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The Strong Approximation by Linear Positive Operator In terms of the Averaged Modulus of Order One

Zainab Esa Abdul Naby
Al – Mustansiriya University-College of Science

Abstract: In this work, we introduce Bernst- ein linear positive operators $B_{n,k}(f,x)$ in the space of all continuous functions C_I where $I=[0,1]$ with some properties of this operator so to find the strong approximation of continuous functions with the averaged modulus of order one.

Keywords : Strong Approximation ,Linear Positive Operator, Averaged Modulus ,Order One

1-Introduction

The strong approximation of function connected with Fourier series was examined in many papers published in last 40 years. The problem of strong approxim- mation with power $q>0$ is well known for 2π - periodic functions and their Fourier series [1], [2]. For example [3], if $S_n(f,x)$ is the n-th partial sum of trigonometric Fourier series of f , then the n-th $(C,1)$ -mean of this series is defined by the formula :

$$\sigma_n(f,x)=\frac{1}{n+1}\sum_{k=0}^{n}S_n(f,x)\,,\ \ n\in N_0$$

where $N_0=\{0,1,...\}$. The n-th strong $(C,1)$ - mean of this series is defined as follows:

$$H_n^q(f,x)=\left\{\frac{1}{n+1}\sum_{k=0}^{n}|S_n(f,x)-f(x)|^q\right\}^{\frac{1}{q}}\,,$$

$n\in N_0$. Where q is a fixed positive number, It is clear that: $|\sigma_n(f,x)-f)x)|\le H_n^1(f,x)$
And $H_n^q(f,x)\le H_n^p(f,x)$, $0<q<p<\infty$ ……………………………. (1.1) In [4] is investigated the strong approxi- imation of functions $f\in C_I$ some linear operators.

Definitions and Lemmas:

In this paper we examine this problem for $f\in C_I(I=[0,1])$ and introduced $B_{n,k}(f,A,x)$ linear positive operators. Let C_I be the space of all functions, continuous and bounded on $f:I\to R$ with the norm: $\|f\|=\sup\{|f(x)|:x\in I\}$ …………. (1.2) Let $r\in N_0$ be a fixed number and let $C_I^r=\{f\in C_I: f^{(r)}\in C_I\}$ and the norm C_I^r is defined by (1.2), where $C_I^0\equiv C_I$. Let $A\in\mathcal{M}$ and $n\in N$. Where $\mathcal{M}$ the set of all infinite matrices $A=[a_{n,k}(x)]$. The Bernstein operators [5]: $B_{n,k}(f,A,x)=\sum_{k=0}^{n}a_{n,k}(x)f\left(\frac{k}{n}\right)$ ……. (1.3)
Defined for continuous f on the interval $I=[0,1]$ with the matrix $A=[a_{n,k}(x)]$ where:
$a_{n,k}(x)=\left\{\binom{n}{k}x^k(1-x)^{n-k}\right\}$…… (1.4)

Lemma (1.1): **[3]**
Let $A=[a_{n,k}(x)], n\in N, k\in N_0$ then
$a_{n,k}(x)\le 0$, for $x\in R, n\in N, k\in N_0$.
$a_{n,k}(x)=\left\{\begin{matrix}\binom{n}{k}x^k(1-x)^{n-k}=1 & if\ k=n\\ \binom{n}{k}x^k(1-x)^{n-k}=0 & if k>n\end{matrix}\right\}$…. (1.5)

Lemma (1.2): **[3]**
Let $A=[a_{n,k}(x)], n\in N, k\in N_0$, $x\in[0,\infty)$ as in (1.4) then:
1-$B_{n,k}(1,A,x)=1$
2-$B_{n,k}\left(\frac{k}{n}1,A,x\right)=x$
3-$B_{n,k}\left((\frac{k}{n})^2,A,x\right)=x^2\left(\frac{n-1}{n}\right)+\frac{x}{n}$

For every matrix $A\in\mathcal{M}$, $p\in N_0$ and $B_{n,k}(f,A,x)$. Then strong deference $H_n^q(f,x)$ is well – defined for every $f\in C_q$,$x\in I=[0,1]$, $n\in N$ with power $q>0$ as follows [6]:
$H_n^q(f,x)=\left\{\sum_{k=0}^{n}a_{n,k}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right|^q\right\}^{\frac{1}{q}}$…… (1.6)

Let the function f be defined and bounded in the interval $[a,b]$ then [4]: $\omega(f,\delta)=\{\sup(|f(x)-f(y)|):x,y\in[a,b],|x-y|\le t\}, t\ge 0$.................. (1.7)
In [5] if $f\in R_{0=}[0,\infty)$, then:
$\omega(f,\lambda t)\le(\lambda+1)\omega(f,t)$, for $\lambda, t\in R_0$ ………………………………………. (1.8)

