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Approximate Solutions for Systems of Volterra Integro-differential Equations Using Laplace –Adomian Method

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Abstract

Some modified techniques are used in this article in order to have approximate solutions for systems of Volterra integro-differential equations. The suggested techniques are the so called Laplace-Adomian decomposition method and Laplace iterative method. The proposed methods are robust and accurate as can be seen from the given illustrative examples and from the comparison that are made with the exact solution.

Keywords:Volterra integro-differential equations; Laplace transform; Adomian decomposition method.

حلول تقريبية لأنظمة معادلات فولتيرا التكاملية التفاضلية باستخدام طريقة لابلاس– ادوميان

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الخلاصة

يتم استخدام تقنيات معدلة في هذا البحث من أجل الحصول على حلول تقريبية لأنظمة المعادلات التفاضلية التكاملية لفولتيرا. تُسمى التقنيات المقترحة طريقة تجزئة لابلاس أدوميان وطريقة لابلاس التكرارية. الأساليب المقترحة قوية ودقيقة والتي يمكن رؤيتها من خلال الأمثلة التوضيحية المقدمة ومن المقارنة التي تم إجراؤها مع الحل الدقيق.

Introduction

Integro-differential equations emerge in different branches of physics and engineering, such as fluid mechanics, thin films, diffusion processes and so on [1-3]. It has encouraged many authors to have numerical and approximate solutions [4-6]. The Laplace-Adomian decomposition method (LADM) combines between two powerful methods, which are the Laplace transform and the Adomian decomposition, and was introduced for the first time by Khuri [7, 8]. Previously presented methods handled a wide class of non-linear problems and showed a great success in terms of obtaining the approximate results. Agadjanov [9] solved the Duffing equation by using LADM, while Hossein zadeh *et al*. applied LADM for solving Klein–Gordon equation [10]. Khan *et al*. employed LADM to obtain a solution for nonlinear coupled partial differential equations [11]. Jafari*et al*. solved non-linear fractional diffusion–wave equations by using LADM [12]. Manafianheris [13] applied the modified LADM for solving integro-differential equations. In this work we employ two techniques to obtain an

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approximate solution for systems of integro-differential equations, which are LADM and Laplace Iterative Method (LIM). This article is ordered as follows: The description of LADM is illustrated in section 1, in section 2 Laplace Iterative Method is given, in section 3 some illustrative examples are given, and at last the conclusions are presented in section 4.

1. Laplace-Adomian Decomposition Method (LADM)

Consider the system of Volterra's integro- differential equations:

$$y_i^{(n)} = f_i + \int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)\, dt, \; i = 1,2,\ldots,m \qquad (1)$$
$$y_i^{(j)}(0) = c_{ij},$$

where c_{ij} , j=0,1,2,….,m-1 are given constants.

The technique consists at first of implementing Laplace transformation to equation (1), hence

$$\mathrm{L}(y_i^{(n)} = f_i + \int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)dt), \; i = 1,2,\ldots,m \qquad (2)$$

Equation (2) can be simplified as :

$$\mathrm{L}(y_i^{(n)}) = F_i(s) + \mathrm{L}(\int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)dt), \; i = 1,2,\ldots,m$$

According to the properties of the Laplace transform we get :

$$s^n Y_i(s) - s^{n-1}y_i(0) - s^{n-2}y_i'(0)\ldots - y_i^{(n-1)}(0) =$$
$$F_i(s) + \mathrm{L}(\int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)dt), \; i = 1,2,\ldots,m \qquad (3)$$

Equation (3) can be written as :

$$Y_i(s) = \frac{1}{s^n}(s^{n-1}y_i(0) - s^{n-2}y_i'(0)\ldots - y_i^{(n-1)}(0)) +$$
$$\frac{1}{s^n}(F_i(s)) + \frac{1}{s^n}(\mathrm{L}(\int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)dt)), \; i = 1,2,\ldots,m \qquad (4)$$

By performing the inverse Laplace transform to equation (4) we get,

$$y_i(t) = \mathrm{L}^{-1}(\frac{1}{s^n}(s^{n-1}y_i(0) - s^{n-2}y_i'(0)\ldots - y_i^{(n-1)}(0))) +$$
$$\mathrm{L}^{-1}(\frac{1}{s^n}(F_i(s))) + \mathrm{L}^{-1}(\frac{1}{s^n}(\mathrm{L}(\int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)dt))), \; i = 1,2,\ldots,m \qquad (5)$$

