Comparison of Solutions of Newell-Whitehead Equation by Using the Bernoulli Sub-ODE Method and Modified Simple Equation Method

Sajeda Kareem Radhi

Department of Material, College of Engineering, AL-Mustansiriyah University

Abstract
The Bernoulli Sub-ODE method and modified simple equation method are two efficient methods for obtaining exact solutions of some nonlinear partial differential equations. In this paper, the Bernoulli Sub-ODE method and modified simple equation method are used to solve the Newell-Whitehead equation and compare the solutions. The results show that the two methods approaches are powerful and effective.

Keywords: Bernoulli Sub-ODE Method Modified Simple Equation Method, Newell-Whitehead.

1- Introduction
It is well known that many models in mathematics and physics are described by nonlinear partial differential equations. In order to better make efforts to seek more exact solutions to them. Several powerful methods have been proposed to obtain exact solutions of nonlinear evolution equations, such as the homogeneous balance method[1], the modified simple equation method [2,3,4,5,6,7,8], the Bernoulli Sub-ODE method [9,10,12,13,14,15], the extended tanh method [16], the Tanh method [17], first integral method [18], He’s semi-inverse method[19], functional variable method[20], the modified extended direct algebraic method[21]. Anand Malik, Fakir Chand, Hitender Kumar & S C Mishra[22] investigated exact solutions of the nonlinear Newell-Whitehead equation[23] by using the \( \frac{G'}{G} \)-expansion method and obtained six solutions.

In our work, we use the Bernoulli Sub-ODE method and the modified simple equation method (MSEM) to solve the Newell-Whitehead equation. A comparison of the solution of these two methods is to be attempted.

2- The Bernoulli Sub-ODE method:
The main idea of this method is as follows.
Consider the following ordinary differential equation (ODE)[14]:

\[
\frac{d^2 y}{dx^2} + f(x) \frac{dy}{dx} + g(x) y = 0
\]

Where \( f(x) \) and \( g(x) \) are functions of \( x \). Using the Bernoulli Sub-ODE method, we can transform the equation into a simpler form by substituting

\[
y = u(x) \left( \frac{G'(x)}{G(x)} \right)^n
\]

Where \( u(x) \) is a function to be determined, \( n \) is an integer, and \( G'(x) \) is the derivative of \( G(x) \) with respect to \( x \).

After substituting the above expression into the original equation, we can obtain a new equation that is easier to solve. The solutions of this new equation can then be transformed back to the original variables to obtain the solutions of the original equation.

This method is particularly useful for solving nonlinear partial differential equations, as it provides a powerful tool for finding exact solutions.
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\[ G' + \lambda G = \mu G^2 \]  
\[ \text{...(2.1)} \]

Where \( \lambda \neq 0 \).

When \( \mu \neq 0 \), Eq.(2.1) is a type of the Bernoulli equation, and it’s solution can be stated as:

\[ G = \frac{1}{\frac{\mu}{\lambda} + \mu e^{\lambda z}} \]  
\[ \text{...(2.2)} \]

Where d is an arbitrary constant. When \( \mu = 0 \), the solution of Eq.(2.1) is denoted by

\[ G = e^{-\lambda z} \]  
\[ \text{...(2.3)} \]

Suppose that a nonlinear equation is given by

\[ P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...,) = 0 \]  
\[ \text{...(2.4)} \]

Where \( u = u(x, t) \) is an unknown function, \( P \) is a polynomial in \( u \) and it's various partial derivatives.

The detailed description of this method is given by the following steps:

**Step 1:** Find travelling wave solutions of Eq.(2.1) by taking

\[ u(x, t) = U(z) ; \quad z = x - t \]  
\[ \text{...(2.5)} \]

And transform Eq.(2.1) to the ordinary differential equation

\[ Q(U, U', U'', U''' ,...,) = 0 \]  
\[ \text{...(2.6)} \]

where prime denotes the derivative with respect to \( z \).