And if $f \in R_0$ are uniformly continuous functions then $\lim_{n\to 0^+} \omega(f,t) = 0$.The k^{th} averaged modulus of smoothness for $f \in R_0$ is defined by [7]: $\tau_k(f,\delta)_p = \|\omega_k(f,\delta)\|_p$
The averaged modulus of order one defined by:
$\tau_1(f,\delta)_p = \|\omega_1(f,\delta)\|_p$.………... (1.9) in [7] 1- if f is measurable bounded function on $[a,b]$, $p \geq 1$ then
$\omega_k(f,\delta)_p \leq \tau_k(f,\delta)_p$
2- If $\delta \geq \delta'$ then
$\omega_k(f,x,\delta) \geq \omega_k(f,x,\delta')$, and $\tau_k(f,x,\delta) \geq \tau_k(f,x,\delta')$ ………. (1.10)
where $\omega_k(f,x,\delta) = \{\sup|\Delta_h f(t)|: t \in \left[x-\frac{h}{2}, x+\frac{h}{2}\right], x \in [0,\infty)\}, k \in N, \delta \in [0,\infty]$.

2- Main results

First we prove some properties of $B_{n,k}(f,A,x)$ and Lemma to using them in the proof of our theorems.

Lemma (2.1):
Let $A = [a_{n,k}(x)], n \in N, k \in N_0$ as in (1.4), $x \in I = [0,1]$ then:
$B_{n,k}\left((\frac{k}{n})^3, A, x\right) = x^3\left(\frac{(n-1)(n-2)}{n^2}\right) + 3x^2 + \frac{x}{n^2}$
Proof:
From (1.3), (1.4) and lemma (1.2), we have:
$B_{n,k}\left((\frac{k}{n})^3, A, x\right) = \sum_{k=0}^{n} a_{n,k}(x).\left(\frac{k}{n}\right)^3$
$= x\sum_{k=0}^{n}\left(\frac{k}{n}\right)^2 \binom{n}{k} x^k(1-x)^{n-k}$
$= x\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^2 \binom{n}{k} x^{k-1}(1-x)^{(n-1)-(k-1)}$
Let $j = k-1$
$= x\sum_{j=0}^{n-1}\left(\frac{j+1}{n}\right)^2 \binom{n-1}{j} x^j(1-x)^{(n-1)-j} = x\sum_{j=0}^{n-1}\left(\frac{j}{n}\right)^2 \binom{n-1}{j} x^j(1-x)^{(n-1)-j} + 2x\sum_{j=0}^{n-1}\frac{j}{n^2}\binom{n-1}{j} x^j(1-x)^{(n-1)-j} + x\sum_{j=0}^{n-1}\frac{1}{n^2}\binom{n-1}{j} x^j(1-x)^{(n-1)-j} = x^2\frac{(n-1)}{n}\sum_{j=1}^{n-2}\frac{j}{n}\binom{n-2}{j} x^{j-1}(1-x)^{(n-2)-j+1} + 2x\frac{(n-1)}{n^2}\sum_{j=1}^{n-2}\frac{j-1}{n}\binom{n-1}{j-1} x^{j-1}(1-x)^{(n-1)-j+1} + x\frac{1}{n^2}$
Let $v = j-1$
$= x^2\frac{(n-1)}{n}\sum_{v=0}^{n-2}\frac{v+1}{n}\binom{n-2}{v} x^v(1-x)^{(n-2)-v} + 2x^2\frac{(n-1)}{n^2}\sum_{v=0}^{n-2}\frac{v+1}{n}\binom{n-2}{v} x^v(1-x)^{(n-2)-v} + x\frac{1}{n^2}$.
$= x^2\frac{n-1}{n}\sum_{v=0}^{n-2}\frac{v+1}{n}\binom{n-2}{v} x^v(1-x)^{(n-2)-v} + 2x^2\frac{(n-1)}{n^2} + x\frac{1}{n^2} = x^2\frac{n-1}{n}\sum_{v=0}^{n-2}\frac{v}{n}\binom{n-2}{v} x^v(1-x)^{(n-2)-v} + x^2\frac{(n-1)}{n^2}\sum_{v=0}^{n-2}\binom{n-2}{v} x^v(1-x)^{(n-2)-v} + 2x^2\frac{(n-1)}{n^2} + x\frac{1}{n^2}$
$= x^3\frac{(n-1)}{n}\sum_{v=1}^{n-3}\frac{v}{n}\binom{n-2}{v} x^{v-1}(1-x)^{(n-2)-v+1} + 3x^2\frac{(n-1)}{n^2} + x\frac{1}{n^2} = x^3\frac{(n-2)(n-1)}{n^2} + 3x^2\frac{2(n-1)}{n^2} + x\frac{1}{n^2}$