According to Adomian decomposition method, we have,

$$\sum_{n=0}^{\infty} y_{in}(t) = \mathrm{L}^{-1}(\frac{1}{s^n}(s^{n-1}y_i(0) - s^{n-2}y_i'(0)\ldots - y_i^{(n-1)}(0))) +$$
$$\mathrm{L}^{-1}(\frac{1}{s^n}(F_i(s))) + \mathrm{L}^{-1}(\frac{1}{s^n}(\mathrm{L}(\int_0^x k_i(x,t)F(\sum_{n=0}^{\infty} y_{1n}, \sum_{n=0}^{\infty} y_{2n}, \ldots, \sum_{n=0}^{\infty} y_{mn})dt)), \; i = 1,2,\ldots,m \qquad (6)$$

This implies that

$$\left\{ \begin{aligned} y_{i0} &= \mathrm{L}^{-1}(\frac{1}{s^n}(s^{n-1}y_i(0) - s^{n-2}y_i'(0)\ldots - y_i^{(n-1)}(0))) + \mathrm{L}^{-1}(\frac{1}{s^n}(F_i(s))) \\ y_{im} &= \mathrm{L}^{-1}(\frac{1}{s^n}(\mathrm{L}(\int_0^x k_i(x,t)F(\sum_{n=0}^{\infty} y_{1n}, \sum_{n=0}^{\infty} y_{2n}, \ldots, \sum_{n=0}^{\infty} y_{mn})dt)), \; i = 1,2,\ldots,m \end{aligned} \right\} \qquad (7)$$

2. Laplace Iterative Method (LIM)

Consider the system of Volterra's integro- differential equations:

$$y_i^{(n)} = f_i + \int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)\, dt, \ i = 1,2,...,m$$
$$y_i^{(j)}(0) = c_{ij}, \tag{8}$$

where c_{ij} , j=0,1,2,….,m-1 are given constants.

The technique consists at first of implementing Laplace transformation to equation (8), hence

$$\text{L}(y_i^{(n)} = f_i + \int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)dt), \ i = 1,2,...,m \tag{9}$$

Equation (9) can be simplified as :

$$\text{L}(y_i^{(n)}) = F_i(s) + \text{L}(\int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)dt), \ i = 1,2,...,m$$

According to the properties of the Laplace transform we get :

$$s^n Y_i(s) - s^{n-1}y_i(0) - s^{n-2}y_i'(0)\ldots - y_i^{(n-1)}(0) =$$
$$F_i(s) + \text{L}(\int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)dt), \ i = 1,2,...,m \tag{10}$$

Equation (10) can be written as :

$$Y_i(s) = \frac{1}{s^n}(s^{n-1}y_i(0) - s^{n-2}y_i'(0)\ldots - y_i^{(n-1)}(0)) +$$
$$\frac{1}{s^n}(F_i(s)) + \frac{1}{s^n}(\text{L}(\int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)dt)), \ i = 1,2,...,m \tag{11}$$

By performing the inverse Laplace transform to equation (11) we get,

$$y_i(t) = \text{L}^{-1}(\frac{1}{s^n}(s^{n-1}y_i(0) - s^{n-2}y_i'(0)\ldots - y_i^{(n-1)}(0))) +$$
$$\text{L}^{-1}(\frac{1}{s^n}(F_i(s))) + \text{L}^{-1}(\frac{1}{s^n}(\text{L}(\int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)dt))), \ i = 1,2,...,m \tag{12}$$
$$= f_i + A_i(y_1(t), y_2(t), \ldots, y_n(t))$$

where

$$f_i = \text{L}^{-1}(\frac{1}{s^n}(s^{n-1}y_i(0) - s^{n-2}y_i'(0)\ldots - y_i^{(n-1)}(0))) + \text{L}^{-1}(\frac{1}{s^n}(F_i(s))) \tag{13}$$

and

$$A_i(y_1(t), y_2(t), \ldots, y_n(t)) = \text{L}^{-1}(\frac{1}{s^n}(\text{L}(\int_0^x k_i(x,t)F(y_1, y_2, \ldots, y_m)dt))), \ i = 1,2,...,m$$

