On ideal supra topological space

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Abstract

In this paper we introduce via of ideal supra topological space classes of concepts (I-supra dense, *-supra dense, *-closed supra, *-perfect supra) in ideal supra topological space. At last we study and investigate some characterization of these concept for there more we introduces some continuous mapping, and there properties.

1- Introduction

The concept of ideal in topological space was first introduced by Kuratowski[10] and Vaidyananthswamy [15]. They also have defined local function in ideal topological space. Further Hamlett and Jankovic in [19] and [5] studied the properties of ideal topological spaces and introduced the notion of I-open set in topological space and they have introduced set operator called ( ) *μ , μ-local function , and introduced μ-codense ideal .[6] Further investigated I-open sets and I-continuous functions. Dontchev [9] introduced the notion of pre-I-open sets and obtained a decomposition of I-continuity. The notion of semi-I-open sets to obtain decomposition of continuity was introduced by Hatir and Noiri [7, 8]. In addition to this, Caksu Guler and Aslim [1] have introduced the notion of b-I-open sets and b-I-continuous functions. Different types of generalized open sets like semi-open[12]; preopen [3]; α-open[13] already are there in literature and these generalized sets have a common Property which is closed under arbitrary union. Mashhour et al [2] put all of the sets in a pocket and defined a generalized space which is supra topological space. In the light of the above results, the purpose of this paper is study b-I-supra open sets and b-I-supra continuous functions and to obtain several characterizations and properties of these concepts.

2- Preliminaries

Definition 2.1 [17]
A nonempty collection I of subset of X is called an ideal on X if:

i. A ∈ I and B ⊆ A implies B ∈ I (heredity);
ii. A ∈ I and B ∈ I implies A ∪ B ∈ I (finite additivity).

Definition 2.2 [4]
A sub family μ of the power set (X) of nonempty set X is called a supra topology on X if μ satisfies the following condition:

1. μ contains ø and X;
2. μ is closed under the arbitrary union.

The pair (X, μ) is called a supra topological space. In this respect, the member of μ is called supra open set in (X, μ). The complement of supra open set is called supra closed set.

Definition 2.3 [4]
The (X, μ, I) is called ideal supra topological space if (X, μ) is supra topological space and I is ideal on X.

Example 2.4
Let X = {a, b, c}, and μ = { ø, X, {a}, {b}, {a, b}, {c}, {a, b, c}}. I = { ø, {a}, {b}, {a, b}}.

Then (X, μ, I) is ideal supra topological space.

Definition 2.5 [17]
Let (X, μ, I) be an ideal supra topological space. A set operator ( ) *μ : p(X) → p(X), is called the μ-local function of I on X with respect to μ, is defined as:

(A) *μ (I, μ) = { x ∈ X: (U ∩ A) ∊ I, for every U ∊ μ(x)}

where μ(X) = {U ⊆ μ: x ∊ U}. This is simply called μ-local function and simply denoted as A *μ.

We have discussed the properties of μ-local function in the following theorem.

Theorem 2.6 [17]
Let (X, μ, I) be an ideal supra topological space, and let A, B, A₁, A₂, ..., Aₙ.

Be subsets of X, then
1. φ *μ = φ;
2. A ∩ B implies A *μ ⊆ B *μ;
3. For another ideal J ⊇ I on X, A *μ(J) ⊇ A *μ(I);
4. A *μ ⊆ cl *μ(A);
5. A *μ is a supra closed set;
6. (A *) *μ ⊆ A *μ;
7. A *μ ∪ B *μ ⊆ (A ∪ B) *μ;
8. ∪ (A *μ) ⊆ (∪(A_i)) *μ
9. (A ∩ B) *μ ⊆ A *μ ∩ B *μ;
10. For V ∈ μ , V ∩ (V ∩ A) *μ ⊆ V ∩ A *μ;
11. For I ⊆ ι, (A ∪ I) *μ = A *μ ∪ (I - A) *μ.

Following example shows that A *μ ∪ B *μ ≠ (A ∪ B) *μ does not hold in general

Example 2.7 [17]
Let X = {a, b, c, d}, μ = { ø, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, c, d}, {b, c, d}}

I = { ø, {c}}, then

Conceder A = {a, c}, B = {b, c}, then A *μ = {a, c} , B *μ = {a, b, c}

Now (A ∪ B) *μ = {a, b, c} ≠ {a, b, c, d}

Hence A *μ ∪ B *μ ≠ (A ∪ B) *μ.

Definition 2.8 [17]
An ideal I in a space (X, μ, I) is called μ-codense ideal if μ ∩ I = { ø }.

Example 2.9
Let X = {a, b, c, d}, μ = { ø, X, {a}, {b}, {c}, {d}, {a, b, c}, {a, b, c}}, I = { ø, {d}}

μ ∩ I = { ø }, then (X, μ, I) is called μ-codense ideal.
Theorem 2.10 [17]
Let \((X, \mu, I)\) be an ideal supra topological space and \(I\) is \(\mu\)-codense with \(\mu\). Then \(X = X^\mu\).

Proof:
It is obvious that \(X^\mu \subseteq X\). For converse, suppose \(x \in X^\mu\). Then there exist \(U \subseteq \mu(x)\) such that \(U \cap X \in I\) that is \(U \in I\), a contradiction to the fact that \(\mu \cap I = \{\phi\}\), hence \(X = X^\mu\).

