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# Quasi Cyclic Phenomena of Operators on Quasi Banach Spaces

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**الظاهرة شبه الدوارية للمؤثرات على فضاءات شبه بناخ**

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**الخلاصة**

قدمنا في هذا البحث مفاهيم جديدة للظاهرة شبه الدوارية للمؤثرات على فضاءات شبه بناخ يطلق عليها شبه الدوارية الدائرية وشبه الدوارية القرصية وشبه الدوارية القرصية المشاركة، كما أعطينا مع البرهان الخصائص والمكافئات الأساسية لكل واحدة من هذه المفاهيم.

## ABSTRACT

In this paper we introduced new concepts of quasi cyclic phenomena of operators on quasi-Banach spaces, which are called the quasi cyclic, quasi disk cyclic and quasi codisk cyclic operator. So, we gave with proves of some basic properties and characteristics of each one of them.

**Keywords:** Quasi Banach spaces, Quasi cyclic, quasi circle cyclic, quasi disk cyclic and quasi codisk cyclic.

## INTRODUCTION

Let X be a separable quasi Banach space and let $T: X \rightarrow X$ be an operator (continuous linear transformation). $T$ is called quasi hypercyclic if $\exists x \in X \ni$ the orbit of x under T, which is denoted by orb $(T, x) = \{x, Tx, T^2x, \ldots\}$ is quasi dense in X. T is called quasi cyclic operator if $\exists x \in X \ni$ the linear span of orb(T, x) is quasi dense in X. also, the concept that is mid-way between quasi cyclicity and qusai hypercyclicity introduced and called a quasi supercyclicity, that is T is quasi supercyclic if $\exists x \in X \ni$ the set $\{\alpha T^n x | n \in N, \alpha \in \mathbb{C}\}$ is quasi dense in X [1].

A new concept of cyclic phenomena, which is circle cyclicity was introduced by [2], where X is a Banach space, and T is called circle cyclic if $\exists x \in X \ni$ the set $\{\alpha T^n x | n \in N, \alpha \in ₵, |\alpha| = 1\}$ is dense in X, and then another concept of cyclic phenomena, which is called disk cyclic were introduced by [3], where T is called disk cyclic if $\exists x \in \mathrm{X} \ni$ the set $\{\alpha \mathrm{T}^n x | n \in N, \alpha \in ₵, |\alpha| \leq 1\}$ is dense in X, in addition to T that called codisk cyclic if $\exists x \in X \ni$ the set $\{\alpha \mathrm{T}^n x | n \in N, \alpha \in ₵, |\alpha| \geq 1\}$ is dense in X [4]. In this paper we give the definitions of the concepts of quasi cyclic phenomena on quasi Banach space, which are called quasi circle cyclic, quasi disk cyclic and quasi codisk cyclic operator, and give with prove some basic properties and theorems about them, and give the relation among them.

## 1. Basic Definitions and Theorems

In this section, we state the basic definitions and theorems, which we need in this work.

**Definition 1.1 [6]:**

Let X be a non empty set. A function $D: \mathrm{X} \times \mathrm{X} \to R$ is said to be a quasi-metric if satisfy the following conditions $(1)\ D(x, y) \geq 0\ \ \forall x, y \in$X and $D(x, y) = 0\ iff\ x = y$. $(2) D(x, y) = D(y, x)\ \forall x, y \in$X. $(3) \exists$ a constant $c \geq 1$ such that $D(x, y) \leq c[D(x, z) + D(z, y)]\ \ \forall x, y, z \in$X. The pair (X, D) is called a quasi-metric space.

It is clear that every metric space is a quasi-metric space but the converse may be not true (it is true if c=1).

**Definition 1.2 [1]:**

Let $(X, D)$ be a quasi-metric space, then the set of all open subset of $\mathrm{X}$ form a topology on $\mathrm{X}$ called the quasi-metric topology (or a quasi-topology) induced on $\mathrm{X}$ by D.

**Definition 1.3 [1]:**

A quasi-metric space in which every Cauchy sequence is convergent is called a complete quasi-metric space.

