Approximate Solutions of Barker Equation in Parabolic Orbits

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Received on:28/6/2009
Accepted on:5/11/2009

Abstract

The basic motivation of this paper is to apply the Horner's method to perform the steps in Newton and improved Newton methods for approximating the Solution of Barker's equation in parabolic motion.

A simple starting value for the iterative solutions is suggested. Some Numerical applications are presented and show that only little iteration is required to obtain approximate solutions which are found to be accurate and efficient.

Keywords: Barker's equation, Horner's method, Newton method, improved Newton method.

حلول تقريبية لمعادلة باركر في مدارات القطع المُكافيء

الخلاصة

الحافز الاساسي لهذا البحث هو تطبيق طريقة هورنر لأنجاز الخطوات في طريقتي نيوتن ونيوتن المحسنه للحل العددي لمعادلة باركر في حركة القطع المُكافيء.

تم اقتراح قيمة اولية بسيطة للحل التكراري، كما قدمنا بعض التطبيقات العددية والتي من خلالها اتضح لنا ان الحل التقريبي يتم الحصول عليه من خلال عدد قليل من التكرارات ووجد ان الحل عادةً يكون دقيق وكفوء.

1. Introduction

Many instances of parabolic orbits occur in the solar system, especially some smaller comments. Also, the intermediate portion of interplanetary mission may be a heliocentric parabola [2].

For the parabolic motion, the basic equation to be solved is known as Barker's equation which is the relation between the true anomaly n and the time t for parabolic motion and is given as [5]:

$$2\bar{n}(t-t) = \tan\frac{n}{2} + \frac{1}{3}\tan^3\frac{n}{2} \quad \ldots(1)$$

where $\bar{n} = \sqrt{\frac{m}{p^3}}$, m is the gravitational constant, and p is twice the pericenter distance ($p = \frac{2p}{n}$, n is the mean motion), t is the time of pericenter passage. The problem is to find the true anomaly n , where t, t, $\bar{n}$ are given.

Eq.(1) can be rearranged as:

$$\tan^3\frac{n}{2} + 3\tan\frac{n}{2} - 6\bar{n}(t-t) = 0 \quad \ldots(2)$$

There are many methods to solve eq (2) [5-7], in this paper both Newton Horner and improved

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https://doi.org/10.30684/etj.28.3.7
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Newton Horner methods are applied to solve eq (2).

Descarte's Rule of signs to Barker's equation

The expected roots can be examined using Descarte's Rule of signs for polynomials of the form $f(x)=0$.

Let $f(x)$ be a polynomial with real coefficients then:

(i) the number of positive zeros of f is either equal to the number of variations in sign of $f(x)$ or less than this by an even number.

(ii) The number of negative real zeros of f is either equal to the number of variations in sign of $f(-x)$ or less than this by an even number.

Note that Barker's equation is a kind of special cubic equation in $\tan^3\frac{n}{2}$,

Hence eq.(2) can be written a

$$x^3 + 3x - b = 0 \quad \ldots(3)$$

where $b = 6\bar{n}(t - t)$.

Now, using (i), $x^3 \underbrace{+3x}_{+\text{ to }-} - b = 0$

There is one sign change so we have at most one positive real root (note $b > 0$).

Using (ii), $(-x^3) + 3(-x) - b = 0$, we have zero sign changes so no negative real roots.

Our conclusion is that eq.(3) has one real positive root and two complex conjugate roots.

We need to solve for the real root.

Module of Horner's Method [1]

Horner's Method for polynomial Evaluation

Assume that

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \quad \ldots(4)$$

and $x = z$ is a number for which $f(z)$ is to be evaluated. Then $f(z)$ can be computed recursively as follows:

set $b_n = a_n$ and $b_k = a_k + z\ b_{k+1}$ for $k = n-1, n-2, \ldots, 2, 1, 0$ then $f(z) = b_0$.

3.2. Horner's Method for First Derivatives [1]

Assume that

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

and $x = z$ is a number for which $f(z)$ and $f'(z)$ are to be evaluated. We have already seen that $f(z) = b_0$ can be computed recursively as follows $b_n = a_n$ and

$b_k = a_k + z\ b_{k+1}$ for $k = n-1, n-2, \ldots, 2, 1, 0$

Now, $f'(z)$ can be computed recursively as follows $c_n = b_n$ and

$c_k = b_k + z\ c_{k+1}$ for $k = n-1, n-2, \ldots, 2, 1$. Then $f'(z) = c_1$

3.3. Horner's Method for Second Derivatives [1]

Assume that

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

and $x = z$ is a number for which $f(z)$, $f'(z)$ and $f''(z)$

are to be evaluated. We have already seen that $f(z)=b_0$ and $f'(z)=c_1$ can be computed recursively as follows $b_n=a_n$,

$b_k = a_k + z\, b_{k+1}$ for $k=n-1, n-2, ..., 2, 1, 0$

$c_n = b_n$ and $c_k = b_k + z\, c_{k+1}$ for $k=n-1, n-2, ..., 2, 1$.

