_2$ $b\tau$-open, we have $\tau_1\tau_2$ $b\tau$_int($\tau_1\tau_2$ $b\tau$-cl(X))= $\tau_1\tau_2$ $b\tau$-int(X)=X.Since A is non $\tau_1\tau_2$ $b\tau$-dense in X, $\tau_1\tau_2$ $b\tau$-int($\tau_1\tau_2$ $b\tau$-cl(A)) = $\emptyset$. Thus $\tau_1\tau_2$ $b\tau$-int($\tau_1\tau_2$ $b\tau$-cl(X))=X ,And $\tau_1\tau_2$ $b\tau$-int($\tau_1\tau_2$ $b\tau$-cl(A))= $\emptyset$. It follows A $\neq$ X ■

Theorem (2-5): The union of finite number of non $\tau_1\tau_2$ $b\tau$-dense set is non $\tau_1\tau_2$ $b\tau$-dense sets.

Proof **:** it suffices to prove that the theorem for the case oftwo non sp-dense sets ,say A and B For simplicity we put G= $\tau_1\tau_2$ $b\tau$-int($\tau_1\tau_2$ $b\tau$-cl(A $\cup$B)) So that G $\subset$ $\tau_1\tau_2$ $b\tau$-cl(A $\cup$ B) =- $\tau_1\tau_2$ $b\tau$_cl(A) $\cup$ $\tau_1\tau_2$ $b\tau$-cl(B). It follows that G ∩ [$\tau_1\tau_2$ $b\tau$-cl(B)]$'$ $\subset$ ($\tau_1\tau_2$ $b\tau$-cl(A) $\cup$ $\tau_1\tau_2$ $b\tau$-cl(B)) ∩ [$\tau_1\tau_2$ $b\tau$-cl(B)]$'$ = [$\tau_1\tau_2$ $b\tau$-cl(A) ∩ ($\tau_1\tau_2$ $b\tau$ cl(B))$'$] $\cup$ [$\tau_1\tau_2$ $b\tau$-cl(B) ∩($\tau_1\tau_2$ $b\tau$-cl(B))$'$] = $\tau_1\tau_2$ $b\tau$-cl(A) ∩ ($\tau_1\tau_2$ $b\tau$-cl(B))$'$ Since [$\tau_1\tau_2$ $b\tau$-cl(B) ∩ ($\tau_1\tau_2$ $b\tau$-cl(B))$'$ =$\emptyset$] $\subset$ $\tau_1\tau_2$ $b\tau$-cl(A). Then $\tau_1\tau_2$ $b\tau$-int(G ∩ (($\tau_1\tau_2$ $b\tau$-cl(B))$'$) $\subset$ $\tau_1\tau_2$ $b\tau$-int($\tau_1\tau_2$ $b\tau$-cl(A)) = $\emptyset$, since A is non $\tau_1\tau_2$ $b\tau$-dense. But $\tau_1\tau_2$ $b\tau$-int [G ∩ ($\tau_1\tau_2$ $b\tau$-cl(B))]$'$ = G ∩ $\tau_1\tau_2$ $b\tau$-cl(B)$'$, Since G ∩ ($\tau_1\tau_2$ $b\tau$-cl(B))$'$ is an sp-open set ,It follows that G ∩ ($\tau_1\tau_2$ $b\tau$-cl(B))$'$ = $\emptyset$, which implies G $\subset$ $\tau_1\tau_2$ $b\tau$-cl(B) then $\tau_1\tau_2$ $b\tau$-int (G) $\subset$ $\tau_1\tau_2$ $b\tau$-int ($\tau_1\tau_2$ $b\tau$-cl(B)) = $\emptyset$,Sinse [B is non $\tau_1\tau_2$ $b\tau$-dense] . But $\tau_1\tau_2$ $b\tau$-int(G) = $\tau_1\tau_2$ $b\tau$-int[$\tau_1\tau_2$ $b\tau$-int($\tau_1\tau_2$ $b\tau$-cl (A $\cup$ B))] = $\tau_1\tau_2$ $b\tau$-int($\tau_1\tau_2$ $b\tau$-cl(A $\cup$ B)). So that $\tau_1\tau_2$ $b\tau$-int ($\tau_1\tau_2$ $b\tau$-cl(A $\cup$ B)) = $\emptyset$. Hense A $\cup$ B is non $\tau_1\tau_2$ $b\tau$-dense ■

Theorem(2-6) : Let A be a subset of abitopological spaces (X,τ_1,τ_2),Then A is non $\tau_1\tau_2$ $b\tau$-dense in X if and only if X- $\tau_1\tau_2$ $b\tau$-cl(A) is $\tau_1\tau_2$ $b\tau$-dense in X.

