Transverse Shear Effect on Deflection of Tapered Beam under Uniformly Distributed Load

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Abstract

The effect of transverse shear as well as bending on deflection of tapered beams is considered with comparison to that of prismatic beams. It is noticed that the deflection due to transverse shear is usually small. An approximate solution is suggested for deflection of tapered beams by assuming the taper beam as a prismatic beam of uniform cross section equal to the smaller cross-section of the taper beam to simplify the solution.

A simply supported taper beam with circular or square cross section and loaded with uniformly distributed load is considered. The suggested approximate solution gives 6% higher deflection ratio with respect to that by exact integration.

The deflection ratio along a taper beam is presented by curves in logarithmic scales for different taper ratios (1.5 to 10) due to shear and bending. The deflection ratio due to the effect of transverse shear alone is also presented. The logarithmic scale is used for clearer presentation of the results.

Both exact integration solution and approximate solution are used. The suggested approximate solution by using the smaller cross section area of a taper beam as an equivalent uniform area is found to be useful to deal with taper beams of different cross sectional shapes and also for beams under axial forces.

تأثير القص العرضي على العتبات المستدقة تحت أحمال موزعة بانتظام

الخلاصة

تمت دراسة تأثير القص العرضي في زيادة مقدار الهطول (الأود) في العتبات المستدقة إضافة إلى الانحناء ومقارنتها بالعتبات الموشورية المنتظمة. لوحظ أن مقدار ما يضيفه القص من تشوه هو مقدار قليل. تم اقتراح حل تقريبي لحساب التشوهات في العتبات المستدقة لتجنب تكامل المعادلات الرياضية للأود في العتبات متغيرة المقطع وذلك بتحويل العتبة المستدقة إلى عتبة موشورية ذات مقطع منتظم مساوي لأصغر مقطع في العتبة المستدقة. أجريت الدراسة إلى

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عتبات مستدقة ذات مقاطع مربعة ومحملة بأحمال منتظمة ومسندة إسنادا بسيطا في النهايتين حيث تم التوصل إلى أن هذا التبسيط في الحل التقريبي المقترح يعطي قيم هطول قصوى لا تزيد عن ٦% عن القيم الحقيقية التي تم حسابها باستعمال التكاملات المضبوطة.

تم عرض مقدار التشوهات كقيم نسبية على طول العتبة المستدقة بالاعتماد على التشوهات في العتبات الموشورية المقترحة بدأ من نسبة الاستدقاق ١,٥ إلى ١٠ وبمقياس لوغاريتمي ذلك لمقارنة القيم في حالة القيم الحقيقية للهطول بتأثير الانحناء والقص معا وآخر للقص بشكل منفصل ثم عرض المقارنة بين القيم الحقيقية والتقريبية بشكل منحني باستعمال القيم القصوى للهطول. وتم رسم منحني آخر للمقارنة بين الهطول بوجود وعدم وجود تأثير القص.

التبسيط (أو الحل التقريبي) المقترح على تقريب مساحة مقطع العتبة المستدقة إلى مساحة ثابتة مساوية للمساحة الصغرى تعطي نتائج تقريبية مفيدة للعتبات المستدقة ذات أشكال مقاطع متنوعة ويمكن استخدام هذه الفكرة لدراسة سلوك العتبة المستدقة تحت تأثير قوى محورية.

Key words: Bending moment, Deflection, Prismatic beam, Taper beam, Transverse shear.

Introduction

Present days, for economy of materials together with enhancement of analysis technique, the element structural strength demands special shapes for beams for preserving the aesthetic and architectural requirements. These shapes exist in taper members. The uniformly distributed load on simply supported taper members will produce deformations due to flexural (bending) and transverse shear effects.

The objective of this study is to obtain the total deflection by double integration of the second order differential equation then applying the boundary conditions at the member ends. The comparison between these deformations with and without the effect of transverse shear at different tapering ratios (u = 1.5, 2, 3, 4, 5, 6, 7, 8, 9 and 10) and different shape factors (m = 1, 2,2.1,2.2,2.3,2.4,2.5, 2.6, 3 and 4) are presented graphically and numerically.

Deflection Curve

When a beam is loaded, the initially straight longitudinal axis is deformed into a curve, called the deflection curve of the beam, which is produced by combined bending and shear deformations.

