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Jordan Higher Bi- Homomorphism and Co- Jordan Higher Bi- Homomorphism on Banach Algebra

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Abstract

The concepts of higher Bi- homomorphism and Jordan higher Bi- homomorphism have been introduced and studied the relation between Jordan and ordinary higher Bi- homomorphism also the concepts of Co- higher Bi- homomorphism and Co- Jordan higher Bi- homomorphism introduced and the relation between them in Banach algebra have also been studied.

Keywords: Banach algebra, Co-Homomorphism ,Homomorphism, Jordan Homomorphism, Jordan Map.

Introduction:

Let H and G are complexes Banach algebras and let $\xi : H \to G$ be an additive map. Then $\xi$is said n-homomorphism if for every $a_1, a_2, .. a_n \epsilon H$

$$\xi(a_1, a_2, \dots, a_n) = \xi(a_1)\xi(a_2)\dots\xi(a_n) \quad (1)$$

Hejazian Sh. and Mirzavaziri M in (1) studied the concept of n-homomorphism on complex algebra. 2- homomorphism being homomorphism, on this sense .We indicate to Bracic J., and Moslehian M.S in (2) on 3- homomorphism on certain attributes . Zelazko W in (3) introduced the definition of an m-Jordan homomorphism. An additive map ξ between A and B which are Banach algebras is said to be m-Jordan homomorphism(for short m-JH) if

$$\xi(x^m) = (\xi(x))^m, \quad \text{for all } x\epsilon A \quad (2)$$

A 2-Jordan homomorphism is said to be Jordan homomorphism. Every m- homomorphism is m-Jordan homomorphism. But generally the converse is false and it is true with some conditions. Zelazko W in (4) shows from Banach algebra into commutative semi simple Banach algebra that every Jordan homomorphism is a homomorphism . Zivari A in (5) refers to the reader to characterizations of 3- Jordan Homomorphism. Zelazko W in (3) shows that every m-Jordan homomorphism between commutative Banach algebras is an m-homomorphism for ∈ {2,3,4} . An G. , Ding Y. and Li J in (6)and Liu L in(7)some results about Jordan centralizers and Jordan derivations done.

In this article, let $\mathcal{U} = A \times B$. Then U with multiplication

$$(a,b)(x,y) = (ax,by) \ , \ (a,b),(x,y)\epsilon\mathcal{U} \quad (3)$$

is Banach algebra. It is unital if and only if A and B are unital and it is commutative if and only if A and B are commutative and semi simple if its radical is zero also refer by e to the unital element of A and B. Suppose D is complex Banach algebras. Bi-linear map is a function $\xi: \mathcal{U} \to D$ such that for any $a\epsilon A$ the map $b \to \xi(a,b)$ is a map from $\mathcal{U}$ to D, and for any $b\epsilon\mathcal{U}$ the map $a \longmapsto \xi(a,b)$ is additive from A to D.

Zivari A in(8) defined Jordan (resp,.Bi-homomorphism)as follows

A bi- additive mapping δ is said Bi-homomorphism if for every $(a,b),(x,y)\epsilon A \times B$,
$\delta(ax,by) = \delta(a,b)\delta(x,y)$

And it is called Bi- Jordan homomorphism if $\delta(a^2,b^2) = (\delta(a,b))^2$

Bi- Homomorphism be Jordan Bi- Homomorphism. In general the converse is not true Zivari A in(8) shows that the converse holds under some conditions .In this article ,the concepts of Jordan (higher Bi- homomorphism) and the relation between them have been introduced and studied. An additive map δ between A and B which are Banach algebra is said to Co- homomorphism if $\delta(xy) = -\delta(x)\delta(y)$,$x,y\epsilon$ A.and it is called a Co- Jordan homomorphism $\delta(x^2) = -(\delta(x))^2$ for all $x\epsilon A$, Zivari A in (5) studied the relation between Co-Jordan homomorphism and Co- homomorphism under certain condition. In this article, the concepts of Co- Bi- higher homomorphism and Co- Bi-Jordan higher homomorphism and the relation between them have been introduced and studied .

