The Equivalence Between T-Stabilities For Some Iterative Procedures

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Abstract

In this paper, we prove that the equivalence between T-stabilities of modified Ishikawa and modified Mann iteration procedures for a self-mapping satisfying a certain contractive conditions. Our results extend several stability results in the literature.

Key words: Fixed points,Iteration procedure, Contractive conditions, T-stability.

1. Introductionand Preliminaries

The concept of stability of a fixed point iteration procedure due to Ostrowski[(1)] as mentioned by [(2)]. In[(1)] proved the stability of Picard iteration using the contraction condition. Note that, this is direct conclusion for Banach'scontraction principle. Since the Picard iteration does not converge to a fixed point for all kind of contraction mappings (such as the non- expansive mappings, for example see [(3,p.481)], to overcome thesedifficult, other fixed point iteration procedures were considered: Mann iteration, Ishikawa iteration…etc, see [(4)]. The stability for Picard and Mann has been systematically studied by [(5)] in her Ph.D. Thesis and published in the papers[(6,7)]. In[(2,8)] extended the results in [(7)]. In[(9)] established the same stability results for the same iteration processes using the same set of contractive definitions as in [(7)]but thesame method of shorter proof as in [(10)].

Let X be a normed space, B be a nonempty subset of X and T be a self mapping on B. Recall some of iteration processes introduced by[(11)].For $u_0 \in B$, the sequence $\{u_n\}_{n=0}^{\infty}$defined by

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T^n u_n,\ n=0,1,2,\ldots \quad (1)$$

where$\{\alpha_n\}$ is a sequence in [0,1], is known as modified Mann iteration process (see[(11)]). For $x_0 \in B$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \qquad (2)$$
$$y_n = (1-\beta_n)x_n + \beta n T^n x_n, n=0,1,2,\ldots$$

where $\{\alpha_n\}$, $\{\beta n\} \subset [0,1]$ and the iteration$\{x_n\}_{n=0}^{\infty}$ is called the modified Ishikawa iteration(see[(11)]). In (2) if $\beta_n = 0$ we get (1). Replacing T^n by T in (1), (2), we obtain ordinary Mann, Ishikawa iteration, respectively.

An important practical feature of given fixed point iteration procedure is numerically stable if, "small" modifications in the initial data or in the data that are involved in the computation process, will produce a "small" in flounce on the computed value of the fixed point.

Now, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by an iteration procedure involving the operator T,

$$x_{n+1} = f(T^n, x_n),\ n = 0,1,2,. \quad (3)$$

where $x_0 \in B$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T.

For example, the modified Picard iteration is obtained from (3) forf $(T^n, x_n) = T^n x_n$,while the modified Mann iteration is obtained for $f(T^n,x_n) = (1-\alpha_n)x_n + \alpha_n T^n x_n$.

Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. when calculating $\{x_n\}_{n=0}^{\infty}$, then we cover the following steps:

1. We choose the initial approximation $x_0 \in B$;

2. We compute $x_1 = f(T^n, x_0)$ but, due to various errors that occur during the computations (rounding errors, numerical approximations of functions, derivatives or integrals etc.), we do not get the exact value of x_1, but a different one, say y_1, which is however close enough to x_1, i.e., $y_1 \approx x_1$.

3. Consequently, when computing $x_2 = f(T^n, x_1)$ we will actually comput x_2 as $x_2 = f(T^n, y_1)$,and so, instead of the theoretical value x_2 , we will obtain in fact another value, say y_2, again close enough to x_2, i.e., $y_2 \approx x_2$, ..., and so on. In this way, instead of the theoretical sequence $\{x_n\}_{n=0}^{\infty}$,defined by the given iterative method, we will practically obtain an approximatesequence $\{y_n\}_{n=0}^{\infty}$. We shall consider the given fixed point iteration method to be numerically stableif and only if, for y_n close enough (in some sense) to x_n at each stage, the approximate sequence $\{y_n\}_{n=0}^{\infty}$ still converges to the fixed point of T.

The aim of this paper is to prove that the stability of modified Ishikawa iteration is equivalent to the stability of modified Mann iteration for more general contractive definitions than those of [12,13,14] and others.

Our results will generalize and extend several equivalent T-stabilities results of [15], and [16-18].