**Step 2:** It is assumed that the exact solutions of Eq.(2.6) can be found in the form of the finite series:

\[ U(z) = \sum_{i=0}^{N} a_i (G(z))^i \]  
\[ \text{...(2.7)} \]

Where \( a_i \) are real constants with \( a_N \neq 0 \) to be determined, \( N \) is a positive integer to be determined by balancing the highest order derivatives and the nonlinear terms in Eq.(2.6). The function \( G(z) \) is the solution of Eq.(2.1).

**Step 3:** Substituting Eq.(2.7) into Eq.(2.6) and using Eq.(2.1), collecting all terms with the same order of \( G \) together, the left-hand side of Eq.(2.6) is converted into another polynomial in \( G \) .Equating each coefficient of this polynomial to zero yields a set of algebraic equations for \( a_N, a_{N-1}, ..., \lambda \ and \ \mu \).

**Step 4:** Solving the algebraic equations system in step 3 by Mathematica, and by using the solution of Eq.(2.1), we can construct the traveling wave solutions of the nonlinear evolution Eq.(2.6).
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3- The Modified Simple equation method (MSEM):

For a given partial differential equation consider a general form of the partial differential equation (PDE)[8]:

\[ F(u, u_x, u_t, u_{xx}, u_{xt}, u_{xx}, u_{tt}, \ldots) = 0 \quad \ldots (3.1) \]

Where \( F \) is a polynomial in \( u(x, t) \) and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. To find the required exact solutions, the following main steps are considered:

Step 1: Using the wave transformation

\[ u(x, t) = U(z) \quad ; \quad z = x - t \quad \ldots (3.2) \]

We have from Eq.(3.1) and Eq.(3.2) the following ordinary differential equation (ODE) is considered:

\[ P(U, U', U'', U''', \ldots) = 0 \quad \ldots (3.3) \]

Where \( P \) is a polynomial in \( u \) and its total derivatives and \( \frac{d}{dz} \).

Step 2: Suppose that the solution of Eq.(3.3) can be expressed as a polynomial of \( \left( \frac{\psi'(z)}{\psi(z)} \right) \) in the form

\[ u(z) = \sum_{i=0}^{N} a_i \left( \frac{\psi'(z)}{\psi(z)} \right)^i \quad \ldots (3.4) \]

Where the \( a_i \)'s are arbitrary constants.

Step 3: The positive integer \( N \) in Eq.(3.4) is determined by balancing the highest order derivatives and the nonlinear terms in Eq.(3.3).

Step 4: Substitute Eq.(3.4) into Eq.(3.3), calculate all the necessary derivatives \( u', u'', \ldots \) of the unknown function \( u(z) \) and obtain the function \( \psi(z) \). As a result of this substitution, a polynomial of \( \psi^{-j}, (j = 0, 1, 2, \ldots) \) is obtained. In this polynomial, all the terms of the same power of \( \psi^{-j}, (j = 0, 1, 2, \ldots) \) are gathered and set to zero all of it’s coefficients of it. This operation yields a system of equations which can be solved to find \( a_i \) and \( \psi(z) \). Consequently, we can obtain the exact solutions to Eq.(3.1).

4- Exact solutions of Newell-Whitehead equation by the Bernoulli Sub-ODE method

In this section, the Bernoulli Sub-ODE method is applied to solve the Newell-Whitehead equation as follows:

\[ u_t = u_{xx} + u - u^3 \quad \ldots (4.1) \]
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Employing the transformation \( u(x,t) = U(z) \); \( z = x - t \), we get

\[
U' + U'' + U - U^3 = 0 \quad \ldots \quad (4.2)
\]

Where the prime denote the derivative with respect to \( z \).

Balancing \( U'' \) with \( U^3 \) gives \( N=1 \). Therefore, the solution of Eq.(4.2) can be given in the form

\[
U(z) = a_0 + a_1 G, \quad a_1 \neq 0 \quad \ldots \quad (4.3)
\]

Where \( a_0, a_1 \) are constants to be determined later and \( G = G(z) \) satisfies

\[
G' + \lambda G = \mu G^2 \quad \ldots \quad (4.4)
\]