Now, $f''(z)$ can be computed recursively as follows

$d_n = c_n$ and

$d_k = c_k + z\, d_{k+1}$ for $n-1, n-2, ..., 2$. Then $f''(z) = 2!\, d_2$

Remarks:

Assume that the coefficients $\{a_{[1,k]}\}_{k=1}^{4}$ of Barker's equation, eq.(3), of degree 3 are stored in the first row of the matrix $[a_{i,j}]_{5\times4}$. Then the polynomial $f(x)$ can be written in the form $f[x] = \sum_{k=0}^{3} a_{[1,k+1]} x^k$.

Given the value $x = z$, the subroutine for computing all the derivatives $\{f^{(i)}[z]\}_{i=0}^{3}$ is

$a[i,4] = a[i-1,4]$ for $i = 2,3,4,5$

$$[a_{i,k}] = a_{[i-1,k]} + z\, a_{[i,k+1]}$$

for $k = 3,2,1$; $i = 2,3,4,5$

and $f^{(i)}[z] = i!\, a_{i+2,i+1]}$

for $i = 0,1,2,3$.

Therefore, the matrix $\lfloor a_{i,j} \rfloor_{5\times4}$ can be constructed as follows:

$$[a_{i,j}]_x = \begin{bmatrix} -b & 3 & 0 & 1 \\ -b+3z+z^3 & 3+z^2 & z & 1 \\ 0 & 3+3z^2 & 2z & 1 \\ 0 & 0 & 6z & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Newton –Horner Method

Assume that $f(x)$ is a polynomial of degree $n \geq 2$ and there exists a number $x \in [a,b]$, where $f(x)=0$. If $f'(x) \neq 0$, then there exists a d >0 such that the sequence $\{x_x\}_{k=0}^{\infty}$ defined by the Newton-Raphson iteration formula [3]

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0,1,...$$

will converge to x for any initial approximation $x_0 \in \{x-d, x+d\}$, [3].

Now, the recursive formula of Newton-Horner iteration can be adapted to compute $f(x_k)$ and $f'(x_k)$ such that $f(x_k) = b_{k,0}$ and $f'(x_k) = c_{k,1}$ and the resulting Newton-Horner iteration formula will be

$$x_{k+1} = x_k - \frac{b_{k,0}}{c_{k,1}}, \quad k = 0,1,... \qquad ...(5)$$

where x_0 is an initial guess.

5. Improved Newton-Horner Method

A modified Newton's method can be developed by keeping the second order terms in Taylar series expansion and the following result will be

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}\left[1 + \frac{1}{2}\frac{f(x_k)f''(x_k)}{(f'(x_k))^2}\right]$$

Now, the recursive formula of improved Newton-Horner iteration can be adopted to compute $f(r_k)$, $f'(r_k)$ and $f''(k)$ as follows

$f(x_k) = b_{k,0}$, $f'(x_k) = c_{k,1}$ and $f''(x_k) = 2!d_{k,2}$

Hence, the resulting improved Newton-Horner iteration formula will be

$$x_{k+1} = x_k - \frac{b_{k,0}}{c_{k,1}}\left[1 + \frac{1}{2}\frac{2b_{k,0}\,d_{k,2}}{(c_{k,1})^2}\right]$$

or

$$x_{k+1} = x_k - \frac{b_{k,0}}{c_{k,1}}\left[1 + \frac{b_{k,0}\,d_{k,2}}{(c_{k,1})^2}\right],$$

$k = 0,1,\ldots$ (6)

and x_0 ia an initial guess.

6. Convergence of Newton-Horner Method

In this section we will show a criterian convergence of Newton-Horner method. The algorithm

$$x_{k+1} = x_k - \frac{b_{k,0}}{c_{k,1}} \qquad k = 0,1,\ldots$$

is of the form $x_{k+1} = g(x_k)$. Successive iterations converge if $|g'(x_k) < 1|$.