Firstly, we recall the definition of concept of stability which idea introduced by [5-7] as the following:

Definition 1.1:

Let X be a normed space, B be a nonempty subset of X and T be a self mapping on B, $x_0 \in$ B and suppose that the iteration procedure(3), that is, the sequence $\{x_n\}_{n=0}^{\infty}$ produced by(3), converges to a fixed point p and f is some function. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in B and set

$$\varepsilon_n = \|y_{n+1} - f(T^n, y_n)\|,\ n = 0,1,2,.. \quad (4)$$

Then, the iteration (3) is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \varepsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

For example about stability when n =1 see [19, p.6] and [20, p.2].

Now, we define some types of successively contraction conditions:

For all x, y∈B, there exist a, $0 \le a < 1$, such that

$$\|T^n x - T^n y\| \le a\|x - y\| \quad (5)$$

For all x, y ∈ B, there exist b, $0 \le b < 0.5$, such that

$$\|T^n x - T^n y\| \le b[\|x - T^n x\| + \|y - T^n y\|] \quad (6)$$

There exist c, $0 \le c < 0.5$, such that

$$\|T^n x - T^n y\| \le c[\|x - T^n y\| + \| y - T^n x\|],\ \text{for all } x, y \in B. \quad (7)$$

There exist real numbers a,b and c satisfying $0 \le a < 1$, $0 \le b < 0.5$ and $0 \le c < 0.5$ such that for each x, y in B, at least one of the following is true :

(Z_1) $\|T^n x - T^n y\| \le a\|x - y\|$;

(Z_2) $\|T^n x - T^n y\| \le$

$b\,[\|x - T^n x\| + \|y - T^n y\|\,];$

(Z_3)$\|T^n x - T^n y\| \leq$

$c\,[\|x - T^n y\| + \|y - T^n x\|\,]$.(8)

For each x, y in B, there exist h, $0 \leq h < 1$ such that

$\|T^n x - T^n y\| \leq h \max \{\|x - T^n y\|, \|y - T^n x\|\}$ (9)

For each x,y in B, there exist h, $0 \leq h < 1$ such that

$\|T^n x - T^n y\| \leq h \max \{\|x - y\|, \|x - T^n x\|, \|y - T^n y\|\}$ (10)

There exist h, $0 \leq h < 1$ such that, $\forall$ x, y $\in$B

$\|T^n x - T^n y\| \leq h \max \{\|x - y\|, \frac{1}{2}[\|x - T^n x\| + \|y - T^n y\|\,], \|x - T^n y\|, \|y - T^n x\|\}$ (11)

For each x, y in B, there exist h, $0 \leq h < 1$ such that

$\|T^n x - T^n y\| \leq h \max \{\|x - y\|, \|x - T^n x\|, \|y - T^n y\|, \frac{1}{2}[\|x - T^n y\| + \|y - T^n x\|\,]\}$ (12)

For all x, y$\in$B, there exist h, $0 \leq h < 1$ such that

$\|T^n x - T^n y\| \leq h \max \{\|x - y\|, \frac{1}{2}[\|x - T^n x\| + \|y - T^n y\|\,], \frac{1}{2}[\|x - T^n y\| + \|y - T^n x\|\,]\}$ (13)

There exist h, $0 \leq h < 1$ such that $\forall$x, y$\in$B

$\|T^n x - T^n y\| \leq h \max \{\|x - y\|, \|x - T^n x\|, \|y - T^n y\|, \|x - T^n y\|, \|y - T^n x\|\}$ (14)

For all x, y in B, there exist C, $0 \leq C < 1$ and for some $L \geq 0$, $\forall$x, y $\in$ B such that

$\|T^n x - T^n y\| \leq C\|x - y\| + L\,\|x - T^n x\|$,(15)

The conditions (5), (6) and (7) are independent sinceT^n is continuous for all n but (6), (7) not necessary continuous (see[12,21], when n = 1 as espial case).Clearly (8) is generalization of (5), (6), (7). Below we prove that (8) implies to (14) and (15) independency. And then, one can prove that (9), (10),(11),(12),(13) and (13) implies (14) by similar way.

Proposition 1.1:

If T is holding the condition (8), then

(i)T satisfies (14) , (ii)T satisfies (15)

Proof:

The proof of part (i) is clearly.