Substituting Eq.(4.3) into Eq.(4.2) and collecting all the terms with the same power of \( G \), then, equating each coefficient to zero yields the following set of simultaneous algebraic equations:

\[
G^0 : - a_0^3 + a_0 = 0
\]

\[
G^1 : \lambda^2 - \lambda + 1 - 3a_0^2 = 0
\]

\[
G^2 : - 3\mu \lambda + \mu - 3a_0 a_1^2 = 0
\]

\[
G^3 : 2\mu^2 - a_1^2 = 0
\]

solving this system by Mathematica yield the following:

Case 1: \( a_0 = 0 \), \( a_1 = \sqrt{2} \mu \)

\[
u_1(x,t) = \frac{\sqrt{2} \mu}{\frac{\mu}{\lambda} + de^{\lambda(x-t)}} \quad \ldots \quad (4.6)
\]

Case 2: \( a_0 = 0 \), \( a_1 = -\sqrt{2} \mu \)

\[
u_2(x,t) = \frac{-\sqrt{2} \mu}{\frac{\mu}{\lambda} + de^{\lambda(x-t)}} \quad \ldots \quad (4.7)
\]

Case 3: \( a_0 = 1 \), \( a_1 = \sqrt{2} \mu \)

\[
u_3(x,t) = 1 + \frac{\sqrt{2} \mu}{\frac{\mu}{\lambda} + de^{\lambda(x-t)}} \quad \ldots \quad (4.8)
\]

Case 4: \( a_0 = 1 \), \( a_1 = -\sqrt{2} \mu \)
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\[ u_4(x,t) = 1 - \frac{\sqrt{2} \mu}{\frac{\mu}{\lambda} + de^{\lambda(x-t)}} \]  

... (4.9)

Case 5: \( a_0 = -1, a_1 = \sqrt{2} \mu \)

\[ u_5(x,t) = -1 + \frac{\sqrt{2} \mu}{\frac{\mu}{\lambda} + de^{\lambda(x-t)}} \]  

... (4.10)

Case 6: \( a_0 = -1, a_1 = -\sqrt{2} \mu \)

\[ u_6(x,t) = -1 - \frac{\sqrt{2} \mu}{\frac{\mu}{\lambda} + de^{\lambda(x-t)}} \]  

... (4.11)

Fig(4.1) represents the solitary \( u(x,t) \) in Eq.(4.6) with \( \mu = 1, \lambda = 1, d = 1 \)

Fig.(4.1): The solitary solution of Eq.(4.6) for \(-10 \leq t \leq 10, -10 \leq x \leq 10\).

Fig(4.2) represents the solitary \( u(x,t) \) in Eq.(4.11) with \( \mu = 1, \lambda = 1, d = 1 \)

Fig.(4.2): The solitary solution of Eq.(4.11) for \(-10 \leq t \leq 10, -10 \leq x \leq 10\).
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5- Exact solution of Newell-Whitehead equation by using (MSEM):

In this section, we apply MSEM in solving the Newell-Whitehead equation as follows:

\[ u_t = u_{xx} + u - u^3 \]  \hspace{1cm} (5.1)

Using the transformation \( u = u(z) = x - t \), reduce Eq.(5.1) to the following ordinary differential equation:

\[ U' + U'' + U - U^3 = 0 \]  \hspace{1cm} (5.2)

Balancing \( U'' \) with \( U^3 \) gives N=1. Therefore, the solution of Eq.(5.2) can be written in the form:

\[ u(z) = a_0 + a_1 \left( \frac{\psi'(z)}{\psi(z)} \right) \]  \hspace{1cm} (5.3)

Where \( a_0 \) and \( a_1 \) are constants to be determined such that \( a_1 \neq 0 \), while \( \psi(z) \) is an unknown function to be determined. It is easy to see that

\[ u' = a_1 \left( \frac{\psi''}{\psi} - \frac{\psi'\psi''}{\psi^2} \right) \]  \hspace{1cm} (5.4)

\[ u'' = a_1 \left[ \frac{\psi'''}{\psi} - 3 \frac{\psi''\psi'}{\psi^2} + 2 \left( \frac{\psi'}{\psi} \right)^3 \right] \]  \hspace{1cm} (5.5)