3- b-I-supra open set.

Definition 3.1
The ideal supra interior of a subset \(A\) denoted by \(\text{int}^\mu(A)\) is the union of ideal supra open sets included in \(A\).

Definition 3.2
The ideal supra closure of a set \(A\) denoted by \(\text{cl}^\mu(A)\) is the intersection of ideal supra closed sets including \(A\).

Theorem 3.3
Let \((X, \mu)\) be an ideal supra topological space and \(A \subseteq X\), then
1. \(\text{int}^\mu(A) \subseteq A\);
2. \(A \in V^\mu\) if and only if \(\text{int}^\mu(A) = A\);
3. \(\text{cl}^\mu(A) \supseteq A\);
4. \(A\) is ideal supra closed set if and only if \(\text{cl}^\mu(A) = A\);
5. \(x \in \text{cl}^\mu(A)\) if and only if every ideal supra open set \(U\) containing \(x\), \(U \cap A \neq \phi\).

Proof:
(1) Proof is obvious from the definition of ideal supra interior.
(2) Since arbitrary union of ideal supra open sets is again an ideal supra open set, then the proof is obvious.
(3) Proof is obvious from the definition of ideal supra closure.
(4) If \(A\) is an ideal supra closed set, then smallest ideal supra closed set containing \(A\) is \(A\). Hence \(\text{cl}^\mu(A) = A\).
(5) Proof: Let \(x \in \text{cl}^\mu(A)\). If possible suppose that \(U \cap A = \phi\), where \(U\) is ideal supra open set containing \(x\). Then \(A \subseteq (X - U)\) and \(X - U\) is an ideal supra closed set containing \(A\). Therefor \(x \in (X - U)\), a contradiction.

Conversely:
Supposed that \(U \cap A = \phi\), for every ideal supra open set \(U\), containing \(x\).

If possible Suppose that \(x \in \text{cl}^\mu(A)\), then \(x \in \text{X- cl}^\mu(A)\).
Then there is a \(U_x \subseteq \mu\) such that \(U_x \in (X - \text{cl}^\mu(A))\), i.e. \(U_x \subseteq (X - \text{cl}^\mu(A)) \subseteq (X - A)\).
Hence \(U_x \cap A = \phi\), a contradiction. So \(x \in \text{cl}^\mu(A)\).

Definition 3.4
Let \((X, \mu, I)\) be an ideal supra topological space and \(x \in \mu\). Then \(\mu\) is said to be an ideal supra neighborhood of a point \(x\) of \(X\) if for some ideal supra open set \(U \subseteq \mu\), \(x \in U \subseteq \mu\).

Definition 3.5
1. A subset \(A\) of \((X, \mu, I)\) is said to be \(I\)-supra -dense for short \((I - \mu\)-dense) if \(A^\mu = X\).
2. A subset \(A\) of \((X, \mu, I)\) is said to be \(\ast\)-supra -dense for short \((\ast - \mu\)-dense) if \(A \subseteq A^\mu\).
3. A subset \(A\) of \((X, \mu, I)\) is said to be \(\ast\)-closed supra for short \((\ast - \mu\)-closed) if \(A^\mu \subseteq A\).
4. A subset \(A\) of \((X, \mu, I)\) is said to be \(\ast\)-perfect supra for short \((\ast - \mu\)-perfect ) if \(A^\mu = A\).

Remark 3.6
There exist an ideal supra topology \(\mu^\ast(I)\), finer than \(\mu\) generated by \(B(I, \mu) = \{U \subseteq I, U \in \mu \text{ and } I \subseteq I\}\) and \(\text{cl}^\mu\ast (A) = \mu^\ast(A)\) defines a kuratowski closure operator for \(\mu^\ast(I)\).

Example 3.7
Let \(X = \{a, b, c, d\}\), \(\mu = \{\phi, X, \{a\}, \{b, c, d\}, \{a, b\}\}\), \(I = \{\phi, \{b\}\}\); \(\mu^\ast(I) = \{\phi, X, \{a\}, \{b, c, d\}, \{a, b\}, \{a, c, d\}\}\), hence \(\mu^\ast\) is finer than \(\mu\).

Definition 3.8
A sub set \(A\) of a supra topological space \((X, \mu)\) and \(A \subseteq X\) and \(A\) is said to
1. supra open if \(A \subseteq \text{int}^\mu(A)\), and a subset \(K\) is called supra closed if it is complement is supra open.
2. \(\beta\)- supra open if \(A \subseteq \text{cl}^\beta(A)\).
3. pre supra open if \(A \subseteq \text{int}^\beta(A)\).
4. semi supra open if \(A \subseteq \text{cl}^\beta(A)\).
5. b- supra open if \(A \subseteq \text{cl}^\beta(\text{int}^\beta(A))\cup \text{int}^\beta(\text{cl}^\beta(A))\), [14]

The class of all semi-supra- open (pre supra-open, \(\beta\)- supra- open, b- supra- open) sets in \(X\) will be denoted by \(SO(X, \mu)\), PO(X, \(\mu\)), \(\alpha O(X, \mu)\) and \(BO(X, \mu)\) respectively.