Now, To proof (ii):

If T is satisfying (8) for all x ,y in B, then at least one of (Z_1), (Z_2) or (Z_3) is true.

If (Z_1) holds then$\|T^n x - T^n y\| \leq a\,\|x - y\|$, thus condition (15) hold where C = a , L = 0. If (Z_2) satisfies then$\|T^n x - T^n y\| \leq$ $b[\|x - T^n x\| + \|y - T^n y\|]$ $\leq b[\|x - T^n x\| + \|y - x\| + \|x - T^n x\| + \|T^n x - T^n y\|]$$\|T^n x - T^n y\| \leq \frac{b}{1-b}\|x - y\| + \frac{2b}{1-b}\|x - T^n x\|$

Since $0 \leq b < 0.5$ therefore we have (15) with $C = \frac{b}{1-b}$ and $L = \frac{2b}{1-b}$. If (Z_3) holds, $0 \leq C < 0.5$ then similarly of (Z_2) we get (15) satisfies. ■

On the other hand, we pose the following question: Are (14) and (15) independent? In fact, we cannot have an exact answer but we give a part of answer in the following proposition and example:

Proposition 1.2:

Any mapping satisfying condition (14) with $0 < h < 1/2$ is also, satisfying condition (15).

Proof: Let T: B → B be a mapping for which satisfying (15) for all x ,y in B.
To prove ,we have five possible cases:

Case 1. When $\|T^n x - T^n y\| \leq h\,\|x - y\|$ then condition (15) is obviously satisfied (with C = h and L = 0).

Case 2. When $\|T^n x - T^n y\| \leq h\,\|x - T^n x\|$ then (15) holds (with C = 0 and L = h).

Case 3. $\|T^n x - T^n y\| \leq h\,\|y - T^n y\|$

$$\leq h[\|y - x\| + \|x - T^n x\| + \|T^n x - T^n y\|]$$

$$(1 - h)\|T^n x - T^n y\| \leq h[\|y - x\| + \|x - T^n x\|]$$

$$\|T^n x - T^n y\| \leq \frac{h}{1-h}\|y - x\| + \frac{h}{1-h}\|x - T^n x\|$$

which is (15) with $C = \frac{h}{1-h} < 1$ (since $h < \frac{1}{2}$) and $L = \frac{h}{1-h} > 0$.

Case 4. $\|T^n x - T^n y\| \leq h\,\|x - T^n y\|$

$$\leq h[\|x - T^n x\| + \|T^n x - T^n y\|]$$

$$\|T^n x - T^n y\| \leq \frac{h}{1-h}\|x - T^n x\|$$

$$\leq c\|x - y\| + \frac{h}{1-h}\|x - T^n x\|$$

Thus, the condition (15) is satisfying with $C = 0$ and $L = \frac{h}{1-h} > 0$.

Case 5. When $\|T^n x - T^n y\| \leq$

$$h\,\|y - T^n x\| \leq h[\|y - x\| + \|x - T^n x\|]$$

which is (15) with C = h and L = h.

This completes the proof . ■

Now ,we give example satisfies condition (15) but not condition (14):

Example 1.1:

Let X = [0,1] be unit interval with usual norm when n = 1 define T:[0,1]⟶[0,1] by $T(x) = \frac{x}{2}$ for all $x \in [0, 1)$ and $T(1) = \frac{1}{2}$, if x = 1. Then T satisfies condition (15), since $\|Tx - Ty\| \leq C\|x - y\| + L\|x - Tx\|$,

then $\left|\frac{x}{2} - \frac{1}{2}\right| \leq C|x - 1| + L\left|x - \frac{x}{2}\right|$,

which is true if we take $C = \frac{1}{2}$ and $L \geq 2$.

For any $0 < h < 1$, if x = h, y = 0 then $T(x) = \frac{h}{2}$, T(0) = 0, hence $\|Tx - Ty\| = \frac{h}{2}$, and $h \max\left\{h, \frac{h}{2}, 0, \frac{h}{2}, h\right\} = h^2$.

Therefore T is not satisfy (14).