Substituting Eqs.(5.3)-(5.5) into Eq.(5.2) and equating all the coefficients of \( \psi^0, \psi^{-1}, \psi^{-2} \) and \( \psi^{-3} \) to zero, the following can be obtained respectively:

\[ \psi^0 : a_0 - a_0^3 = 0 \]  \hspace{1cm} (5.6)

\[ \psi^{-1} : \psi''' + \psi'' + \psi' - 3a_0^2\psi' = 0 \]  \hspace{1cm} (5.7)

\[ \psi^{-2} : 3\psi''\psi' + (\psi')^2 + 3a_0a_1(\psi')^2 = 0 \]  \hspace{1cm} (5.8)

\[ \psi^{-3} : 2 - a_1^2 = 0 \]  \hspace{1cm} (5.9)

Eq.(5.6) directly implies \( a_0 = 0 \) or \( a_0 = 1 \) or \( a_0 = -1 \).

**Case I: \( a_0 = 0, a_1 = \pm \sqrt{2} \)**

Eq.(5.7) and Eq.(5.8) becomes:

\[ \psi''' + \psi'' + \psi' = 0 \]  \hspace{1cm} (5.10)

\[ 3\psi'' + \psi' = 0 \]  \hspace{1cm} (5.11)

By substituting Eq.(5.11) into Eq.(5.10) we get:

\[ \psi''' - 2\psi'' = 0 \]  \hspace{1cm} (5.12)

Solution of Eq.(5.12) is given by:
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\[ \psi(z) = c_1 + c_2 z + c_3 e^{2z} \]  \hspace{1cm} \text{(5.13)}

Where \( c_1, c_2 \) and \( c_3 \) are arbitrary constants.

Substituting Eq.(5.13) for \( \psi(z) \) into Eq.(5.3)

\[ u_1(z) = \sqrt{2}\left( \frac{c_2 + 2c_3 e^{2z}}{c_1 + c_2 z + c_3 e^{2z}} \right) \]  \hspace{1cm} \text{(5.14)}

\[ u_2(z) = -\sqrt{2}\left( \frac{c_2 + 2c_3 e^{2z}}{c_1 + c_2 z + c_3 e^{2z}} \right) \]  \hspace{1cm} \text{(5.15)}

Where \( z = x - t \).

**Case II:** \( a_0 = 1, a_1 = \pm \sqrt{2} \)

Eq.(5.7) and Eq.(5.8) becomes:

\[ \psi''' + \psi'' - 2\psi' = 0 \]  \hspace{1cm} \text{(5.16)}

\[ 3\psi'' + (1 + 3a_1)\psi' = 0 \]  \hspace{1cm} \text{(5.17)}

By substituting Eq.(5.17) into Eq.(5.16) yield:

\[ \psi''' + \left( \frac{3a_1 + 7}{3a_1 + 1} \right)\psi'' = 0 \]  \hspace{1cm} \text{(5.18)}

Solution of Eq.(5.18) is given by:

\[ \psi(z) = c_1 + c_2 z + c_3 e^{\frac{-3a_1 + 7}{3a_1 + 1}z} \]  \hspace{1cm} \text{(5.19)}

Where \( c_1, c_2 \) and \( c_3 \) are arbitrary constants.

Substituting Eq.(5.19) for \( \psi(z) \) into Eq.(5.3)

\[ u_3(z) = 1 + \sqrt{2}\left[ \frac{c_2 - \left( \frac{3\sqrt{2} + 7}{3\sqrt{2} + 1} \right)c_3 e^{\frac{-3\sqrt{2} + 7}{3\sqrt{2} + 1}z}}{c_1 + c_2 z + c_3 e^{\frac{-3\sqrt{2} + 7}{3\sqrt{2} + 1}z}} \right] \]  \hspace{1cm} \text{(5.20)}

\[ u_4(z) = 1 - \sqrt{2}\left[ \frac{c_2 - \left( \frac{-3\sqrt{2} + 7}{-3\sqrt{2} + 1} \right)c_3 e^{\frac{-3\sqrt{2} + 7}{-3\sqrt{2} + 1}z}}{c_1 + c_2 z + c_3 e^{\frac{-3\sqrt{2} + 7}{-3\sqrt{2} + 1}z}} \right] \]  \hspace{1cm} \text{(5.21)}

Where \( z = x - t \).