Proposition 3.9
For a space \((X, \mu)\) the following are equivalent:
1. \(PO(X, \mu) \cup SO(X, \mu) = BO(X, \mu)\)
2. For each subset \(A \subseteq X\), \(\text{int}^\mu(\text{cl}^\mu(A)) \subseteq \text{cl}^\mu(\text{int}^\mu(A)) \subseteq \text{cl}^\mu(\text{cl}^\mu(A))\).

Proof: (1) \(\Rightarrow\) (2): Let \(\exists X\) and let \(B = (A \cap \text{int}^\mu(A)) \cup \text{cl}^\mu(\text{int}^\mu(A))\).
It easily checked that \(\text{int}^\mu(\text{cl}^\mu(A)) = \text{int}^\mu(\text{cl}^\beta(B))\).
Since \(A \cap \text{int}^\mu(\text{cl}^\mu(A)) \subseteq PO(X, \mu)\) and \(\text{cl}^\mu(\text{int}^\mu(A)) \subseteq SO(X, \mu)\), we have that \(B \subseteq BO(X, \mu)\).
By hypothesis, \(B \subseteq PO(X, \mu) \cup SO(X, \mu)\).
If \(B \subseteq PO(X, \mu)\), then \(\text{cl}^\mu(\text{int}^\mu(A)) \subseteq \text{cl}^\mu(\text{cl}^\mu(B)) = \text{cl}^\mu(\text{cl}^\mu(A))\).
If \(B \subseteq SO(X, \mu)\), then \(\text{cl}^\mu(\text{int}^\mu(A)) \subseteq \text{cl}^\mu(\text{cl}^\mu(A)) = \text{cl}^\mu(\text{int}^\mu(A)) = \text{cl}^\mu(\text{int}^\mu(A)) = \text{cl}^\mu(\text{cl}^\mu(A))\).

(2) \(\Rightarrow\) (1): Let \(S \subseteq BO(X, \mu)\).
If \(\text{int}^\mu(\text{cl}^\mu(S)) \subseteq \text{cl}^\mu(\text{cl}^\mu(S))\) and so \(S \subseteq PO(X, \mu)\).

Definition 3.10
A subset \(A\) of an ideal supra topological space \((X, \mu, I)\) is said to be
1. I-supra open if $A \subseteq \text{int}^* (A^s)$, and a subset $K$ is called I-supra closed if it is complement is I-supra open.

2. $\alpha$-I-supra open if $A \subseteq \text{int}^* (\text{cl}^* (\text{int}^* (A)))$.

3. $\beta$-pre-I-supra open if $A \subseteq \text{int}^* (\text{cl}^* (A))$.

4. semi-I-supra open if $A \subseteq (\text{cl}^* (\text{int}^* (A)))$.

5. $\beta$-I-supra open if $A \subseteq \text{cl}^* (\text{int}^* (A)) \cup \text{int}^* (\text{cl}^* (A))$.

The family of all I-supra opens (resp., $\beta$-I-supra open, semi-I-supra open, pre-I-supra open, $\alpha$-I-supra open) sets of $(X, \mu)$ is denoted by $I\mu(X)$, $\text{BIO}(X, \mu), \text{SIO}(X, \mu), \text{PIO}(X, \mu), \alpha I\mu(X, \mu)$.

**Example 3.11**

Let $X = \{a, b, c\}$, $\mu = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $I = \{\phi, \{b\}\}$. Then $A = \{a, c\}$, and $A$ is a $\beta$-I-supra open set for $\text{int}^* (\text{cl}^* (A)) \cup \text{cl}^* (\text{int}^* (A))$.

**Proposition 3.12** [14]

Every semi-I-supra open set is $b$-supra open.

**Proof:**

Let $A$ be a semi-I-supra open set in $(X, \mu)$. Then $A \subseteq \text{cl}^* (\text{int}^* (A))$.

Hence $A \subseteq \text{cl}^* (\text{int}^* (A)) \cup \text{int}^* (\text{cl}^* (A))$, and $A$ is $b$-supra open in $(X, \mu)$.

The converse of above proposition need not be true as shown by the following example:

**Example 3.13**

Let $(X, \mu)$ be a supra topological space, where $X = \{1, 2, 3\}$, $\mu = \{\phi, X, \{1\}, \{1, 2\}, \{2, 3\}\}$ and $A = \{1, 3\}$. Then $A$ is $b$-supra open set for $\text{int}^* (\text{cl}^* (A)) \cup \text{cl}^* (\text{int}^* (A))$.

**Proposition 3.14**

Every pre-I-supra open set is $b$-supra open.

**Proof:**

Let $A$ be a pre-I-supra open set in $(X, \mu)$. Then $A \subseteq \text{cl}^* (\text{int}^* (A))$.

Hence $A \subseteq \text{cl}^* (\text{int}^* (A)) \cup \text{int}^* (\text{cl}^* (A))$, and $A$ is $b$-supra open in $(X, \mu)$.

The converse of above proposition need not be true.

**Example 3.15**

Let $(X, \mu)$ be a supra topological space, where $X = \{a, b, c\}$, $\mu = \{\phi, X, \{a\}, \{a, b\}\}$ and $A = \{b\}$, then $A$ is a $b$-supra open set for $\text{int}^* (\text{cl}^* (A)) \cup \text{cl}^* (\text{int}^* (A))$.

**Proposition 3.16**

Every semi-I-supra open set is $b$-I-supra open.