Lemma (2.2):
Let $A = [a_{n,k}(x)], n \in N, k \in N_0$ as in (1.4), $x \in I = [0,1]$ then:
$B_{n,k}\left((\frac{k}{n})^4, A, x\right) = x^4\left(\frac{(n-1)(n-2)(n-3)}{n^3}\right) + 3x^2\frac{(n-1)(n-2)}{n^3} + 2x^2\frac{(n-1)(n-2)}{n^3} + 7x^2\frac{(n-1)}{n^3} + x\frac{1}{n^3}$
Proof:
By (1.3), (1.4) and lemma (1.2) we get
$B_{n,k}\left((\frac{k}{n})^4, A, x\right) = \sum_{k=0}^{n} a_{n,k}(x).\left(\frac{k}{n}\right)^4$
$= x\sum_{k=0}^{n}\left(\frac{k}{n}\right)^3 \binom{n}{k} x^k(1-x)^{n-k} = x\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^3 \binom{n-1}{k-1} x^{k-}(1-x)^{(n-1)-(k-1)}$
As in the proof of the lemma (1.2) and (2.1) we have the following $= x^4\left(\frac{(n-1)(n-2)(n-3)}{n^3}\right) + 3x^2\frac{(n-1)(n-2)}{n^3} + 2x^2\frac{(n-1)(n-2)}{n^3} + 7x^2\frac{(n-1)}{n^3} + x\frac{1}{n^3}$

Lemma (2.3):
Let $k,n,x \in [0,b], and \in \lambda \geq 0$ then $\left|f\left(\frac{k}{n}\right) - f(x)\right| \leq (1 + \left(\frac{k}{n}-x\right)^2\lambda^{-1})\omega(f,\lambda)$
Proof:
If $\left|\frac{k}{n}-x\right| \leq \lambda$, by (1.10) we have $\omega(f, \left|\frac{k}{n}-x\right|) \leq \omega(f,\lambda)$
If $\left|\frac{k}{n}-x\right| \geq \lambda$ then $\omega(f, \left|\frac{k}{n}-x\right|) \leq \omega(f, \frac{\left|\frac{k}{n}-x\right|^2}{\lambda}$
Let $\frac{k}{n}, x \in [0,b]$, from (1.10), (1.8) we have
$\left|f\left(\frac{k}{n}\right) - f(x)\right| \leq \omega(f, \left|\frac{k}{n}-x\right|) \leq \omega(f, \frac{\left|\frac{k}{n}-x\right|^2}{\lambda} \leq 1 + \left(\frac{k}{n}-x\right)^2\lambda^{-1})\omega(f,\lambda)$

Theorem (2.1):
For every matrix $A \in \mathcal{M}$, and $s \in N$ there exists a positive constant $M_1(A,x,2s)$ independent on $x \in [0,1]$ and $n \in N$ such that : $B_{n,k}(A,x,2s) = \sum_{k=0}^{n} a_{n,k}(x).\left(\frac{k}{n} - x\right)^{2s}$ ……………………………….... (2.1)
Then
$\|B_{n,k}(A,x,2s)\| \leq \frac{M_1(A,x,2s)}{n^s}$, $n \in N$. (2.2)
Proof:
By (2.2) and (2.1), we get
$\|B_{n,k}(A,x,2s)\| = \left|\sum_{k=0}^{n} a_{n,k}(x).\left(\frac{k}{n}-x\right)^{2s}\right|$
$= \sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^{2s}\binom{n}{k}x^k(1-x)^{n-k}$
If $s = 1$ from lemma (2.1), (2.3) and (1.2), weget
$B_{n,k}(A,x,2s) = \sum_{k=0}^{n}(\frac{k}{n}-x)^2\binom{n}{k}x^k(1-x)^{n-k}$

$= \sum_{k=0}^{n}((\frac{k}{n})^2 - 2x\frac{k}{n} + x^2)\binom{n}{k}x^k(1-x)^{n-k} = \sum_{k=0}^{n}(\frac{k}{n})^2\binom{n}{k}x^k(1-x)^{n-k} - 2x\sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}x^k(1-x)^{n-k} + x^2\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}$

$\frac{x^2(n-1)}{n} + \frac{x}{n} - 2x^2 + x$

$= \frac{M_1(A,x,2s)}{n^s} \quad 0 \le x \le 1$

Now we prove the strong approximation of the functions by using the linear positive operators$B_{n,k}(f,A,x)$.