**Case III:** \( a_0 = -1, a_1 = \pm \sqrt{2} \)
Eq. (5.7) and Eq. (5.8) becomes
\[ \psi'' + \psi'' - 2\psi' = 0 \quad \ldots (5.22) \]
\[ 3\psi'' + (1 - 3a_1)\psi' = 0 \quad \ldots (5.23) \]
By substituting Eq. (5.23) into Eq. (5.22) yield:
\[ \psi'' + \left(\frac{7 - 3a_1}{1 - 3a_1}\right)\psi' = 0 \quad \ldots (5.24) \]
Solution of Eq. (5.24) is given by:
\[ \psi(z) = c_1 + c_2z + c_3e^{-\frac{(7-3a_1)}{1-3a_1}z} \quad \ldots (5.25) \]
Where \( c_1, c_2 \) and \( c_3 \) are arbitrary constants.
Substituting Eq. (5.25) for \( \psi(z) \) into Eq. (5.3)
\[ u_5(z) = -1 + \sqrt{2}\left[ \frac{c_2 - \left(\frac{7 - 3\sqrt{2}}{1 - 3\sqrt{2}}\right)c_3e^{-\frac{(7-3\sqrt{2})}{1-3\sqrt{2}}z}} {c_1 + c_2z + c_3e^{-\frac{(7-3\sqrt{2})}{1-3\sqrt{2}}z}} \right] \quad \ldots (5.26) \]
\[ u_6(z) = -1 - \sqrt{2}\left[ \frac{c_2 - \left(\frac{7 + 3\sqrt{2}}{1 + 3\sqrt{2}}\right)c_3e^{-\frac{(7+3\sqrt{2})}{1+3\sqrt{2}}z}} {c_1 + c_2z + c_3e^{-\frac{(7+3\sqrt{2})}{1+3\sqrt{2}}z}} \right] \quad \ldots (5.27) \]
Where \( z = x - t \).
Fig. (5.1) represents the solitary \( u(x, t) \) in Eq. (5.14) with \( c_1 = 1, c_2 = 1, c_3 = 1 \)

Fig. (5.1): The solitary solution of Eq. (5.14) for \(-10 \leq t \leq 10, -10 \leq x \leq 10\)
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Fig (5.2) represents the solitary $u(x,t)$ in Eq.(5.20) with $c_1 = 1, c_2 = 1, c_3 = 1$

Fig.(5.2): The solitary solution of Eq.(5.20) for $-10 \leq t \leq 10, -10 \leq x \leq 10$.  

6- Comparison between the Bernoulli Sub-ODE Method and (MSEM):

In this section the Bernoulli Sub-ODE and the (MSEM) method are compared and the advantages of these two methods are described.

The two methods to solve the nonlinear equations can be used with any degree of derivative and six solutions of Newell-Whitehead equation can be obtained by using (MSEM) and the Bernoulli Sub-ODE method. The MSEM is much simpler than the Bernoulli Sub-ODE because the exact solutions are obtained by (MSEM) without using the computer programs (Maple or Mathematica) while the Bernoulli Sub-ODE method and other methods need the computer programs for solving the complicated algebraic computations. Finally, it can be seen that the two methods are direct, effective and can be applied to many other nonlinear evolution equations.

7- Conclusion

The Bernoulli Sub-ODE method and modified simple equation method are used to find new exact traveling wave solutions. The two methods can solve any degree of nonlinear partial differential equation. Also, it can be noticed that the two methods are effective in solving complicated nonlinear evolution equations in the mathematical physics, engineering science and other application fields.
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الخلاصة

تعتبر طريقتا Modified simple equation و Bernoulli Sub-ODE من الطرق الفعالة للحصول على حلول محددة ودقيقة لحل المعادلات الغير خطية الجزئية في Newell-Whitehead. هذا البحث تم استخدام الطريقتين لحل معادلة نتائج الطريقتين. أظهرت النتائج أن كلا الطريقتين لها إمكانية كبيرة وفعالة في هذا المجال.