For equivalence between T-stabilities, suppose that $p \in F(T)$, let $\{u_n\}, \{x_n\}, \{y_n\} \subset B$ be such that $u_0 = x_0 = y_0 \in B$, let the sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, Satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n, \sum_{n=1}^{\infty} \alpha_n = \infty$. (16)

The following non-negative sequences are well defined for all n

$$\varepsilon_n = \|u_{n-1} - (1 - \alpha_n)u_n - \alpha_n T^n u_n\| \quad (17)$$

We consider $y_n = (1 - B_n)X_n + B_n T^n x_n$, and

$\xi_n = \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n y_n\|$ (18)

Definition 1.2:

a. The modified Mann iteration (1) is said to be T-stable if

$\lim_{n\to\infty} \varepsilon_n = 0, \Longrightarrow \lim_{n\to\infty} u_n = p.$

b. The modified Ishikawa iteration (2) is said to be T-stable if

$\lim_{n\to\infty}\xi_n = 0 , \Longrightarrow \lim_{n\to\infty} x_n = p$.

precisely the following conditions equivalence:

c- For all $\{\alpha_n\},\{\beta_n\} \subset (0,1)$, satisfying (16) the modified Ishikawa iteration is T-stable .

d-For all $\{\alpha_n\},\{\beta_n\} \subset (0,1)$, satisfying(16),

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n y_n\| = 0$$

implies that $\lim_{n\to\infty} x_n = p$.

Also, for modified Mann iteration

e- For all $\{\alpha_n\} \subset (0,1)$ satisfying (16), the Mann iteration is T-stable .

f- For all $\{\alpha_n\} \subset (0,1)$satisfying (16),

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \|u_n + (1-\alpha_n)u_n - \alpha_n T^n u_n\| = 0$$

$\Longrightarrow \lim_{n\to\infty} u_n = p.$

2. Main Results

We give the following results:

Theorem 2.1:

Let X be a normed space, B be a nonempty convex subsetof X . Let T be a self-mapping satisfying a condition (14) with a fixed point p. For $x_0 \in B$, let$\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$defined by (1) and (2) respectively with$\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$which satisfying (16). Then, the following assertions are equivalent:

m-The modified Ishikawa iteration is T-stable ;

n-The modified Mann iteration is T-stable .

Proof :

From (17) and (18) show that (m) $\Leftrightarrow$ (n) is mean that (d) $\Leftrightarrow$(f).

i.e. suppose that the modified Ishikawa iteration is T-stable.

Then, we prove that modified Mann iteration is T-stable.

Now , by using (17) and (14) with $x := u_n$, $y := y_n$, we obtain

$$\|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| \le \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n\|T^n y_n - T^n u_n\|. \quad (19)$$

Since

$$\|T^n y_n - T^n u_n\| \le h \max \{\|y_n - u_n\|, \|y_n - T^n y_n\|, \|u_n - T^n u_n\|, \|y_n - T^n u_n\|, \|u_n - T^n y_n\|\}$$

$$\le h \max \{\|y_n - u_n\|, \|y_n - T^n y_n\|, \|u_n - y_n\| + \|y_n - T^n y_n\| + \|T^n y_n - T^n u_n\|, \|y_n - T^n y_n\| + \|T^n y_n - T^n u_n\|, \|u_n - y_n\| + \|y_n - T^n y_n\|\}$$

$$= \max\{h\|y_n - u_n\|, h\|y_n - T^n y_n\|, \frac{h}{1-h}[\|y_n - u_n\| + \|y_n - T^n y_n\|],$$

$\frac{h}{1-h}\|y_n - T^n y_n\|, h[\|y_n - u_n\| + \|y_n - T^n y_n\|]\}$

$$= \frac{h}{1-h}[\|y_n - u_n\| + \|y_n - T^n y_n\|] \quad (20)$$

Hence (20) implies to

$$\|T^n y_n - T^n y_n\| \le \frac{h}{1-h}\|y_n - u_n\| + \frac{h}{1-h}[\|y_n - u_n\| + \|u_n - T^n u_n\| + \|T^n u_n - T^n y_n\|]$$

$$\left(1 - \frac{h}{1-h}\right)\|T^n y_n - T^n y_n\| = \frac{h}{1-h}\|y_n - u_n\| + \frac{h}{1-h}[\|y_n - u_n\| + \|u_n - T^n u_n\|]$$