**Proof:**

Let $A$ be a semi-I-supra open set in $(X, \mu, I)$. Then $A \subseteq (\text{cl}^* (\text{int}^* (A)))$. Hence $A \subseteq \text{cl}^* (\text{int}^* (A)) \cup \text{int}^* (\text{cl}^* (A))$, and $A$ is $b$-I-supra open in $(X, \mu)$. The converse of above proposition need not be true as shown by the following example:

**Example 3.17**

Let $(X, \mu, I)$ be a supra topological space, where $X = \{a, b, c\}$, $\mu = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}\}$, $I = \{\phi, \{b\}\}$, and $A = \{a, c\}$ then $A$ is $b$-I-supra open set, but it is not semi-I-supra open.

**Example 3.18**

Let $X = \{a, b, c, d\}$, $\mu = \{\phi, X, \{a\}, \{a, b\}, \{a, b, d\}\}$, $I = \{\phi, \{b\}\}$, and $A = \{b, d\}$ is $b$-I-supra open but it is not semi-I-supra open.

Because $\mu (\text{cl}^* (\text{int}^* (A))) \cup \text{cl}^* (\text{int}^* (A)) = \mu (\{b, d\}) \cup \text{cl}^* (\text{int}^* (\{b, d\})) = \mu (\text{int}^* (X)) \cup \{b\} \cup \{b\} = X \cup \{b\} = X \supseteq A$ and hence $A$ is $b$-I-supra open set.

And since $\text{cl}^* (\mu (\text{int}^* (A))) \cup \text{cl}^* (\mu (\text{int}^* (\{b, d\}))) = \{b\} \cup \{b\} = \emptyset \cup \{b\} = \{b\} \nsubseteq A$ and hence $A$ is not semi-I-supra open.

**Proposition 3.19**

For a subset of an ideal supra topological space, the following condition hold:

a. Every $b$-I-supra open set is $b$-supra open.

**Proof:**

Let $A$ be a $b$-I-supra open then $A \subseteq \text{int}^* (\text{cl}^* (A)) \cup \text{cl}^* (\text{int}^* (A))$.

Hence $A \subseteq \text{cl}^* (\text{int}^* (A)) \cup \text{int}^* (\text{cl}^* (A))$, and $A$ is $b$-supra open in $(X, \mu)$.

b. Every pre-I-supra open is $b$-I-supra open set.

**Proof:**

Let $A$ be a pre-I-supra open then $A \subseteq \text{int}^* (\text{cl}^* (A)) \cup \text{cl}^* (\text{int}^* (A))$.

Hence $A \subseteq \text{int}^* (\text{cl}^* (A)) \cup \text{cl}^* (\text{int}^* (A))$, and $A$ is $b$-I-supra open set.

c. Every semi-I-supra open is $b$-I-supra open set.

**Proof:**

Let $A$ be a semi-I-supra open then $A \subseteq \text{int}^* (\text{cl}^* (A)) \cup \text{cl}^* (\text{int}^* (A))$, and $A$ is $b$-I-supra open set.

d. SIO $(X, \mu) \subseteq \text{PIO}(X, \mu) \subseteq \text{BIO}(X, \mu)$.

**Proof:**

The proof is obvious.

**Example 3.20**

Let $X = \{1, 2, 3, 4\}$ be the ideal supra topological space by setting $\mu = \{X, \emptyset, \{2, 3, 4\}, \{2, 3, 4\}\}$, $I = \{\emptyset, \{3\}, \{3, 4\}\}$, and $A = \{1, 2\}$ is $b$-I-supra open set but it is not pre-I-supra open.

**Proposition 3.21**

Let $S$ be a $b$-I-supra open set such that $\text{int}^* (S) = \emptyset$ then $S$ is pre-I-supra open set.

**Proof:**

Since $S \subseteq \text{int}^* (\text{cl}^* (S)) \cup \text{cl}^* (\text{int}^* (S)) = \text{int}^* (\text{cl}^* (\emptyset)) \cup \text{cl}^* (\text{int}^* (\emptyset)) = \text{int}^* (\text{cl}^* (S))$, then $S$ is pre-I-supra open.

**Lemma 3.22**

Let $(X, T)$ be an ideal topological space and $A, B$ subsets of $X$. Then, the following properties hold:

a. If $A \subseteq B$, then $A^{\mu\mu} \subseteq B^{\mu\mu}$.

b. If $U \in T$, then $U \cap A^{\mu\mu} \subseteq (U \cap A)^{\mu\mu}$.
A subset A of an ideal supra topological space (X, μ, I) is said to be b-I-supra closed if its complement is b-I-supra open.

\[ \text{Proposition 3.30} \]
Let (X, μ, I) be an ideal supra topological space and let X = \{a, b, c\}, μ = \{ϕ, X, {a, b}, {b, c}\}, I = \{ϕ, {c}\}, A = \{b, c\} is a
Semi-I-supra open but it is not α-I-supra open.

\[ \text{Proposition 3.31} \]
If a subset A of an ideal supra topological space (X, μ, I) is closed.