We are looking for the solution,

$$y_i(t) = \sum_{j=0}^{\infty} y_{ij}(t), \ i = 1,2,\ldots,n \tag{14}$$

The nonlinear operators A_i can be decomposed according to Daftardar-Gejji [14],

$$A_i(y_1(t), y_2(t), \ldots, y_n(t)) = A_i(y_{10}(t), y_{20}(t), \ldots, y_{n0}(t)) +$$
$$\sum_{j=0}^{\infty}\left[A_i(\sum_{k=0}^{j} y_{1k}(t), \ldots, \sum_{k=0}^{j} y_{nk}(t)) - A_i(\sum_{k=0}^{j-1} y_{1k}(t), \ldots, \sum_{k=0}^{j-1} y_{nk}(t))\right] \tag{15}$$

According to equation (14) and equation (15), then equation (12) is equivalent to

$$\sum_{j=0}^{\infty} y_{ij}(t) = f_i + A_i(y_{10}(t), y_{20}(t), \ldots, y_{n0}(t)) +$$

$$\sum_{j=0}^{\infty}\left[A_i(\sum_{k=0}^{j} y_{1k}(t), \ldots, \sum_{k=0}^{j} y_{nk}(t)) - A_i(\sum_{k=0}^{j-1} y_{1k}(t), \ldots, \sum_{k=0}^{j-1} y_{nk}(t))\right]$$

Hence we have,

$$y_{i0} = f_i \quad , i = 1, 2, \ldots, m$$

$$y_{i1} = A_i(y_{10}(t), y_{20}(t), \ldots, y_{n0}(t)) = \mathrm{L}^{-1}(\frac{1}{s^n}(\mathrm{L}(\int_0^x k_i(x,t)F(y_{10}(t), y_{20}(t), \ldots, y_{n0}(t))dt))),\ i = 1, 2, \ldots, m$$

$$y_{i(j+1)} = \mathrm{L}^{-1}(\frac{1}{s^n}(\mathrm{L}(\int_0^x k_i(x,t)F(\sum_{k=0}^{j} y_{1k}(t), \ldots, \sum_{k=0}^{j} y_{nk}(t)))dt)))$$

$$-\mathrm{L}^{-1}(\frac{1}{s^n}(\mathrm{L}(\int_0^x k_i(x,t)F(\sum_{k=0}^{j-1} y_{1k}(t), \ldots, \sum_{k=0}^{j-1} y_{nk}(t)))dt)))\ ,\ i = 1, 2, \ldots, m$$

For the uniqueness and convergence, see refer to a previous work [14].

3. Illustrative Examples

In this section, two non-linear examples will be given in order to demonstrate the applicability and accuracy of the suggested methods.

Example 1: consider the following equations

$$\left.\begin{aligned} y_1'' &= 1 - 2\cos t + \sin t + t^2 - \int_0^x (y_1 + y_2)dt \\ y_2'' &= 1 - 2\sin t - \cos t - \int_0^x (y_2 - y_1)dt \end{aligned}\right\} \quad (16)$$

with respect to $y_1(0) = 1,\ y_2(0) = 0,\ y_1'(0) = 1,\ y_2'(0) = 2$ (17)

The exact solution is given by an earlier report [15] as $y_1(t) = t + \cos t, y_2(t) = t + \sin t$

$$y_{10}(t) = 2t + 2\cos(t) - \sin(t) + \frac{t^2}{2} + \frac{t^4}{12} - 1$$

$$y_{11}(t) = 3\sin(t) - \cos(t) - 3t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{12} - \frac{t^5}{6} - \frac{t^7}{2520} + 1$$

$$y_{12}(t) = 4t - 2\cos(t) - 4\sin(t) - t^2 - \frac{2t^3}{3} + \frac{t^4}{12} + \frac{t^5}{30} - \frac{t^6}{360} + \frac{t^8}{20160} + 2$$

$$y_{20}(t) = \frac{t^2}{2} + 2\sin(t) + \cos(t) - 1$$

$$y_{21}(t) = t - 3\cos(t) - \sin(t) - \frac{3t^2}{2} + \frac{t^4}{12} + \frac{t^7}{2520} + 3$$

$$y_{22}(t) = 2t + 4\cos(t) - 2\sin(t) + 2t^2 - \frac{t^3}{3} - \frac{t^4}{6} + \frac{t^5}{60} + \frac{t^6}{360} - \frac{t^7}{1260} - \frac{t^8}{20160} - \frac{t^{10}}{907200} - 4$$

$$y_1(t) = y_{10}(t) + y_{11}(t) + y_{12}(t)$$

$$y_2(t) = y_{20}(t) + y_{21}(t) + y_{22}(t)$$

Table 1-demonstrates a numerical comparison between the approximate solution of problems (16)-(17) using LADM and LIM with the exact solutions.