$$\frac{1-2h}{1-h}\|T^n y_n - T^n u_n\| = \frac{h}{1-h}\|y_n - u_n\| + \frac{h}{1-h}[\|y_n - u_n\| + \|u_n - T^n u_n\|]$$

$$\|T^n y_n - T^n u_n\| = \frac{h(1-h)}{(1-2h)(1-h)}\|y_n - u_n\| + \frac{h(1-h)}{(1-2h)(1-h)}[\|y_n - u_n\| + \|u_n - T^n u_n\|$$

$$= \frac{h}{1-2h}\|y_n - u_n\| + \frac{h}{1-2h}[\|y_n - u_n\| + \|u_n - T^n u_n\|]$$

$$= \frac{2h}{1-2h}\|y_n - u_n\| + \frac{h}{1-2h} \times \|u_n - T^n u_n\| \quad (21)$$

Now, substituting (21) in (19), we have

$$\|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| \le \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n[\frac{2h}{1-2h}\|y_n - u_n\| + \frac{h}{1-2h}\|u_n - T^n u_n\|]$$

$$= \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n[\frac{2h}{1-2h}\|[(1-\beta_n)u_n + \beta_n T^n u_n] - u_n\| + \frac{h}{1-2h}\|u_n - T^n u_n\|]$$

$$\le \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n[\frac{2h}{1-2h}((1-\beta_n)\|u_n - u_n\| + \beta_n\|u_n - T^n u_n\|) + \frac{h}{1-2h}\|u_n - T^n u_n\|]$$

$$= \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n \frac{2h\beta_n + h}{1-2h}\|u_n - T^n u_n\| \to 0 \text{ as } n \to \infty.$$

By condition (d) thus $\lim_{n\to\infty}\|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| = 0$, implies that $\lim_{n\to\infty} u_n = p$.

Hence $\lim_{n\to\infty}\|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| \to 0$, yields $\lim_{n\to\infty} u_n = p$.

Conversely, we show that (f) $\Longrightarrow$ (d) i.e. Assume that the modified Mann iteration is

T-stable . Then, we shown that the modified Ishikawa iteration is T-stable.

Now , by using (14) and (18) with $x := x_n$, $y := y_n$, we have .

$$\|x_{n+1} - (1-\alpha)x_n - \alpha_n T^n y_n\|$$
$$\le \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| + \alpha_n\|T^n x_n - T^n y_n\|$$
$$\le \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| + \alpha_n\left[\frac{h}{1-h}\|x_n - y_n\| + \frac{h}{1-h}\|x_n - T^n x_n\|\right]$$
$$= \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| + \alpha_n[\frac{h}{1-h}\|x_n - [(1-\beta_n)x_n + \beta_n T^n x_n]\| + \frac{h}{1-h}\|x_n - T^n x_n\|$$
$$\le \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| + \alpha_n[\frac{h}{1-h}((1-\beta_n)\|x_n - x_n\| + \beta_n\|x_n - T^n x_n)\| + \frac{h}{1-h}\|x_n - T^n x_n\|]$$
$$= \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| + \alpha_n\frac{h\beta_n + h}{1-h}\|x_n - T^n x_n\| \to 0\,, \text{as} n \to 0.$$

Hence condition (f) show that $\lim_{n\to\infty}\|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| = 0$ implies to $\lim_{n\to\infty} x_n = p$.

Thus $\lim_{n\to\infty}\|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n y_n\| \to 0$,

yields $\lim_{n\to\infty} x_n = p$. ■

Remark 2.1: As consequence of Theorem (2.1) we have Theorem (3.6) and corollary (3.7) in [15, p.1889-1890] directly.

Corollary 2.1: Let $X, B, \{x_n\}_{n=0}^{\infty}, \{u_n\}_{n=0}^{\infty}$, $\{\alpha_n\}, \{\beta_n\}$ and p be as in Theorem (2.1). Let $T: B \to B$ be a mapping satisfying condition (8) such that the conclusion of theorem (2.1) satisfies.

Proof: From (17) and (18) show that (m)⟺ (n) is equivalent (d) ⟺(f).

To prove that (d) ⟹ (f). i.e. Suppose that the modified Ishikawa iteration is T-stable. Then, we prove that modified Mann iteration is T-stable.