Theorem (2.2):

Suppose that $A \in \mathcal{M}$, then for $n \in N, x \in [0,1], p > 0$ we have:

$|B_{n,k}(f,A,x) - f(x)| \le H_n^1(f,x)$... (2.3)

And

$H_n^p(f,x) \le H_n^q(f,x)$ If $0 < p < q < 0$ ……………………………..….. (2.4)

Proof:

By using (1.3) and (1.6) we get

$|B_{n,k}(f,A,x) - f(x)| \le \left|\sum_{k=0}^{n} a_{n,k}(x)(f\left(\frac{k}{n}\right) - f(x))\right| \le \sum_{k=0}^{n} a_{n,k}(x)\left|f\left(\frac{k}{n}\right) - f(x)\right|$

For $0 \le x \le 1$ and lemma (1.2) ($B_{n,k}(1,A,x) - 1 = 0$), which by (1.6) yield (2.3) let $g_x\left(\frac{k}{n}\right) = f\left(\frac{k}{n}\right) - f(x)$. Applying by the holder inequality and lemma (1.1), we get

$\left(B_{n,k}\left(\left|g_x\left(\frac{k}{n}\right)\right|^p, A, x\right)\right)^{\frac{1}{p}} \le \left(B_{n,k}\left(\left|g_x\left(\frac{k}{n}\right)\right|^q, A, x\right)\right)^{\frac{1}{q}}, x \in [0,1], n \in N$ ………………………………….. (2.5)

For every $g \in C_I, 0 < p < q < \infty$ and from (1.6), (2.5) immediately follows (2.4).

Theorem (2.3):

Let $A \in \mathcal{M}, f \in C_I^1$ and$p > 0$, then there exists $M_2(A,x,2s)$ such that:

$\|H_n^p(f,A,x)\| \le \frac{M_2(A,x,2s)\|f'(x)\|}{n^{2s}}$ for all $x \in [0,1]$ and $n \in N$.

Proof:

For $f \in C_I^1$ and $t, x \in [0,1]$ we have

$|f(t) - f(x)| \le \|f'(x)\||t-x|$

From this we get

$\|H_n^p(f,A,x)\| \le \left\{\sum_{k=0}^{n} a_{n,k}(x)\left|f\left(\frac{k}{n}\right) - f(x)\right|^p\right\}^{\frac{1}{p}}, x \in [0,1], n \in N.$

$\le \|f'(x)\| (B_{n,k}(\left|f\left(\frac{k}{n}\right) - f(x)\right|^p)^{\frac{1}{p}}$

For all $x \in [0,1]$ and$n \in N$.

Which by (2.2), (2.1) and from inequality:

$\left\{L_n(\left|\frac{k}{n} - x\right|^p, A, x\right\}^{\frac{1}{p}} \le \left\{L_n(\left|\frac{k}{n} - x\right|^s, A, x\right\}^{\frac{1}{s}}$

$x \in [0,1], n \in N, 0 < p < s < \infty$

Then obtain $p \le 2s$ we have

$\|H_n^p(f,A,x)\| \le \left\{\sum_{k=0}^{n} a_{n,k}(x)\left|f\left(\frac{k}{n}\right) - f(x)\right|^p\right\}^{\frac{1}{p}}, x \in [0,1], n \in N$

$\le \|f'(x)\|\left(B_{n,k}\left(\left|f\left(\frac{k}{n}\right) - f(x)\right|^{2s}, A, x\right)\right)^{\frac{1}{2s}}$

$\le \|f'(x)\|\left(B_{n,k}\left(\left|g_x\left(\frac{k}{n}\right)\right|^{2s}, A, x\right)\right)^{\frac{1}{2s}}$

By (2.3), (2.5) and (2.2) we get

$\|H_n^p(f,A,x)\| \le \frac{M_2(A,x,2s)\|f'(x)\|}{n^{2s}}$

Theorem (2.4):

Let $A \in \mathcal{M}, f \in C_I$ and $p > 0$, then there exists $M_3(A,p,2) > 0$ for all $x \in [0,1]$and $n \in N$ such that :

$\|H_n^p(f,A,x)\| \le \frac{M_3(A,p,2)}{\sqrt{n}}\tau(f, \frac{1}{\sqrt{n}})$

Proof:

For all $f \in C_I$ and $n \in N, p > 0$ we get from (1.5)