\[ \text{Theorem 3.32} \]
If A is b-I-supra closed, then [int^μ(cl^μ(A))] \cap cl^μ(int^μ(A)) \subseteq A

\[ \text{Corollary 3.33} \]
Let A be a subset of an ideal supra topological space (X, μ, I) such that
X \subseteq [int^μ(cl^μ(A))] \cap cl^μ(int^μ(A)) and X \subseteq [cl^μ(int^μ(A))] \cap cl^μ(int^μ(A)). Hence A \subseteq cl^μ(int^μ(A)).

\[ \text{Proposition 3.33} \]
Let A be a subset of an ideal supra topological space (X, μ, I) such that
X \subseteq [int^μ(cl^μ(A))] \cap cl^μ(int^μ(A)) and X \subseteq [cl^μ(int^μ(A))] \cap cl^μ(int^μ(A)). Hence A \subseteq cl^μ(int^μ(A)).
1. Arbitrary union of b-I- supra open set is always b-I- supra open set.
2. Finite intersection of b-I- supra open sets may fail to be b-I- supra open set.
3. X is b-I- supra open set.

Proof:
1. Let A and B be two b-I- supra open sets. Then, $A \subseteq \text{int}^\mu_{(\alpha)}(A)$ and $B \subseteq \text{int}^\mu_{(\beta)}(B)$, and $A \cap B \subseteq \text{int}^\mu_{(\alpha)}(A) \cap \text{int}^\mu_{(\beta)}(B)$. Then $A \cup B \subseteq \text{int}^\mu_{(\alpha)}((A \cup B)) \cap \text{int}^\mu_{(\beta)}((A \cup B))$, therefore $A \cup B$ is b-I- supra open set.

2. In the example $X = \{a, b, c\} \subseteq \{ \phi, \{a\}, \{b\}, \{b, c\}\}$, $I = \{\phi, \{b\}\}$, and let $A = \{a\}, B = \{b\}$ be b-I-supra closed sets but their union $\{a, b\}$ is not b-I-supra open set.

Proposition 3.34
1. Arbitrary intersection of b-I- supra closed sets is always b-I- supra closed set.
2. Finite union of b-I- supra closed set may fail to be b-I- supra closed set.

Example 3.35
Let $(X, \mu, I)$ be an ideal supra topological space and $X = \{a, b, c\}, \mu = \{\phi, X, \{a\}, \{b, c\}\}$, $I = \{\phi, \{b\}\}$, and let $A = \{a\}, B = \{b\}$ be b-I-supra closed sets but their union $\{a, b\}$ is not b-I-supra closed set.

Definition 3.36
• The b-I-supra closure of a set $A$, denoted by $\text{cl}^\mu_{(\alpha)}(A)$, is the intersection of b-I- supra closed set including $A$.
• The b-I-supra interior of a set $A$, denoted by $\text{int}^\mu_{(\alpha)}(A)$, is the union of b-I- supra open sets included in $A$.

Proposition 3.37
1. $A \subseteq \text{cl}^\mu_{(\alpha)}(A)$, and $A = \text{cl}^\mu_{(\alpha)}(A)$ iff $A$ is a b-I-supra closed set.
2. $\text{int}^\mu_{(\alpha)}(A) \subseteq A$, and $\text{int}^\mu_{(\alpha)}(A) = A$ iff $A$ is a b-I-supra open set.
3. $X \setminus \text{int}^\mu_{(\alpha)}(X) = \text{cl}^\mu_{(\alpha)}(X) - A$.
4. $X \setminus \text{cl}^\mu_{(\alpha)}(A) = \text{int}^\mu_{(\alpha)}(X) - A$.

Proposition 3.38
a. $\text{int}^\mu_{(\alpha)}(A) \cap \text{int}^\mu_{(\beta)}(B) \subseteq \text{int}^\mu_{(\alpha)}(A \cup B)$
b. $\text{cl}^\mu_{(\alpha)}(A \cap B) \subseteq \text{cl}^\mu_{(\alpha)}(A) \cap \text{cl}^\mu_{(\beta)}(B)$.

Proof:
a. Let $A \subseteq A \cup B \subseteq \text{AUB}$ and $B \subseteq \text{AUB}$
implies $\text{int}^\mu_{(\alpha)}(A) \subseteq \text{int}^\mu_{(\alpha)}(A \cup B)$ and $\text{int}^\mu_{(\beta)}(B) \subseteq \text{int}^\mu_{(\alpha)}(B)$.
b. Let $\text{AUB} \subseteq \text{AUB}$
implies $\text{cl}^\mu_{(\alpha)}(A \cap B) \subseteq \text{cl}^\mu_{(\alpha)}(A) \cap \text{cl}^\mu_{(\beta)}(B)$.

Proposition 3.39
For an ideal supra topological space $(X, \mu, I)$ and $\text{AUB}$ we have:

a. If $\phi = \phi$, then $A$ is b-I-supra open if and only if $A$ is b- supra open.
b. If $I = \phi$, then $A$ is b-I-supra open if and only if $\phi \subseteq \mu$.
c. If $I = N$, then $A$ is b-I-supra open if and only if $A$ is b- supra open.

Proof:

a- Necessity is easy as diagram for Sufficiency note that in case of the minimal ideal $A^\mu = \text{cl}^\mu(\alpha)$. b- Necessity : If $A$ is a b-I-supra open set, then $A \subseteq \text{int}^\mu(A) \cap \text{int}^\mu(A) = \text{int}^\mu(A) \cup \text{int}^\mu(A) = \text{int}^\mu(A) \cup \text{int}^\mu(A)$.