Proof: By theorem (1-21) $\tau_1\tau_2$ $b\tau$-int(A) = X – $\tau_1\tau_2$ $b\tau$-cl(X-A) and $\tau_1\tau_2$ $b\tau$-cl(A) = X – $\tau_1\tau_2$ $b\tau$-int (X-A) it follows that $\tau_1\tau_2$ $b\tau$-int($\tau_1\tau_2$ $b\tau$-cl(A))= X- $\tau_1\tau_2$ $b\tau$-cl(X- $\tau_1\tau_2$ $b\tau$-cl(A)). Sinse A is non $\tau_1\tau_2$ $b\tau$-dense then $\tau_1\tau_2$ $b\tau$_int($\tau_1\tau_2$ $b\tau$-cl(A))=$\emptyset$ then X- $\tau_1\tau_2$ $b\tau$-cl(X- $\tau_1\tau_2$ $b\tau$-cl(A)) = $\emptyset$ then $\tau_1\tau_2$ $b\tau$-cl(X- $\tau_1\tau_2$ $b\tau$-cl(A)) = X then $\tau_1\tau_2$ $b\tau$-cl(X- $\tau_1\tau_2$ $b\tau$-cl(A)) is $\tau_1\tau_2$ $b\tau$-dense■

Definition (2-7) : A subset A of a bitopological spaces (X, τ_1 , τ_2) is called **-$\tau_1\tau_2$ $b\tau$-dense- in-itself** if A $\subseteq \tau_1\tau_2$ $b\tau$- D(A) that is every points of A is $\tau_1\tau_2$ $b\tau$-limit point of A .

Example (2-8): let X={a,b,c,d,e} , with τ_1={ Ø,X,{b},{d,e},{b,d,e},{a,c,d,e}}, τ_2={ Ø,X,{b}} are a topology on X .consider the subset A={a,c},then a is $\tau_1\tau_2$ $b\tau$- limit point of A since the α-nhds of a are {a,c,d,e} and X each of which contains a point of A other than a , also c is $\tau_1\tau_2$ $b\tau$-limit point of A since the α-nhds of c are {a,c,d,e} and X each of which contains a point of A other than c .hence A is $\tau_1\tau_2$ $b\tau$-dense- in-itself .

Theorem (2-9): If A is $\tau_1\tau_2$ $b\tau$-dense -in -itself set then $\tau_1\tau_2$ $b\tau$-cl(A) is $\tau_1\tau_2$ $b\tau$-dense-in-itself.

Proof :By theorem (let A be a subset of a topological space bitopological spaces (X, τ_1 , τ_2)then $\tau_1\tau_2$ $b\tau$-cl (A) = A $\cup \tau_1\tau_2$ $b\tau$-D(A) [2],[3]) since A is $\tau_1\tau_2$ $b\tau$-dense - in -itself that is (every point of A is $\tau_1\tau_2$ $b\tau$- limit point of A) then A $\cup \tau_1\tau_2$ $b\tau$- D(A) = A hence $\tau_1\tau_2$ $b\tau$-cl(A) = A then $\tau_1\tau_2$ $b\tau$-cl(A) is $\tau_1\tau_2$ $b\tau$-dense-in-itself■

Theorem (2-10) : The union of any family of $\tau_1\tau_2$ $b\tau$--dense-in-itself sets is $\tau_1\tau_2$ $b\tau$--dense -in-itself .

Proof : let {A_i}, i∈I, be a family of $\tau_1\tau_2$ $b\tau$--dense-in-itself sets . so $A_i \subseteq \tau_1\tau_2$ $b\tau$-D(A_i) ∀ i∈I, Let p∈∪A_i then p∈A_i .for some i∈I. Hence for each $\tau_1\tau_2$ $b\tau$--pen set U with p∈U, A_i∩(U-{p}) ≠ Ø. Thus (∪A_i) ∩(U-{p}) ≠Ø , hence p ∈ ($\tau_1\tau_2$ $b\tau$-D(UA_i)) therefore ∪A_i⊆($\tau_1\tau_2$ $b\tau$-D(∪A_i)) ; hence ∪A_i is $\tau_1\tau_2$ $b\tau$--denes-in-itsef■

References:

[1] D. Andrijevic, *"On b -open sets"*, Mat. Vesnik, Vol. 48, 1996, 59-64.

[2]G. B. Navalagi"Definition Bank In General Topology", Topology Atlas Survey Articles Section URL: http:at.yorku.ca/t/a/i/c/32htm ,2000.

[3]Jeli's , M., " T_i-pairwise continuous Functions and Bitopological Separation Axioms" , Mat. Vesnik 41, 3(1989) , 155-159 .

[4] J. N. Sharma and A.R. Vasishtha ," topology", meerut ,1976.

[5] Kelly, J. C. "Bitopological spaces", proc. London Math. Soc 13 (1963), 71-89.

[6] Maximilian Ganster, Markus Steiner, *"On $b\tau$-closed sets"*, Applied General Topology, Vol. 8, No. 2, (2007), 243–247.

[7]O. Ravi and 2M. LellisThivagar "A Bitopological $(1,2)^*$ Semi-generalised Continuous Maps"Bull. Malays. Math. Sci. Soc. (2) 29(1) (2006), 79–88

[8] O. Ravi and M. L. Thivagar, On Stronger forms of $(1, 2)*$ quotient mappings in bitopologicalspaces, Internat. J. Math. Game Theory and Algebra, to appear.