Since

$g(x) = x - \dfrac{b_0}{c_1}$,Therefore

$$g'(x) = 1 - \frac{c_1\,b_0' - b_0\,c_1'}{c^2{}_1}$$

(note that $b_0 = f(x)$ and $c_1 = f'(x)$) since $b_0' = c_1$, then

$$g'(x) = \frac{c^2{}_1 - c_1^2 + b_0\,c_1'}{c^2{}_1} = \frac{b_0\,c_1'}{c_1^2}$$

Hence if $\left|\dfrac{b_0\,c_1'}{c_1^2}\right| < 1$ an interval about the root x , the method will converge for any initial value in the interval.

In Barker's equation, we have $b_0 = x^3 + 3x - b$, $c_1 = 3x^2 + 3$, so:

$(d_0 = f''(x))$ so,

$$\left|\frac{b_0\,c_1'}{c_1^2}\right| = \left|\frac{(x^3 + 3x - b)(6x)}{(3x^2 + 3)^2}\right| = \frac{2}{3}\left|\frac{x^4 + 3x^3 - bx}{x^4 + 2x^2 + 1}\right| < 1$$

for convergence

if

$$x_0 = \frac{b}{5} \Rightarrow g'(x_o) = \frac{2}{3}\left|\frac{x_0^4 + 3x_0^2 - bx_0}{x^4 + 2x_0^2 + 1}\right| = 0.19006214535 < 1$$

if

$$x_0 = \frac{b}{4} \Rightarrow g'(x_o) = \frac{2}{3}\left|\frac{x_0^4 + 3x_0^2 - bx_0}{x^4 + 2x_0^2 + 1}\right| = 0.0812937789 < 1$$

if

$$x_0 = \frac{b}{3} \Rightarrow g'(x_o) = \frac{2}{3}\left|\frac{x_0^4 + 3x_0^2 - bx_0}{x^4 + 2x_0^2 + 1}\right| = 0.1173861768 < 1$$

if

$$x_0 = \frac{b}{2} \Rightarrow g'(x_o) = \frac{2}{3}\left|\frac{x_0^4 + 3x_0^2 - bx_0}{x^4 + 2x_0^2 + 1}\right| = 0.41286829 < 1$$

7. Convergence of Improved Newton-Horner Method

The algorithm

$$x_{k+1} = x_k - \frac{b_{k,0}}{c_{k,1}}\left[1 + \frac{b_{k,0}\, d_{k,2}}{(c_{k,1})2}\right] \quad for \;\; k = 0,1,\ldots$$

is of the form $x_{k+1} = g(x_k)$. Successive iterations converge if $|g'(x_k)| < 1$

$$g(x) = x - \frac{b_0}{c_1}\left[1 + \frac{b_0 d_2}{c_1^2}\right]$$ there

fore

$$g'(x) = \frac{1}{2}\left(\frac{b_0^2}{c_1^2}\right)\left(-0.5\, c_1 d_2' + 3 d_2^2\right)$$

(note that $b_0 = f(x)$, $c_1 = f'(x)$ and $d_2 = f''(x)$)

Hence

if

$$g'(x) = \frac{1}{2}\left(\frac{b_0^2}{c_1^2}\right)\left(-0.5\, c_1 d_2' + 3 d_2^2\right)$$

an interval about the root X, the method will converge for any initial value in the interval.

if

$$x_0 = \frac{b}{5} \Rightarrow g'(x_o) = \frac{3}{2}\left|\frac{(x_0^3 + 3x_0 - b)}{(3x_0^2 + 3)^2}\left(-(3x_0^2 + 3) + (6x - 6)^2\right)\right| = 0.1049725984 < 1$$

if

$$x_0 = \frac{b}{4} \Rightarrow g'(x_o) = \frac{3}{2}\left|\frac{(x_0^3 + 3x_0 - b)}{(3x_0^2 + 3)^2}\left(-(3x_0^2 + 3) + (6x - 6)^2\right)\right| = 0.01570247962 < 1$$

if

$$x_0 = \frac{b}{3} \Rightarrow g'(x_o) = \frac{3}{2}\left|\frac{(x_0^3 + 3x_0 - b)}{(3x_0^2 + 3)^2}\left(-(3x_0^2 + 3) + (6x - 6)^2\right)\right| = 0.0281981971 < 1$$

if

$$x_0 = \frac{b}{2} \Rightarrow g'(x_o) = \frac{3}{2}\left|\frac{(x_0^3 + 3x_0 - b)}{(3x_0^2 + 3)^2}\left(-(3x_0^2 + 3) + (6x - 6)^2\right)\right| = 0.05657337689 < 1$$

Numerical Examples

Here, Barker's equation is solved using both Newton-Horner and improved Newton-Horner methods.