The slope of the deflection curve is the first derivative of the deflection function w = w(x). The slope due to bending only is equal to the tangent of the angle of rotation of the cross section (1):

$$dw_b/dx = \tan\theta \qquad (1)$$

where w_b is the deflection due to bending only (no transverse shear deformation).

Equation (1) is based upon geometric considerations and it is applied to a beam of any material.

Most beams undergo only very small rotations when they are loaded; hence, their deflection curves are very flat and have extremely small curvature [1]. Under these conditions, the angle θ is a very small quantity, then some approximations can be made to simplify the work (small-angle bending theory) $\tan\theta \approx \theta$ when θ is a small quantity (in radian), then

$$\theta = \tan\theta = \frac{dw_b}{dx} \qquad (2)$$

Thus with small rotations of a beam, the angle of rotation and the slope are equal. Taking the derivative of θ with respect to x, the following equation is obtained:

$$\frac{d\theta}{dx} = \frac{d^2w_b}{dx^2} \qquad (3)$$

If the material of the beam is linearly elastic following Hooke's law, the curvature is:

$$\frac{d^2w_b}{dx^2} = -\frac{M}{EI} \qquad (4)$$

which is the basic differential equation of the deflection curve of a beam due to bending moment only as shown in Fig. (1). But the slope of the deflection curve of the beam due to shear alone is approximately equal to the shear strain at the neutral axis, as shown in Fig.(2)[1,2].

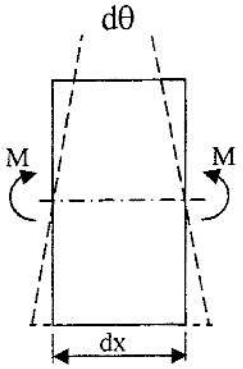
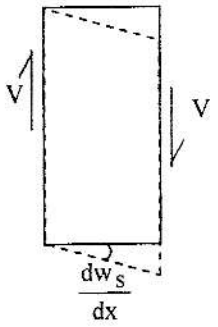
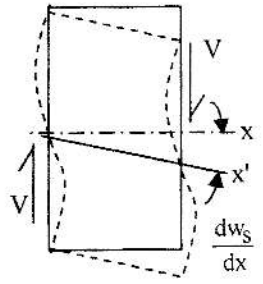


Fig. (1): Cross-section deformation by pure bending moment



a. Constant shear stress



b. Varying shear stress

Fig. (2): Cross-section deformation by pure transverse shear force

Thus denoting by w_s the deflections due to shear alone, the following expression for the slope is obtained:

$$\frac{dw_s}{dx} = \gamma_c = \frac{\alpha_s V}{GA} \qquad (5)$$

in which α_s is a numerical factor as stated in Table (1). This shear shape factor corrects for the assumed uniform shear over the cross-section.

Table (1): Numerical values of shear coefficient [4]

α_s	Shape
1.2	Rectangular
1.1	Circular
A/A_w	I-section

A: total area of the cross-section,
A_w: area of the web

When there is a continuously distributed load q (per unit length) acting on the beam, the shearing force V is a continuous function that may be differentiated with respect to x. The curvature caused by the transverse shear alone is:

$$\frac{d^2 w_s}{dx^2} = \frac{\alpha_s}{GA}\frac{dV}{dx} = -\frac{\alpha_s q}{GA} \quad (6)$$

as $q = -\frac{dV}{dx}$.

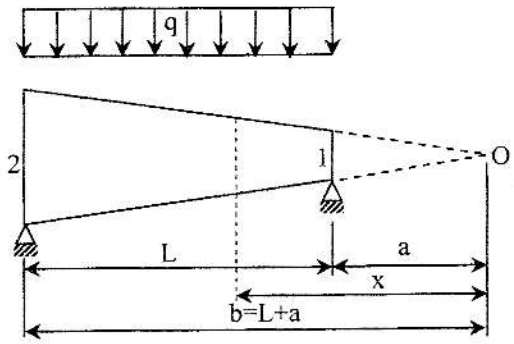


Fig. (3): Simply supported beam with tapering cross-section under uniform load

The total curvature $\frac{d^2w}{dx^2}$ of the prismatic beam due to combined bending and shear effect is the sum of the bending curvature $\frac{d^2w_b}{dx^2}$ and the shear curvature $\frac{d^2w_s}{dx^2}$

$$\frac{d^2w}{dx^2}=\frac{d^2w_b}{dx^2}+\frac{d^2w_s}{dx^2} \quad (7)$$

Then

$$\frac{d^2w}{dx^2}=-\frac{M}{EI}-\frac{\alpha_s q}{GA} \quad (8)$$

The total curvature in a tapered beam will be: -

$$\frac{d^2w}{dx^2}=-\frac{M}{EI(x)}-\frac{\alpha_s q}{GA(x)} \quad (9)$$

as I = I(x) and A = A(x) are functions of position x.