Now, by using (8) and (17) with $x := u_n$ and $y := y_n$, we get

$$\|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| \le \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n\|T^n y_n - T^n u_n\| \qquad (22)$$

Since T is holding the condition (8), then the following condition

$$\|T^n y_n - T^n u_n\| \le \delta\|y_n - u_n\| + 2\delta\|y_n - T^n y_n\| \qquad (23)$$

holds $\forall\, x_n, y_n$ in B, where

$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$, where $0 \le \delta < 0.5$.

From(23) we have

$$\|T^n y_n - T^n u_n\| \le \delta\|y_n - u_n\| + 2\delta[\|y_n - u_n\| + \|u_n - T^n u_n\| + \|T^n u_n - T^n y_n\|]$$
$$\le \frac{\delta}{1-2\delta}\|y_n - u_n\| + \frac{2\delta}{1-2\delta}[\|y_n - u_n\| + \|u_n - T^n u_n\|]$$

$$= \frac{3\delta}{1-2\delta}\|y_n - u_n\| + \frac{2\delta}{1-2\delta}\|u_n - T^n u_n\|.$$

The proof completes by a same way of Theorem (2.1). ■

Corollary2.2: Let $X, B, \{x_n\}_{n=0}^{\infty}, \{u_n\}_{n=0}^{\infty}$, $\{\alpha_n\}, \{\beta_n\}$ and p be as in Theorem (2.1) and $T: B \to B$ be a mapping satisfying (13) such that the conclusion of Theorem (2.1) is satisfying.

Proof: From (17) and (18) the equivalent between (m)⟺(n) is mean that (d)⟺(f). To prove that (d) ⟹ (f). i.e. If modified Ishikawa iteration is T-stable, then the modified Mann iteration is T-stable.
By using condition (13) and (17) with $x := u_n$ and $y := y_n$, we get

$$\|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| \le \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n\|T^n y_n - T^n u_n\| \quad (24)$$

Observe that

$$\|T^n y_n - T^n u_n\| \le h\max\Big\{\|y_n - u_n\|, \frac{1}{2}[\|y_n - T^n y_n\| + \|u_n - T^n u_n\|], \frac{1}{2}[\|y_n - T^n u_n\| + \|u_n - T^n y_n\|]\Big\}$$

$$\le h\max\Big\{\|y_n - u_n\|, \frac{1}{2}[\|y_n - T^n y_n\| + \|u_n - y_n\| + \|y_n - T^n y_n\| + \|T^n y_n - T^n u_n\|], \frac{1}{2}[\|y_n - T^n y_n\| + \|T^n y_n - T^n u_n\| + \|u_n - y_n\| + \|y_n - T^n y_n\|]\Big\}$$

$$= \max\Big\{h\|y_n - u_n\|, \frac{h}{2-h}[2\|y_n - T^n y_n\| + \|y_n - u_n\|], \frac{h}{2-h}[2\|y_n - T^n y_n\| + \|y_n - u_n\|]\Big\}$$

$$= \lambda\|y_n - u_n\| + 2\lambda\|y_n - T^n y_n\|. \quad (25)$$

where $\lambda = \max\Big\{h, \frac{h}{2-h}\Big\}$, where $0 \le \lambda < 1$.
Thus (25) implies to

$$\|T^n y_n - T^n u_n\| \le \lambda\|y_n - u_n\| + 2\lambda[\|y_n - u_n\| + \|u_n - T^n u_n\| + \|T^n u_n - T^n y_n\|]$$

$$= \frac{\lambda}{1-2\lambda}\|y_n - u_n\| + \frac{2\lambda}{1-2\lambda}[\|y_n - u_n\| + \|u_n - T^n u_n\|]$$

$$= \frac{3\lambda}{1-2\lambda}\|y_n - u_n\| + \frac{2\lambda}{1-2\lambda}\|u_n - T^n u_n\|$$

The proof follows by a same way of Theorem (2.1) . ■

Theorem 2.2: Let $X, B, \{x_n\}_{n=0}^{\infty}, \{u_n\}_{n=0}^{\infty}$, $\{\alpha_n\}, \{\beta_n\}$ and p be as in theorem (2.1) and T be a self mapping on B satisfying (15) such that the following are equivalent :

m- The modified Ishikawa iteration is T-stable ;

n- The modified Mann iteration is T-stable.