**Definition 1.4 [1]:**

Let $(\mathrm{X}, \mathrm{D})$ be a quasi metric space and $\mathrm{A}$ be a subset of $\mathrm{X}$ then

1) The closure of $A$, which is denoted by $\bar{A}$, is a closed set containing $A$, which is formed by adding all limit points of sequences of $A$.

2) $A$ is said to be quasi dense in $X$ if $\bar{A} = X$.

**Definition 1.5 [1]:**

A quasi-metric space is called separable if it contains a countable quasi dense subset

**Theorem 1.1 [1]:**

A subset A of a quasi topological space X is quasi dense $iff$ every open subset of X contains some point of A.

**Theorem 1.2 [1]:**

Let A be a non empty subset of a quasi-metric space $(X, D)$, then $x \in \bar{A}$ $iff$ $\exists$ a sequence $\{x_n\}$ in A $\ni$ $x_n \to x$ as $n \to \infty$.

**Theorem 1.3 (Quasi Bair Category Theorem) [1]:**

Let $(X, D)$ be a non empty complete quasi-metric space. If $u_1, u_2, \dots$ are collection of countable quasi dense and open subsets of X, then $\bigcap_{n\in N} u_n$ is quasi-dense in X

**Definition 1.6 [5]**

If X is a vector space over a field F. A quasi-norm on X is a function ${}_q\|.\|: X \to R$ such that:

1) ${}_q\|x\| \geq 0\ \forall x \in X,\ {}_q\|x\| = 0 \Leftrightarrow x = 0$
2) ${}_q\|\lambda x\| = |\lambda|\ {}_q\|x\|\ \forall x \in X, \forall \lambda \in F$
3) $\exists$ a constant $c \geq 1 \ni {}_q\|x + y\| \leq c[\,{}_q\|x\| + {}_q\|y\|]\ \forall x, y \in X$.
   The pair (X, ${}_q\|.\|$) is called a quasi-normed space. We say simply that $X$ is a quasi-normed space.

**Remark 1.1 [6], [7]:**

Every quasi-normed space is a quasi-metric space by (defined $D(x, y) = {}_q\|x - y\|$), also, every normed space is a quasi normed space but the converse may not be true (it is true only if $c = 1$).

**Definition 1.7 [6], [7]:**

A quasi-normed space X is said to be a quasi Banach space if every Cauchy sequence in X is convergent.

**Note 1.1:**

1) We denote by $G_\delta$-set the countable intersection of open set, and by $N(x, r)$ the neighborhood of $x$ of radius $r$.

2) Throughout this research X will always denote a separable quasi Banach space and T is bounded linear transformation.

## 2. Quasi-Circle Cyclicity

In this section, we introduce a new concept of a quasi cyclic phenomenon and give some characterization about it.

**Definition 2.1:**

An operator T on a quasi-Banach space $X$ is called a quasi circle cyclic if there is a vector $x$ in X $\ni$ the set $\{\lambda T^k x|\ k \in N, \lambda \in ₵ \ni\ |\lambda| = 1\}$ is quasi-dense in X and vector $x$ is called quasi circle cyclic vector for T.

**Remark 2.1:**

Every quasi hypercyclic operator is quasi circle cyclic.

**Theorem 2.1:**

Let $T: X \to X$ be an operator on a quasi-Banach space X, then the following statements are equivalent:

1) T is a quasi circle cyclic.
2) For every non empty open sets $u, v$ in $X$ there are $n \in N$ and a non-zero $\alpha \in ₵ \ni\ |\alpha| = 1 \ni T^n(\alpha u) \cap v \neq \emptyset$.
3) The set of all quasi circle cyclic vectors for T is a quasi dense $G_\delta$-set.

**Proof:**

$(1) \Rightarrow (2)$ let $x$ be a quasi circle cyclic vector for T, and let $u, v$ be non empty open subsets of X, then there are $n, k \in N$, $n \geq k$ and $\mu, \lambda \in ₵, |\mu| = |\lambda| = 1 \ni T^n(\mu x) \in u$ and $T^k(\lambda x) \in v$. Put $\gamma = \mu/\ \lambda$, then $|\gamma| = 1$ and $T^{n-k}(\gamma v) \cap u \neq \emptyset$ otherwise, for all $y \in v, T^{n-k}\gamma y \notin u$. By taking $y = T^k(\lambda x)$ we get $T^n(\mu x) = T^{n-k}\gamma(T^k(\lambda x)) \notin u$ which is contradiction.