The bending moment of a simply supported beam subjected to a uniformly distributed load q at distance x is: -

$$M(x)=\frac{qL}{2}(x-a)-\frac{q}{2}(x-a)^2 \quad (10)$$

where a is the distance from the nearest support to the origin O, Fig. (3).

The second order differential equation for combined bending and shear curvature for the simply supported beam shown in Fig. (3)becomes:-

$$\frac{d^2w}{dx^2}=-\frac{q}{2EI(x)}(x-a)(b-x)-\frac{\alpha_s q}{GA(x)} \quad (11)$$

as b = L + a, the distance of other support from the origin O.

The variable moment of inertia with respect to x for a tapered beam can be given as:

$$I(x) = I_1\left(\frac{x}{a}\right)^m \qquad (12)$$

where I_1 is the value of the moment of inertia at nearest support to the origin at $x = a$ and m is a shape factor of the tapered beam. The tapering ratio u is the ratio of the larger to smaller ends depth d_2 / d_1 and which is equal to the ratio of the distance of the farthest to the nearest support (b / a) from the origin O.

Also, the variable cross-sectional area with respect to x is:

$$A(x) = A_1\left(\frac{x}{a}\right)^n \qquad (13)$$

Here n is another shape factor of the tapered beam. The shape factors m and n are given by[3]:

$$m = \frac{\log(I_2 / I_1)}{\log u} \qquad (14)$$

$$n = \frac{\log(A_2 / A_1)}{\log u} \qquad (15)$$

Different symmetrical cross-sectional shapes are considered here for beams under uniformly distributed load as shown in Table (2).

The maximum values of deflection and rotation are different depending on the taper ratio of the tapered beam. The determination from the analytical solution of a beam subjected to uniformly distributed load and having square or circular cross-sectional area as shown in Fig. (4) is compared with those of a prismatic beam of uniform cross section equal to the cross section at end 1 of the taper beam in Figs. (5) to (8) for different taper ratios (u). The case u = 1 indicates uniform cross-section.

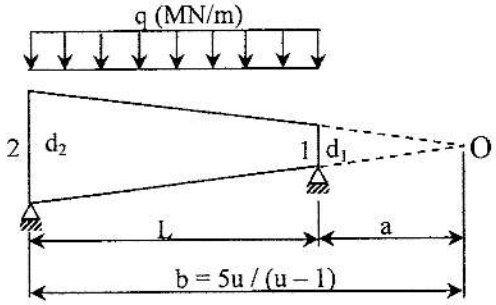
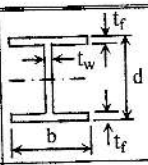
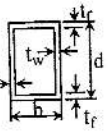
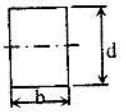
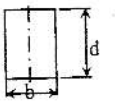
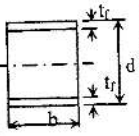
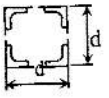
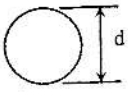
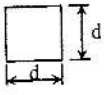


Figure (4): The dimensions and properties of considered beam under uniformly distributed load