Jordan Higher Bi- Homomorphism

Definition 2.1:

Let $=\{\delta_i\}_{i\epsilon N}$, N is the natural number, be the family of bi- Linear map $\delta_i = R\times R \to R$, (a,b) , $(x,y)\epsilon U$, $U = R\times R$ is called higher bi-homomorphism $\delta_n(ax,by) = \sum_{i=1}^{n}\delta_i\,(a,b)\delta_i(x,y)$ and is said Jordan higher Bi-homomorphism(JHBH, for short) if
$\delta_n(a^2,b^2) = \sum_{i=1}^{n}(\delta_i\,(a,b))^2$, $a,b\epsilon U$.

Lemma 2.2: Suppose that $\delta: U \to R$ is a Jordan higher Bi-homomorphism. Then

1. $\delta_n(xy+yx,b^2) = \sum_{i=1}^{n}\delta_i(xy,b^2)\delta_i(yx,b^2)$
2. $\delta_n(a^2,xy+yx) = \sum_{i=1}^{n}\delta_i(a,\mathrm{xy})\delta_i(a,yx)$

Proof: it is straight forward

Lemma 2.3: Let δ be a Jordan higher Bi-homomorphism. If U commutative and unital , then

1. $\delta_n(ax,e) = \sum_{i=1}^{n}\delta_i(a,e)\delta_i(x,e)$
2. $\delta_n(e,by) = \sum_{i=1}^{n}\delta_i(e,b)\delta_i(e,y)$.

Proof: From lemma 2.2,if suppose that y=a and use commutativity .

Lemma 2.4: Let U be unital and let δ be a non-trivial and non-empty family of Jordan higher Bi-homomorphism(for short JHBH). Then $\delta(e,e)\neq 0$.

Proof: Since δ is a Jordan higher Bi-homomorphism. Then
$\forall(a,b)\epsilon U$, we get
$\delta(a^2,b^2) = \sum_{i=1}^{n}(\delta_i(a,b))^2$ (4)
Replacing a by $x+e$, by $y+e$ in (4),Gives
$\delta_n(x^2+2x+e,y^2+2y+e) = \sum_{i=1}^{n}(\delta_i(x+e,y+e))^2$ (5)
Suppose that $\delta_n(e,e) = 0$. Then by lemma 2.3
$\delta_n(x,e) = \delta_n(e,y) = 0 \quad \forall(x,y)\epsilon U$
(6)
It follows from (5) and (6) that
$\delta_n(2x,2y) + \delta_n(2x,y^2) + \delta_n(x^2,2y) = 0$
(7)
For all $(x,y)\epsilon U$. By lemma2.3
$$\left.\begin{array}{l}\delta_n(2x,y^2) = \sum_{i=1}^{n}2\delta_i(x,y)\delta_i(e,y) = 0\\ \delta_n(x^2,2y) = \sum_{i=1}^{n}2\delta_i(x,e)\delta_i(x,y) = 0\end{array}\right\}$$
(8)
Then by (7) and (8), we get
$\delta_n(x,y) = 0 \;\; \forall(x,y)\,\epsilon U$, which is contradicted with δ non-trivial.

Theorem 2.5: Let U commutative and unital and δ be a Jordan Higher Bi-homomorphism from U into a semi simple Banach commutative algebra D. Then δ is higher than Bi-homomorphism.