Proof: By (17) and (18) we known the equivalence (m)⟺(n) means that(d)⟺(f). So, we will prove that (d) ⟹ (f). i.e. Suppose that the modified Ishikawa iteration is T-stable. To show that the modified Mann iteration is T-stable.

By using condition (15) and (17) with $x := u_n$ and $y := y_n$, we have

$$\|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| \le \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n \|T^n y_n - T^n u_n\| \quad (26)$$

Since

$$\|T^n y_n - T^n u_n\| \le C\|y_n - u_n\| + L\|y_n - T^n y_n\| \quad (27)$$

From (27), we get

$$\|T^n y_n - T^n u_n\| \le C\|y_n - u_n\| + L[\|y_n - u_n\| + \|u_n - T^n u_n\| + \|T^n u_n - T^n y_n\|]$$

$$= \frac{C}{1-L}\|y_n - u_n\| + \frac{L}{1-L}[\|y_n - u_n\| + \|u_n - T^n u_n\|]$$

$$= \frac{C+L}{1-L}\|y_n - u_n\| + \frac{L}{1-L}\|u_n T^n u_n\|. \quad (28)$$

Substitution (28) in (26), we obtain

$$\|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| \le \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n\left[\frac{C+L}{1-L}\|y_n - u_n\| + \frac{L}{1-L}\|u_n - T^n u_n\|\right]$$

$$= \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n\left[\frac{C+L}{1-L}\|((1-\beta_n)u_n + \beta_n T^n u_n) - u_n\| + \frac{L}{1-L}\|u_n - T^n u_n\|\right]$$

$$\le \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n\left[\frac{C+L}{1-L}((1-\beta_n)\|u_n - u_n\| + \beta_n\|T^n u_n - u_n\|) + \frac{L}{1-L}\|u_n - T^n u_n\|\right]$$

$$= \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n\left[\frac{C+L}{1-L}\beta_n\|T^n u_n - u_n\| + \frac{L}{1-L}\|u_n - T^n u_n\|\right]$$

$$= \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n\| + \alpha_n\left[\frac{C+L}{1-L}\beta_n + \frac{L}{1-L}\right]\|T^n u_n - u_n\| \longrightarrow 0,$$

$n \longrightarrow \infty$.

From condition (d), we get that, $\lim_{n\to\infty}\| u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n y_n \| = 0$, yields $\lim_{n\to\infty} u_n = p$.

Hence $\lim_{n\to\infty}\|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| \longrightarrow 0$,

implies that $\lim_{n\to\infty} u_n = p$.

Conversely, we prove that (f) ⟹ (d). By using condition (15) and (18) with $x := x_n$ and $y := y_n$, we have

$$\|x_{n+1} - (1-\alpha)x_n - \alpha_n T^n y_n\|$$

$$\leq \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\|$$
$$+ \alpha_n[C\|x_n - y_n\|$$
$$+ L\|x_n - T^n x_n\|]$$
$$= \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\|$$
$$+ \alpha_n\big[C\big\|x_n$$
$$-\big((1-\beta_n)x_n + \beta_n T^n x_n\big)\big\|$$
$$+ \ L\|x_n - T^n x_n\|\big]$$
$$\leq \| \ x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\|$$
$$+ \alpha_n\big[\ C\big((1-\beta_n)\|x_n - x_n\|$$
$$+ \qquad \beta_n\|x_n - T^n x_n\|\big)$$
$$+ L\|x_n - T^n x_n\|\big]$$
$$= \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| +$$
$$\alpha_n(C\beta_n + L)\|x_n - T^n x_n\| \to 0, \text{ as } n \to \infty.$$

Since condition (f) yields $\lim_{n\to\infty}\|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| = 0$, implies that $\lim_{n\to\infty} x_n = p$.

Then, we get $\lim_{n\to\infty}\|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n y_n\| \to 0$,

implies that $\lim_{n\to\infty} x_n = p$. ■

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التكافؤ بين T – استقرارية لبعض العمليات التكرار

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الخلاصة

في هذا البحث نبرهن التكافؤ لT – استقرارية لعمليات أيشيكاو االتكرارية المطورة ولعمليات مان التكرارية المطور لتطبيق ذاتي يحقق شروط انكماشية . نتائجنا تعميم لنتائج الاستقرارية في البحوث المنشورة سابقا .