i	LADM $y_1(t)$	LIM $y_1(t)$	LADM $y_2(t)$	LIM $y_2(t)$	Exact $y_1(t)$	Exact $y_2(t)$
0	1	1	0	0	1	0
0.1	1.095	1.095	0.2	0.2	1.095	0.2
0.2	1.18	1.18	0.399	0.399	1.18	0.399
0.3	1.255	1.255	0.596	0.596	1.255	0.596
0.4	1.321	1.321	0.789	0.789	1.321	0.789
0.5	1.378	1.378	0.979	0.979	1.378	0.979
0.6	1.425	1.426	1.165	1.165	1.425	1.165
0.7	1.465	1.465	1.344	1.344	1.465	1.344
0.8	1.497	1.498	1.517	1.517	1.497	1.517
0.9	1.522	1.525	1.683	1.683	1.522	1.683

Table -1 Numerical comparison between the approximate solutions of problems (16)-(17) using LADM with LIM and the exact solutions.

Figures-(1) and (2) represent a numerical comparison between the Approximate of $y_1$and y_2 of problems (16) – (17) using LADM and LIM with the exact solutions.

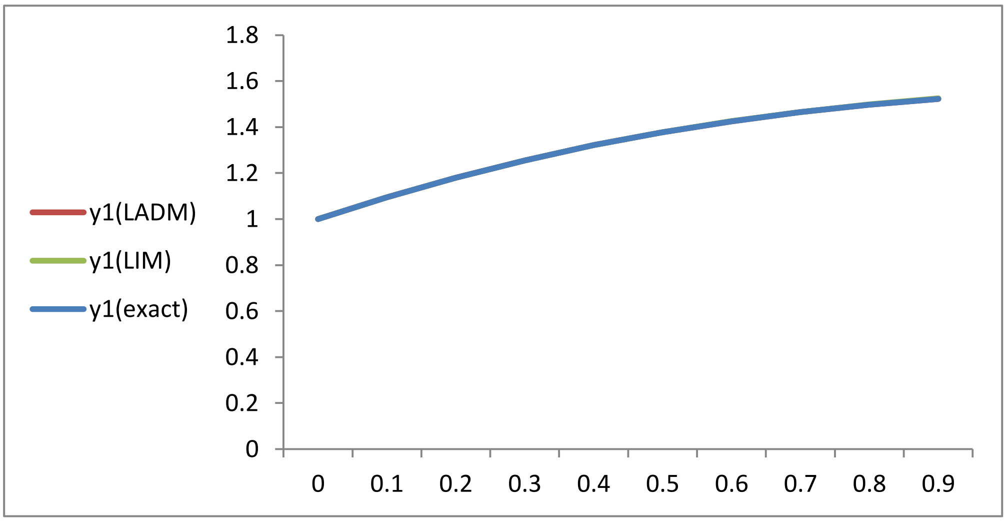

Figure 1-Numerical comparison between the approximate solutions of y_1 using LADM and LIM with the exact solutions.