Table (2): Curvature Differential equations for different shape factors

Description	Shape	Shape factor[3, 5]		$\frac{d^2w}{dx^2}$
		m	n	
Wide-flange or I-section of varying depth d		2.1-2.6	Variable	$\frac{-qa^m}{2EI_1x^m}(x-a)(b-x)-\frac{\alpha_s qa^n}{GA_1x^n}$
Closed box section of varying depth d		2.1-2.6	Variable	$\frac{-qa^m}{2EI_1x^m}(x-a)(b-x)-\frac{\alpha_s qa^n}{GA_1x^n}$
Solid rectangular section of varying depth d		3	1	$\frac{-qa^3}{2EI_1x^3}(x-a)(b-x)-\frac{\alpha_s qa}{GA_1x}$
Solid, rectangular section of varying width b		1	1	$\frac{-qa}{2EI_1x}(x-a)(b-x)-\frac{\alpha_s qa}{GA_1x}$
Open-web section of varying depth d		2	0	$\frac{-qa^2}{2EI_1x^2}(x-a)(b-x)-\frac{\alpha_s q}{GA_1}$
Tower section of varying depth d		2	0	$\frac{-qa^2}{2EI_1x^2}(x-a)(b-x)-\frac{\alpha_s q}{GA_1}$
Solid circular section of varying diameter d		4	2	$\frac{-qa^4}{2EI_1x^4}(x-a)(b-x)-\frac{\alpha_s qa^2}{GA_1x^2}$
Solid square section of varying dimension d		4	2	$\frac{-qa^4}{2EI_1x^4}(x-a)(b-x)-\frac{\alpha_s qa^2}{GA_1x^2}$

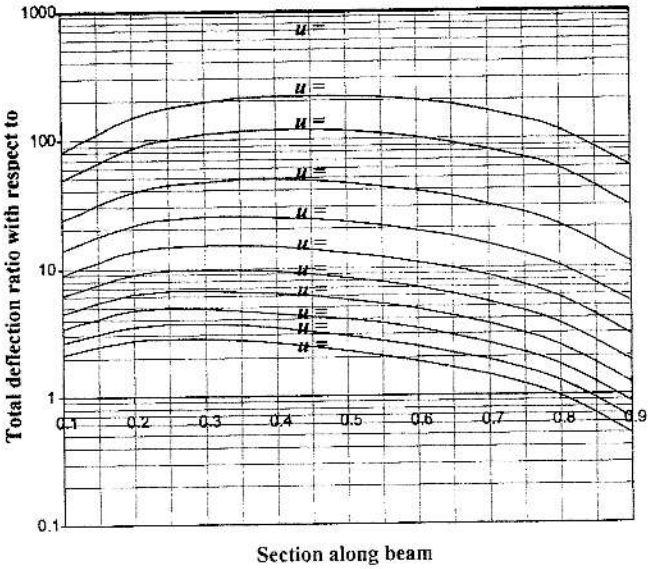


Figure (5): Deflection ratio due to combined bending and shear along beam for different taper ratios

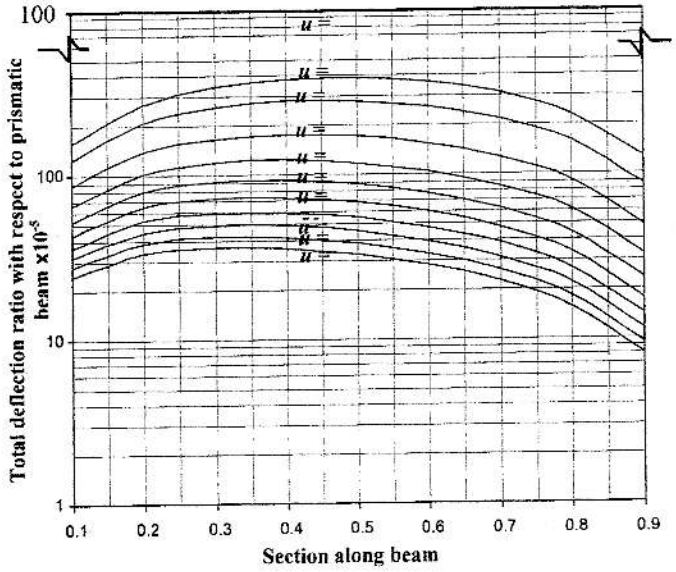


Figure (6): Deflection ratio due to transverse shear along beam length for different taper ratios