Proof: Let $D = \mathbb{C}$ and let $\delta = \{\delta_i\}_{i\epsilon N}$, $\delta_i: U \to \mathbb{C}$ be a Jordan higher Bi-homomorphism. So for every $(a,b)\epsilon U$, we get
$\delta_n(a^2,b^2) = \sum_{i=1}^{n}(\delta_i(a,b))^2$
(9)
Replacing a by $x+e$, and b by $y+e$ in(9)
$\delta_n(a^2,b^2) = \sum_{i=1}^{n}(\delta_i(a,b))^2$
$\delta_n(x^2+2x+e,y^2+2y+e) = \sum_{i=1}^{n}(\delta_i(x+e,y+e))^2$ (10)
By lemma 2.4, $\delta_n(e,e)\neq 0$, so (10) gives
$\delta_n(e,e) = 1$ (11)
From lemma 2.2, we get
$$\left.\begin{array}{l}\delta_n(2x,y^2) = 2\sum_{i=1}^{n}\delta_i(x,y)\delta_i(e,y)\\ \delta_n(x^2,2y) = 2\sum_{i=1}^{n}\delta_i(x,e)\delta_i(x,y)\end{array}\right\}$$
(12)
Thus by (10), (11) and (12), we get
$\delta_n(x,y) = \sum_{i=1}^{n}\delta_i(x,e)\delta_i(e,y)$ for all $(x,y)\epsilon U$
(13)
Replace x by ax and y by by in (13) gives
$\delta_n(ax,by) = \sum_{i=1}^{n}\delta_i(ax,e)\delta_i(e,by)$ (14)
By (14) and lemma 2.4, we get
$\delta_n(a,b)\delta_n(x,y) = \sum_{i=1}^{n}\delta_i(a,b)\delta_i(x,y) = \sum_{i=1}^{n}[\sum_{i=1}^{n}\delta_i(a,e)\delta_i(x,e)]\,.[\sum_{i=1}^{n}\delta_i(e,b)\delta_i(e,y)] = \sum_{i=1}^{n}\delta_i(ax,e)\delta_i(e,by)$ (15)
By (14) and (15)
$\delta_n(ax,by) = \sum_{i=1}^{n}\delta_i(a\;,b)\delta_i(x,y)$ (16)
For all (a,b), $(x,y)\epsilon U$ and so δ is a Bi- higher homomorphism.

Suppose $\mathcal{D}$ is commutative semi simple , $\mathcal{M}(\mathcal{D})$ be maximal ideal space of $\mathcal{D}$, with each f$\epsilon\,\mathcal{M}(\mathcal{D})$, associate a function $\delta_f: U \to \mathbb{C}$ defined by
$\delta_f(a,b) = f(\delta(a,b))$, $(a,b)\epsilon U$ (17)
Pick f$\epsilon\mathcal{M}(\mathcal{D})$ arbitrary. δ_f is a Jordan Higher Bi-homomorphism so it is a higher Bi-homomorphism. So from definition of δ_{nf}, we get
$$f(\delta_n(ax,by)) = f(\sum_{i=1}^{n}\delta_i(a,b)\delta_i(x,y)$$
$$= f(\sum_{i=1}^{n}\delta_i(a,b))f(\sum_{i=1}^{n}\delta_i(x,y))$$
$$= f(\delta_n(a,b))f(\delta_n(x,y)\,)$$
Since $f\epsilon\mathcal{M}(\mathcal{D})$ was arbitrary and D assuming to be semi simple, we obtain
$\delta_n(ax,by) = \sum_{i=1}^{n}\delta_i(a,b)\delta_i(x,y)$, $\forall(x,y),(a,b)\in U$

Co- Jordan Higher bi- Homomorphism

Definition 3.1: A family $\delta = \{\delta_i\}$, $\delta_i: A\times A \to B$ where A and B are Banach algebra is called a Co Bi-higher homomorphism if $\delta_n(ab,cd) = -\sum_{i=1}^{n}\delta_i(a,c)\,\delta_i(b,d)$
And is called a Co- Jordan higher Bi-homomorphism if $a=b, c=d$.