Consider eq.(3), where $(t - t) = 1.2025\; TU$ (time unit) in a parabolic orbit where $p = 2\;AU$ (angular distance unit) and $m = 1$.

Then

$$\bar{n} = \sqrt{\frac{m}{p^3}} = \sqrt{\frac{1}{2^3}} = \frac{1}{2\sqrt{2}}$$

and hence $b = 6\bar{n}(t - t) = 2.55088771$

We use Newton-Horner method to solve eq.(3) to get:

$$[a_{i,j}]^{(k)} = \begin{bmatrix} -2.55088771 & 3 & 0 & 1 \\ -2.55088771 + 3x_k + x_k^3 & 3 + x_k^2 & x_k & 1 \\ 0 & 3 + 3x_k^2 & 2x_k & 1 \\ 0 & 0 & 6x_k & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for $k = 0,1,\ldots$

In this paper, a simple starting value is considered for the proposed methods of Barker's equation is presented to be $x_0 = \frac{b}{4} = 0.6377213$ as an initial guess yields

$$[a_{i,j}]^{(0)}_{5,4} = \begin{bmatrix} -2.55088771 & 3 & 0 & 1 \\ -0.37836728 & 3.40668925 & 0.63772193 & 1 \\ 0 & 4.22006770 & 1.27544385 & 1 \\ 0 & 0 & 3.826331561 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here

$$a^{(0)}_{21} = b_{0,0} = f(x_0) = -0.3783672$$

$$a^{(0)}{}_{32} = c_{0,1} = f'(x_0) = 4.40668925$$

$$a^{(0)}{}_{42} = d_{0,2} = f''(x_0) = 1.91316578$$

Hence, the results are:
Newton-Horner:

$$x_1 = x_0 - \frac{b_{0,0}}{c_{0,1}} = 0.72738364$$

Improved Newton-Horner:

$$x_1 = x_0 - \frac{b_{0,0}}{c_{0,1}}(1 + \frac{b_{0,0}\ d_{0,2}}{(c_{0,1})^2}) = 0.72373930$$

Then the iterations of Barker's equation, (1) will converge to the root $x = 0.72386802$.

Table (1-4): show the number of iterations required for convergence of Barker's equation (1) using different initial guesses.

In order to find the true anomaly n , we use

$$x = \tan\frac{n}{2} \Rightarrow n = 2\tan^{-1} x \Rightarrow n = 71.7991536\ \deg$$

Discussion

Newton-Horner is improved and applied to find the solution of Barker's equation in parabolic motion. Some starting values was suggested, convergence had been demonstrated from the suggested starting values. It was found that the starting value $x_0 = \frac{3\bar{n}(t - t)}{2}$, produced the lowest iteration number. The processes is repeated until the following convergence test is satisfied $|f(x)| < \in$ where $\in$ is the convergence tolerance, here $\in = 10^{-9}$. The improved Newton- Horner's method usually converges more rapidly than Newton-Horner and should make it the preferred method for the solution of Barker's equation.

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Table (1) Number of iterations required for convergence with $x_0 = \frac{b}{5}$

x_k	Newton-Horner	Improved Newton-Horner
x_0	0.510177542	0.510177542
x_1	0.7449308528	0.72262199393167
x_2	0.72407592667	0.723865335887
x_3	0.72386535739	0.723865336333
x_4	0.72386533633	0.723865336333
x_5	0.72386533633	

Table (2) Number of iterations required for convergence with $x_0 = \frac{b}{4}$

x_k	Newton-Horner	Improved Newton-Horner
x_0	0.6377219275	0.6377219275
x_1	0.727380978	0.7237366168585
x_2	0.72387120635	0.723865336333
x_3	0.7238653363	0.723865336333
x_4	0.7238653363	

Table (3) Number of iterations required for convergence with $x_0 = \frac{b}{3}$

x_k	Newton-Horner	Improved Newton-Horner
x_0	0.8502959033	0.8502959033
x_1	0.731362748098	0.72438219390135
x_2	0.72389202896	0.72386533636571
x_3	0.7238653366	0.72386533636571
x_4	0.72386533633	
x_5	0.72386533633	

Table (4) Number of iterations required for convergence with $x_0 = \frac{b}{2}$

x_k	Newton-Horner	Improved Newton-Horner
x_0	1.275443855	1.275443855
x_1	0.85029590333	0.76253084921
x_2	0.73136274809	0.7238793341
x_3	0.7238920289	0.72386533633
x_4	0.723865336	0.72386533633
x_5	0.723865336	