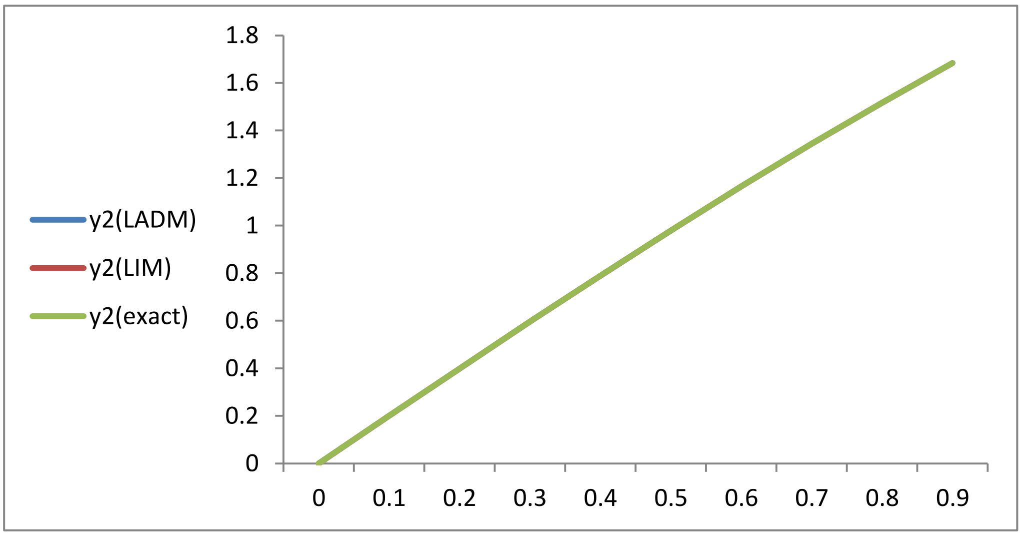

Figure 2-Numerical comparison between the approximate solutions of y_2 using LADM and (LIM with the exact solutions.

Example 2: Consider the nonlinear equations

$$\left.\begin{aligned} y_1' &= 3t^2\frac{t^3(3t^4+7)}{21}-\int_0^x (y_1^2(t)+y_2^2(t))dt \\ y_2' &= 1-\frac{t^2(t^2-2)}{4}-\int_0^x (y_2(t)-y_1(t))dt \end{aligned}\right\} \tag{18}$$

subject to $y_1(0)=0,\ y_2(0)=0$ (19)

The exact solution is given by $y_1(t)=t^3,\ y_2(t)=t$

$$y_{10}(t)=\frac{t^8}{56}+\frac{t^4}{12}+t^3$$

$$y_{11}(t)=\frac{7t^{10}}{64800}-\frac{t^{14}}{61152}-\frac{t^{13}}{4368}-\frac{t^{12}}{52800}-\frac{t^{18}}{959616}-\frac{t^9}{432}-\frac{167t^8}{10080}-\frac{t^6}{90}-\frac{t^4}{12}$$

$$y_{12}(t)=\frac{t^{18}}{959616}+\frac{t^{14}}{61152}+\frac{t^{13}}{4368}+\frac{t^{12}}{52800}+\frac{7t^{10}}{64800}+\frac{t^9}{432}+\frac{167t^8}{10080}+\frac{t^6}{90}+\frac{t^4}{12}$$

$$y_{20}(t)=\frac{t^3}{6}-\frac{t^5}{20}+t$$

$$y_{21}(t)=\frac{t^{10}}{5040}+\frac{t^7}{840}+\frac{t^6}{360}+\frac{t^5}{24}-\frac{t^3}{6}$$

$$y_{22}(t)=\frac{t^3}{6}-\frac{t^7}{840}-\frac{t^6}{360}-\frac{t^5}{24}-\frac{t^{10}}{5040}$$

$$y_1(t)=y_{10}(t)+y_{11}(t)+y_{12}(t)$$

$$y_2(t)=y_{20}(t)+y_{21}(t)+y_{22}(t)$$

Table 2-provides a numerical comparison between the approximate solution of problems (18)-(19) using LADM and LIM with the exact solutions.

i	LADM $y_1(t)$	LIM $y_1(t)$	LADM $y_2(t)$	LIM $y_2(t)$	Exact $y_1(t)$	Exact $y_2(t)$
0	0	0	0	0	0	0
0.1	$1.008\text{x}10^{-3}$	$1.008\text{x}10^{-3}$	0.1	0.1	$1\text{x}10^{-3}$	0.1
0.2	$8.134\text{x}10^{-3}$	$8.134\text{x}10^{-3}$	0.2	0.2	$8\text{x}10^{-3}$	0.2
0.3	0.028	0.028	0.3	0.3	0.027	0.3
0.4	0.066	0.066	0.4	0.4	0.064	0.4
0.5	0.13	0.13	0.5	0.5	0.125	0.5
0.6	0.227	0.227	0.6	0.6	0.216	0.6
0.7	0.343	0.343	0.7	0.7	0.343	0.7
0.8	0.512	0.512	0.801	0.801	0.512	0.8
0.9	0.729	0.729	0.901	0.901	0.729	0.9
1	1	1	1.003	1.003	1	1

Table -2 Numerical comparison between the approximate solution of problems (18)-(19) using LADM and LIM with the exact solutions

Figures-(3) and (4) represent numerical comparisons between the approximate solutions of $y_1$and y_2 of problems (18) – (19) using LADM and LIM with the exact solutions.