Proposition 3.40
The b-I-supra open set is b-I-supra open set.

Lemma 3.40
Let $(X, \mu, I)$ be an ideal supra topological space and let $\text{AUB}$ then $\subseteq \text{U}$.

Proposition 3.41
The intersection of $\alpha$-I-supra open set and is b-I-supra open set.

Proposition 3.42
Each b-I-open and $\mu^*$-closed is semi-I-supra closed.

Proof:
Let $A = \text{b-I- open and } \mu^*$-closed set then $A \subseteq \text{int}^\mu(A) \cup \text{cl}^\mu(A) = \text{int}^\mu(A) \cup \text{int}^\mu(A) = \text{int}^\mu(A)$. 

Proposition 4.1
If $A = \text{b-I- open and } \mu^*$-closed set then $A \subseteq \text{int}^\mu(A) \cup \text{cl}^\mu(A) = \text{int}^\mu(A) \cup \text{int}^\mu(A) = \text{int}^\mu(A)$. 

Proposition 4.3
Let $A, B$ be a subset of a space $(X, \mu, I)$ such that $\text{AUB}$ $\subseteq \text{AUB}$. 

4- b-I-supra continuous mapping

Definition 4.1
a. A function $f : (X, \mu) \rightarrow (Y, \sigma)$ is called b-supra continuous if the inverse image of each supra open set in $Y$ is b- supra open set in $X$.
b. A function $f : (X, \mu) \rightarrow (Y, \sigma)$ is called pre-supra continuous if the inverse image of each supra open set in $Y$ is pre-supra open set in $X$. 

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c. A function $f: (X, \mu, I) \rightarrow (Y, \sigma)$ is called pre-I- supra continuous if the inverse image of each supra open set in $Y$ is pre-I- supra open set in $X$.

d. A function $f: (X, \mu) \rightarrow (Y, \sigma)$ is called semi- supra continuous if the inverse image of each supra open set in $Y$ is semi- supra open set in $X$.

e. A function $f: (X, \mu, I) \rightarrow (Y, \sigma)$ is called semi-I- supra continuous if the inverse image of each supra open set in $Y$ is semi-I- supra open set in $X$.

f. A function $f: (X, \mu) \rightarrow (Y, \sigma)$ is called $\alpha$- supra continuous if the inverse image of each supra open set in $Y$ is $\alpha$- supra open set in $X$.

g. A function $f: (X, \mu, I) \rightarrow (Y, \sigma)$ is called $\alpha$-I- supra continuous if the inverse image of each supra open set in $Y$ is $\alpha$-I- supra open set in $X$.

h. A function $f: (X, \mu, I) \rightarrow (Y, \sigma)$ is called b-I- supra continuous if the inverse image of each supra open set in $Y$ is b-I- supra open set in $X$.

The following examples show that b-I- supra continuous function do not need to be pre-I- supra continuous and semi-I- supra continuous and b- supra continuous function does not need to be $\alpha$-I- supra continuous.

**Example 4.2**

Let $X = Y = \{a, b, c, d\}$ be the supra topological space by setting $\mu = \sigma = \{\{a\}, \{d\}, \{a, d\}, \emptyset, X\}$ and $I = \{\emptyset, \{c\}\}$ on $X$. Define a function $f: (X, \mu) \rightarrow (Y, \sigma)$ as $f(a) = f(c) = d$ and $f(b) = f(d) = b$. Then $f$ is a b-I- supra continuous but it is not pre-I- supra continuous.

Since $A = \{a, c\}$ is b-I-supra open, $A \subset \text{int}^\mu (\text{cl}^\mu (A)) \cup \text{cl}^\mu (f^{-1}(\text{cl}^\mu (A)))$

$= \text{int}^\mu (\{a, c\}) \cup \{a\} \cup \text{cl}^\mu (\{a\}) = \text{int}^\mu (\{a, b, c\} \cup \{a\}) \cup \text{cl}^\mu (\{a\}) = \text{int}^\mu (\{a, b, c\} \cup \{a\} \cup \{a\} = \{a, b, c\} \supseteq A$.

Hence $f$ is b-I- supra continuous.

And $A$ is not pre-I-supra open $A \subset \text{int}^\mu (\text{cl}^\mu (A)) = \text{int}^\mu (\{a, b, c\} \cup \{a\}) = \text{int}^\mu (\{a, b, c\} \cup \{a\} \supseteq A$.

**Example 4.3**

Let $(X, \mu)$ be the real line with the indiscrete supra topology and $(Y, \sigma)$ the real line with the usual topology, then the identity function $f: (X, \mu) \rightarrow (Y, \sigma)$ is b-continuous but not b-I- supra continuous.

**Example 4.4**

Let $X = Y = \{a, b, c\}$ be the supra topological space by setting $\mu = \sigma = \{\emptyset, X, \{a, b\}\}$ and $I = \{\emptyset, \{c\}\}$, define a function $f: (X, \mu, I) \rightarrow (Y, \sigma)$ as follows:

$f(a) = a$, $f(b) = c$,

And $f(c) = b$, then $f$ is b-I- supra continuous but not semi-I- supra continuous.

**Remark 4.5**

If a function $f: (X, \mu, I) \rightarrow (Y, \sigma)$ is semi-I- supra continuous (pre-I- supra continuous), then $f$ is b-I- supra continuous.