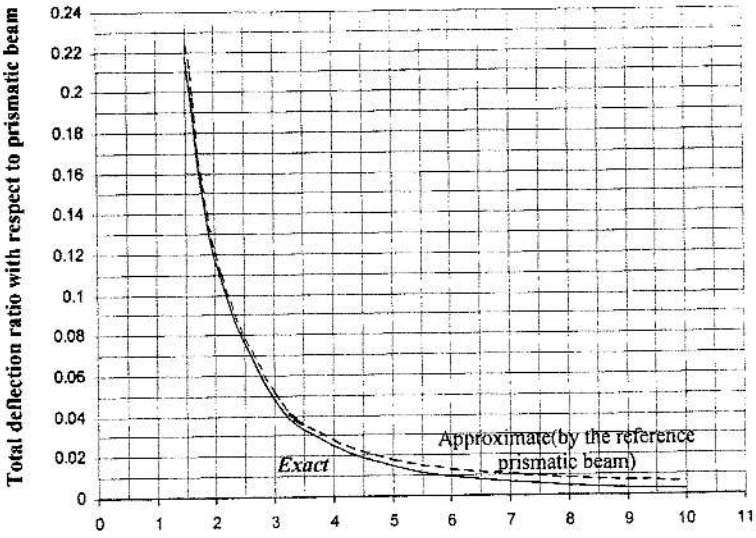


Figure (7): Comparison of exact and approximate maximum deflection ratio for different taper ratios

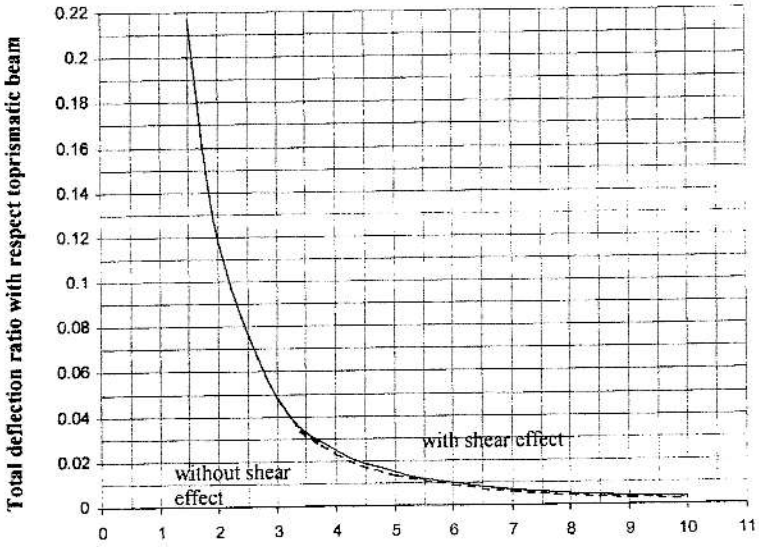


Figure (8): Comparison of maximum deflection ratio by including and excluding transverse shear effect for different taper ratios

Application:

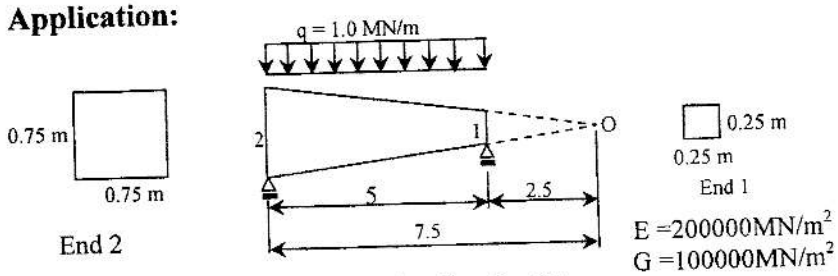


Figure (9): Application (1)

The solution of a tapered beam having a square cross section and a taper ratio 3 ($m = 4$ and $n = 2$) and the dimensions as shown in Fig. (9) is considered.

A prismatic beam having the same properties of the tapered beam shown in Fig. (9) but its cross section is the same as the smaller end section (0.25 m x 0.25 m) and loaded also with (1 MN/m) is compared with the tapered beam for case of maximum deflection due to bending and transverse shear effects.

The central deflection of the prismatic beam is: -

1. due to bending [1]: -

$$w_b = \frac{5qL^4}{384EI} \qquad (16)$$

2. due to transverse shear [1]:

$$w_s = \frac{\alpha_s qL^2}{8GA} \qquad (17)$$

Thus, the total deflection due to combined bending and transverse shear effect is $w = w_b + w_s = 0.1256\,m$.