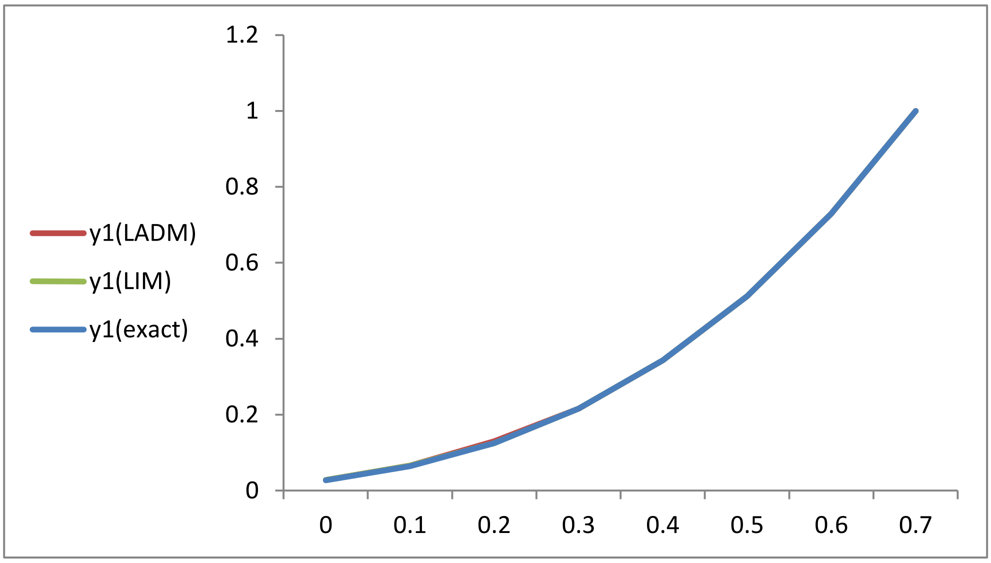

Figure 3-Numerical comparison between the approximate solution of y_1 using LADM and LIM with the exact solutions.

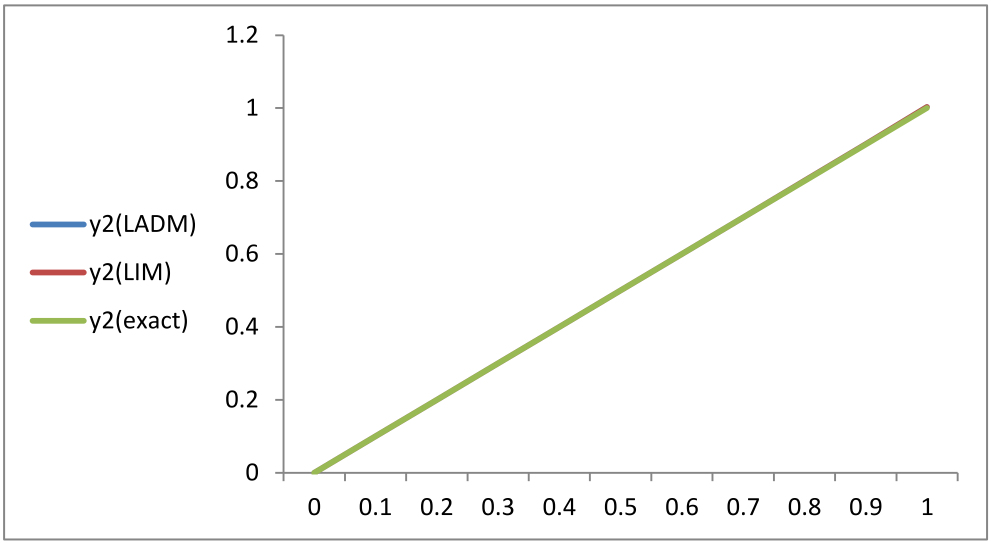

Figure 4-Numerical comparisons between the approximate solution of y_2 using LADM and LIM with the exact solutions.

4. Conclusions

In this study, the Laplace-Adomian decomposition and Laplace Iterative techniques were successfully employed to investigate a solution of systems of Volterra integro-differential equations. From the numerical results, one can observe that these techniques are powerful and acceptable for solving such types of equations .

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