**Proposition 4.6**

If the function $f: (X, \mu, I) \rightarrow (Y, \sigma)$ is b supra continuous, then $f$ is b- supra continuous.

**Proof:**

Let $A$ be b-I- supra open set, then we have

$A \subset \text{int}^\mu (\text{cl}^\mu (A)) \cup \text{cl}^\mu (\text{int}^\mu (A))$

$\subset \{\text{int}^\mu (A) \cup \text{int}^\mu (A) \cup \text{int}^\mu (A\cup A) = \text{cl}^\mu (\text{int}^\mu (A)) \cup \text{int}^\mu (A) \cup \text{int}^\mu (A) \cup \text{int}^\mu (A) \subset \text{cl}^\mu (\text{int}^\mu (A)) \cup \text{int}^\mu (A)$(A).

Hence $A$ is b-supra open set, and hence $f$ is b- supra continuous.

**Proposition 4.7**

Every supra continuous function $f: (X, \mu, I) \rightarrow (Y, \sigma)$ is pre-I- supra continuous.

The converse is not true in general as shown in the following example.

**Example 4.8**

Consider a classical example: The Dirichlet function $f: \mathbb{R} \rightarrow \mathbb{R}$

$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

Let $F$ be the ideal of all finite subsets of $\mathbb{R}$. The Dirichlet function $f: (\mathbb{R}, \mu, F) \rightarrow (\mathbb{R}, \mu)$ is pre-I- supra continuous, since every point of $\mathbb{R}$ belongs to the $\mu$-local function of the rationales with respect to $F$ and $T$ as well as to the $\mu$- local function of the irrationals. Hence $f$ is even I- supra continuous. But on the other hand The Dirichlet function is not supra continuous at any point of its domain.

**Proposition 4.9**

Every I- supra continuous function $f: (X, \mu, I) \rightarrow (Y, \sigma)$ is pre-I- supra continuous.

The converse is again not true in general as shown in the following example.

**Example 4.10**

Let $X = \{1, 2, 3, 4\}$, $\mu = \{\emptyset, X, \{1, 3\}, \{4\}, \{1, 3, 4\}\}$, and let $\sigma = \{\emptyset, Y, \{1, 3, 4\}\}$.

Let $A = \{1, 2, 4\}$. Then the identity function $f: (X, \mu, I) \rightarrow (Y, \sigma)$ is pre-I- supra continuous but not I-supra continuous.

**Example 4.11**

Every pre- I- supra continuous function $f: (X, \mu, I) \rightarrow (Y, \sigma)$ is pre- supra continuous.

**Example 4.12**

A pre-supra continuous function need not be pre-I- supra continuous.

Let $(X, \mu)$ be the real line with the indiscrete supra topology and $(Y, \sigma)$ the real line with the usual topology. The identity function $f: (X, \mu, P(X)) \rightarrow (Y, \sigma)$ is supra continuous but not pre-I-supra continuous.

**Proposition 4.13**

A function $f: (X, \mu, I) \rightarrow (Y, \sigma)$ is satisfy following:

1. Every pre- I- supra continuous function is b-I- supra continuous.
2. Every b- I- supra continuous function is b-supra continuous.
3. Every pre-supra continuous function b-supra continuous.
4. Every $\alpha$-I-supra continuous function is pre-I- supra continuous.
5. Every $\alpha$-I-supra continuous function is $\alpha$-supra continuous.
6- Every α- I-supra continuous function is semi-I-supra continuous.

7- Every semi-I-supra continuous function f: (X, μ, I) → (Y, σ) is semi-supra continuous.

\[
\begin{align*}
\alpha - \mu - \text{continuous} \\

\implies \mu - \text{continuous} & \implies \alpha - I - \mu - \text{continuous} \implies \text{semi-I-} \mu - \text{continuous} \implies \text{semi-}\mu - \text{continuous} \\

\Downarrow \\

I - \mu - \text{continuous} & \implies \text{pre-I-} \mu - \text{continuous} \implies \text{b-I-} \mu - \text{continuous} \implies \text{b-} \mu - \text{continuous}
\end{align*}
\]

Definition 4.15
A function f: (X, μ) → (Y, σ) is said to be b-supra irresolute if f^I (V) is b-supra open in (X, μ) for every b-supra open set V of (Y, σ).

Example 4.16
Let X = {1, 2, 3} = Y, μ = {ϕ, X, {1, 2}}, and let σ = {ϕ, Y, {1}, {1, 2}}, then the identity function f: (X, μ) → (Y, σ) is b-supra irresolute.

Definition 4.17
A function f: (X, μ, I) → (Y, σ, J) is said to be

- be I-supra irresolute if f^I (V) is I-supra open in (X, μ, I) for every I-supra open set V of (Y, σ, J),
- Semi-I-supra irresolute, if f^I (V) is semi-I-supra open in (X, μ, I) for every semi-I-supra open set V of (Y, σ, J).
- α-I-supra irresolute, if f^I (V) is α-I-supra open in (X, μ, I) for every α-I-supra open set V of (Y, σ, J).
- Pre-I-supra irresolute, if f^I (V) is pre-I-supra open in (X, μ, I) for every pre-I-supra open set V of (Y, σ, J).
- b-I-supra irresolute, if f^I (V) is b-I-supra open in (X, μ, I) for every b-I-supra open set V of (Y, σ, J).