Theorem 3.2: Suppose that A is a Banach algebra need not to be commutative. Then each Co- Jordan higher Bi-homomorphism a Co- higher Bi-homomorphism**.**

Proof: Suppose that δ is a Co- Jordan higher Bi-homomorphism, So
$\delta_n(a^2,c^2) = -\sum_{i=1}^{n}\delta_i(a,c)\,\delta_i(a,c) \;\; \forall a,c\epsilon A$,
replaced a by $a+b$ gives $\delta_n(ab+ba,c^2) = -2\sum_{i=1}^{n}\delta_i(a,c^2)\,\delta_i(b,c^2)\;\;\forall\, a,b,c\epsilon A$ (18)
then by (18),

$2\delta_n(aba, c^3) = \delta_n((ab + ba)a + a(ab + ba), c^3) - \delta_n(a^2b + ba^2, c^3)$
$= -2(\sum_{i=1}^{n} \delta(_i(ab + ba, c^2)\delta_i(a, c) - \delta_i(a^2, c^2)\delta_i(b, c))$
$=$
$-2[-2\sum_{i=1}^{n} \delta_i (a, c)\delta_i (b, c) + \delta_i (a, c)\delta_i (a, c)]\delta_i (b, c))$
$2\sum_{i=1}^{n} \delta_i(a, c)\delta_i(a, c)\delta_i(b, c)$, therefore
$\delta_n(aba, c^3) = \sum_{i=1}^{n} \delta_i(a, c)\delta_i(a, c)\delta_i(b, c) \ \ \forall a, b, c\epsilon A$ (19)
Let a and $b\epsilon A$ and put
$2t = \delta_n(ab - ba, c^2)$ (20)
It is clear from (18) and (20) that
$\delta_n(ab, c^2) - t = -\sum_{i=1}^{n} \delta_i(a, c)\delta_i(b, c)$
$\delta_n(ba, c^2) + t = -\sum_{i=1}^{n} \delta_i(a, c)\delta_i(b, c)$ (21)
By (19)-(21)
$4t^2 = (\delta_n(ab - ba, c^2))^2 = \delta_n(((ab)^2 + (ba)^2 - ab^2a - ba^2b), (c^2)^2$
$=$
$\delta_n((ab)^2, c^4) + \delta_n((ba)^2, c^4) - \delta_n(ab^2a, c^4) - \delta_n(ba^2b, c^4)$
$= [t - \sum_{i=1}^{n} \delta_i(a, c)\delta_i(b, c)]^2 + [-t - \sum_{i=1}^{n} \delta_i(a, c)\delta_i(b, c)]^2 - [2\sum_{i=1}^{n}(\delta_i(a, c))^2(\delta_i(b, c))^2$
$= 2t^2$
Then $t = 0$ which proves that $\delta_n(ab, c^2) = \delta_n(ba, c^2)$
Therefore, by (18), $\delta_n(ab, c^2) = -\sum_{i=1}^{n} \delta_i(a, c)\delta_i(b, c)$.

Conclusion:

In this article, Jordan higher Bi- homomorphism on Banach algebra is higher than Bi-homomorphism also Jordan Co- higher Bi-homomorphism on Banach algebra is Co- higher Bi- homomorphism under appropriate conditions.

Author's declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Al-Qadisiyah.

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تشاكل جوردان الثنائي وتشاكل جوردان الثنائي العكسي على جبر بناخ

رجاء جفات شاهين

قسم الرياضيات، كلية التربية، جامعة القادسية، القادسية، العراق.

الخلاصة:

قدمت مفاهيم التشاكل الثنائي من الرتب العليا وتشاكل جوردان الثنائي من الرتب العليا ودرست العلاقة بين تشاكل جوردان الثنائي والعادي من التشاكل الثنائي من الرتب العليا وكذلك قدمت مفاهيم تشاكل الثنائي العكسي من الرتب العليا وتشاكل جوردان الثنائي العكسي من الرتب العليا وكذلك درست العلاقة بينهمافي جبر بناخ.

الكلمات المفتاحية: التشاكل، التشاكل العكسي، تطبيق جوردان، تشاكل جوردان، جبر بناخ.