The general deflection equation of the tapered beam is:

1. due to bending (by exact integration of the taper beam bending equation): -

$$w_b = \frac{qa^4}{4EI}\left[\frac{ab}{6x^2} - \frac{a+b}{2x} - \ln x + \frac{x}{L}\left(\frac{a}{3b} - \frac{u}{3} + \ln u\right) + \frac{b\ln a - a\ln b}{L} - \frac{1}{bL}\left(u\left(\frac{a^2}{2} - \frac{b^2}{3}\right) + \left(\frac{a^2}{3} - \frac{b^2}{2}\right)\right)\right] \qquad (18)$$

2. due to transverse shear (by integration of the taper beam shear deflection equation): -

$$(w_s)_{exact} = \frac{\alpha_s qa^2}{GA_1}\left(\ln x + \frac{a\ln b - b\ln a - x\ln u}{L}\right) \qquad (19)$$

3. due to transverse shear (by an approximate solution by assuming a prismatic beam of uniform cross section equal to the smaller cross section (at end 1) of the taper beam): -

$$(w_s)_{app} = \frac{\alpha_s qL^2}{8GA_1} \qquad (20)$$

From the previous figures at taper ratio u = 3, the following results are obtained:

- From Fig. (5), the maximum deflection ratio due to bending and shear is 0.048367 by exact integration.
- From Fig. (6), the maximum deflection ratio due to transverse shear effect only is 0.001772 by exact integration.
- From Fig. (7), the maximum deflection ratio due to combined bending and transverse shear effect is 0.051536 in the approximate solution.

From the above results, the difference between the approximate and the exact maximum deflection values is +6%.

Conclusions:

The effect of transverse shear force on the maximum deflection is considered and calculated for beams with different cross sectional area values in various parts of a tapered beam.

1. Deflection from transverse shear force effect is small when compared with the deflection from bending effect. The ratio (3.8 %) of maximum deflection due to bending to that due to shear is 4.8 %.
2. To simplify the solution of the differential equation when including the transverse shear effect, an approximation for the shear curvature term is assumed where a constant cross section area along the taper beam equals to smaller area at end 1 is taken. The approximation results are found very close to the accurate values for a taper ratio u = 1.5 up to u = 3.5 as noticed before and still the results are close when this ratio is above u = 3.5. As a useful note, this approximation would be important in studying the behavior of tapered beams subjected to constant axial force with or without transverse shear effect (as noticed in some research work).

The maximum deflection in a prismatic beam always occurs at the cross section of maximum bending moment because the stress varies along the axis of the beam in the same manner as the bending moment. However, this conclusion does not apply to tapered beams because in such beams, the stresses vary along the axis not only proportional to bending moment but also in an inverse proportion to the

moment of inertia of the cross-section.

The deflection curve due to transverse shear effect in a taper beam is different from that in a prismatic beam. The deflections obtained for a tapered beam of cross section from one end to the other by exact integration are compared to those obtained for a prismatic beam of constant cross section area equal to the smaller area at end 1 in the approximate solution.

On the other hand, the difference in deflection due to transverse shear effects between a taper and a prismatic beam depends on the cross sectional area variation from one end of a tapered beam to the other. The comparisons are presented graphically for deflection due to transverse shear effect in two cases: the first is the deflection for a taper beam with varying cross sectional area and the second is the deflection for a prismatic beam with constant cross section equal to the smaller area.

List of symbols:

A :cross-sectional area $A = bd$

A_1 :cross-sectional area at end 1,smaller area $A_1 = bd_1$

A_2 :cross-sectional area at end 2, larger area $A_2 = bd_2$

E :modulus of elasticity

EI :flexural (bending) rigidity of cross section

G :modulus of rigidity (in shear)

I :moment of inertia of cross section, $I = bd^3/12$

I_1 moment of inertia at end 1, smaller area $I = bd_1^3/12$

I_2 moment of inertia at end 2, larger area $I = bd_2^3/12$

M bending moment

V transverse shear force

b beam cross section width (constant)

d depth of cross section

d_1 depth at end 1

d_2 depth at end 2

m, n shape factors (for I and A respectively)

q uniformly distributed load on beam

u taper ratio (u = depth at end 2 / depth at end 1 = d_2/d_1)

w :total deflection

w_b :deflection due to bending

w_s :deflection due to transverse shear

α_s :numerical factor (shear shape factor)

θ :rotation

GA :shearing rigidity of the cross section

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