Definition 4.18
Let N be a subset of a space (X, μ, I) and let x ∈ X, then N is called b-I-supra neighborhood of x if there exists a b-I-supra open set U containing x such that U ⊆ N.

Proposition 4.19
Let f: (X, μ, I) → (Y, σ, J) and g: (Y, σ, J) → (Z, ν) be two functions where I and J are ideals on X and Y, respectively. Then, gof is b-I-supra continuous if f is b-I-supra continuous and g is supra continuous.

Proof:
Let W ⊆ Z is supra open in ν since g is supra continuous then g\(^{-1}\) (W) is supra open in Y. Since f is supra continuous then f\(^{-1}\) (g\(^{-1}\) (W)) is supra open in X. Then (gof)\(^{-1}\) (W) is b-I-supra open in X. Then gof is b-I-supra continuous.

Proposition 4.20
Let f: (X, μ, I) → (Y, σ, J) and g: (Y, σ, J) → (Z, ν) be two functions where I and J are ideals on X and Y, respectively. Then:

8- Every semi-supra continuous function f: (X, μ) → (Y, σ) is b-supra continuous.

Remark 4.14
The following diagram holds for a function f: (X, μ, I) → (Y, σ):

1. gof is pre-I-supra continuous, if f is pre-I-supra continuous and g is supra continuous.
2. g is pre-supra continuous, if f is pre-I-supra continuous and g is supra continuous.
3. g is semi-I-supra continuous, if f is semi-I-supra continuous and g is supra continuous.
4. g is semi-supra continuous, if f is semi-I-supra continuous and g is supra continuous.
5. g is α-I-supra continuous, if f is α-I-supra continuous and g is supra continuous.
6. g is α-supra continuous, if f is α-I-supra continuous and g is supra continuous.

Proof: Obvious

Lemma 4.21
For any functions f: (X, μ, I) → (Y, σ), f (I) is an ideal on Y.

Definition 4.22
A functions f: (X, μ, I) → (Y, σ, J) is called b-I-supra open (resp., b-I-supra closed) if for each U ∈ μ (resp., supra closed set F) f (U) (resp., f (F)) is b-I-supra open (resp., b-I-supra closed).

Remark 4.23
Every b-I-supra open (resp., b-I-supra closed) function is b-supra open (resp., b-supra closed) and the converses are false in general.

Example 4.24
Let X = {1, 2, 3}, μ₁ = {ϕ, X, {2, 3}}, μ₂ = {ϕ, X, {1}, {2, 3}}, and I₁ = {ϕ, {3}}. Then the identity function f: (X, μ₁) → (X, μ₂, I₁) is b-I-supra open but not b-I-supra open.

Example 4.25
Let X = {a, b, c}, μ₁ = {ϕ, X, {a}}, μ₂ = {ϕ, X, {3}, {2, 3}}, and I₁ = {ϕ, {3}}. Define a function f: (X, μ₁) → (X, μ₂, I₁) as follows: f(a) = a, f(b) = f(c) = b.

Then, f is b-supra closed but not b-I-supra closed.

Definition 4.27
A functions f: (X, μ, I) → (Y, σ, J) is called semi-I-supra open (resp., semi-I-supra closed) if for each U ∈ μ (resp., supra closed set F) f (U) (resp., f (F)) is semi-I-supra open (resp., semi-I-supra closed).

b. A functions f: (X, μ, I) → (Y, σ, J) is called pre-I-supra open
There exists \( V \in \mathcal{I} \) such that \( \forall x \in \mathcal{F} \), there exists \( U \in \mathcal{B}(Y, \sigma) \) containing \( f(x) \). Hence \( f(U) = \{ V : x \in U \} \) and hence by \( f(U) \in \mathcal{B}(Y, \sigma) \).

This shows that \( f \) is b-I- supra open.

**Theorem 4.30**

Let \( f : (X, \mu, I) \rightarrow (Y, \sigma, J) \) be b-I- supra open (resp., b-I- supra closed). If \( W \) is any subset of \( Y \) and \( F \) is a supra closed (resp., supra open) set of \( X \) containing \( f^{-1}(W) \), then there exists a b-I- supra closed (resp., b-I- supra open) subset \( H \) of \( Y \) containing \( W \) such that \( f^{-1}(H) \subset F \).

**Proof**

Suppose that \( f \) is a b-I- supra open function. Let \( W \) is any subset of \( Y \) and \( F \) is a supra closed subset of \( X \) containing \( f^{-1}(W) \).

Then \( X-F \) is supra open and since \( f \) is a b-I- supra open.

**References**


حول مثاليات الفضاءات التبولوجية الفوقية

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الملخص

في هذا البحث، قمنا من خلال مفهوم مثاليات الفضاءات التبولوجية الفوقية قمنا صفو من المفاهيم (المجموعة الكثيفة الفوقية من النمط I) والمجموعة المغلقة الفوقية من النمط (المجموعة الكثيفة الفوقية من النمط I و المجموعة المغلقة الفوقية من النمط I) فنثالتين البتولوجية الفوقية واخيراً درسنا وتحرينا بعض خصائص تلك المفاهيم بالإضافة إلى ذلك قمنا بعض الدوال المستمرة وخصائصها.