$\|H_n^p(f,A,x)\| \le \left\{\sum_{k=0}^{n} a_{n,k}(x)\left|f\left(\frac{k}{n}\right) - f(x)\right|^p\right\}^{\frac{1}{p}}$ by (1.6), (1.7), lemma (2.3) we get

$\left|f\left(\frac{k}{n}\right) - f(x)\right| \le \omega(f, \left|\frac{k}{n} - x\right|) \le (\sqrt{n}\left|\frac{k}{n} - x\right|^2 + 1) \le \omega(f, \frac{1}{\sqrt{n}})$

for all $x \in [0,1], n \in N$. Consequently

$\|H_n^p(f,A,x)\| \le \omega(f, \frac{1}{\sqrt{n}})\left\{\sum_{k=0}^{n} a_{n,k}(x)\left|\sqrt{n}\left|\frac{k}{n} - x\right|^2 + 1\right|^p\right\}^{\frac{1}{p}}$

Applying the Minkowski inequality for sum we get

$\|H_n^p(f,A,x)\| \le \omega(f, \frac{1}{\sqrt{n}})\left\{\sum_{k=0}^{n} a_{n,k}(x)\left|\sqrt{n}\left|\frac{k}{n} - x\right|^2 + 1\right|^p\right\}^{\frac{1}{p}}$

$\le \omega\left(f, \frac{1}{\sqrt{n}}\right)\left\{\sum_{k=0}^{n} a_{n,k}(x)\left|\sqrt{n}\left|\frac{k}{n} - x\right|^2\right|^p\right\}^{\frac{1}{p}} + 1$

From (1.10) and theorems (2.3), (2.1) we have:

$\|H_n^p(f,A,x)\| \le \omega\left(f, \frac{1}{\sqrt{n}}\right)\sqrt{n}\,\frac{M_2(A,p,2)}{n}$

$\le \frac{M_3(A,p,2)}{\sqrt{n}}\omega(f, \frac{1}{\sqrt{n}})$

$\le \frac{M_3(A,p,2)}{\sqrt{n}}\tau(f, \frac{1}{\sqrt{n}})$

Corollary (1):

For all $f \in C_I$ and $n \in N, p > 0$ we have

$\mathrm{Lim}_{x\to\infty} \|H_n^p(f,A,.)\| = 0$

Implies that $\lim_{x\to\infty} H_n^p(f,A,x) = 0$ at every$x \in [0,1]$.

Corollary (2):

Let $A \in \mathcal{M}, n \in N$ and $p > 0$, then there exists $M_4(A,x,2)$ such that for every $f \in C_I$

$\|B_{n,k}(f,A,.) - f(.)\| \le \|H_n^1(f,A,.)\| \le \frac{M_4(A,.)}{\sqrt{n}}\tau\left(f, \frac{1}{\sqrt{n}}\right).$

Conclusions:

1-We prove lemma (2.1), (2.2) about the linear positive operate.

2- We fined the strong approximations by using the linear positive operators in terms of the averaged modulus of order one.

References:

1- B. S. (2009).On The Degree of Strong Approximation of Continuous Functions by Special Matrix . J. Inequal. Pure and Appl. Math. V .10: N. 4,132-139.

2- S.M. (2003).Strong Approximation by Fourier Series. International J. of pure and Appl. Math. V.7 : N.1, 59-66.

3- L. R. & M. S. (2004). On The Strong Approximation of Functions by Certain Operators. Math. J. Okayama Univ. V.46 :N.1, 153-161.

4- L. R. & M. S. (2006).On Strong Approximation by Modified Meyer-Konig and Zeller Operators. Tamkang J. of Math. V.37 : N.2 ,123-130.

5- S. B. & P. V. A. (1983). The Averaged Moduli of Smoothness.1-20,series 519.4 , A Wiley-Interscience Publication,USA.

6- L.R.&M.S.(2006).The strong approximation of functions of two variables by certain operators .Soochow J. of. Math. V.32 :N.1 ,37-49.

7- L. A. & D. D. (1989). One sided Multidimensional Approximation by Entire Functions and Trigonometric Polynomials in L_p – Metric, $0 < p \leq \infty$. Math. Balkanica V.3 , 215-224.

التقريب الاقوى بواسطة المؤثر الخطي الموجب في ضوء معدل القياس من الرتبة الاولى

زينب عيسى عبد النبي
E-mail: esazainab@yahoo.com

الخلاصة:

في بحثنا هذا قدمنا المؤثر الخطي الموجب (برنشتاين) في فضاء كل الدوال المستمرة $C_I = [0,1]$ مع بعض الخواص لهذا المؤثر وذلك لإيجاد أقوى الفروق للدوال معتمدين في ذلك على معدلات القياس من الرتبة الاولى .