$(2) \Rightarrow (3)$ let $\{u_k\}_{k=1}^{\infty}$ be a countable base for the quasi topology on X. The vector $x$ is quasi circle cyclic for T if and only if $\{\lambda T^n|n \in N, \lambda \in ₵, |\lambda| = 1\}$ is quasi dense in X if and only if for all $k \geq 1$ there is $\lambda \in ₵, |\lambda| = 1$ and $n \in N \ni \lambda T^n x \in u_k$ if and only if $x \in \cap_k \left[\cup_{\substack{\alpha \in ₵ \\ |\alpha|=1}} \cup_n T^{-n}(\alpha u_k)\right]$, where $\alpha = 1/\lambda$. Thus $\{x \in X|x$ is quasi circle cyclic vector for $T\} = \cap_k \left[\cup_{\substack{\alpha\in₵\\|\alpha|=1}} \cup_n T^{-n}(\alpha u_k)\right]$. And since $u_k$ is open and T is continuous, then $\cap_k \left[\cup_{\substack{\alpha\in₵\\|\alpha|=1}} \cup_n T^{-n}(\alpha u_k)\right]$ is a countable intersection of open sets, then it is a $G_\delta$-set. Now, since for all $k \geq 0, \cup_{\substack{\alpha\in₵\\|\alpha|=1}} \cup_n T^{-n}(\alpha u_k)$ is quasi dense in $X$ by (2). Then by theorem (1.3), the set of all quasi circle cyclic vectors for T is a quasi dense.

$(3) \Rightarrow (1)$ clear

Now, we give another characterization of a quasi circle cyclic operator.

**Theorem 2.2:**

Let $T: X \longrightarrow X$ be an operator on a quasi-Banach space X. Then the following statements are equivalent:

1) T is a quasi circle cyclic operator.
2) For each $x, y \in$X, there are sequences $\{x_k\}$ in X, $\{n_k\}$ in $N$, $\{\alpha_k\}$ in ₵, $|\alpha_k| = 1$ for all $k \ni x_k \to x$ and $T^{n_k}(\alpha_k x_k) \to y$.
3) For each $x, y \in$X and each neighborhood $w$ for zero in X, there are $z \in$X, $n \in N$ and $\alpha \in ₵ \ni |\alpha| = 1 \ni x - z \in w$ and $T^n(\alpha z) - y \in w$.

**Proof:**

$(1) \Rightarrow (2)$ let $x, y \in$X, let $B_1 = N(x, 1)$ and $B'_1 = N(y, 1)$. By theorem ((2.1)-2), there are $\lambda_1 \in ₵, |\lambda_1| = 1$ and $n_1 \in N \ni T^{n_1}(\lambda_1 B_1) \cap B_1' \neq \emptyset$, then there is $x_1 \in B_1 \ni T^{n_1}(\lambda_1 x_1) \in B_1'$. Now, let $B_2 = N(x, 1/2), B_2' = N(y, 1/2)$. By theorem ((2.1)-2), there are $\lambda_2 \in ₵, |\lambda_2| = 1$ and $n_2 \in N \ni T^{n_2}(\lambda_2 B_2) \cap B_2' \neq \emptyset$, then there is $x_2 \in B_2 \ni T^{n_2}(\lambda_2 x_2) \in B_2'$ … … and so on. Thus we get sequences $\{x_k\}$ in X, $x_k \in B_k$ , $\{\lambda_k\}, \lambda_k \in ₵, |\lambda_k| = 1$ and $\{n_k\}, n_k \in N$ for all $\geq 1 \ni T^{n_k}(\lambda_k x_k) \in B_k'$. Thus ${}_q\|x_k - x\| < 1/k$ and ${}_q\|T^{n_k}(\lambda_k x_k) - y\| < 1/k$. Hence, $x_k \to x$ and $T^{n_k}(\lambda_k x_k) \to y$.

$(2) \Rightarrow (3)$ let $x, y \in$X and $w$ be a neighborhood for zero in X, by (2) there are sequences $\{x_k\}$ in $X$, $\{n_k\}$ in $N, \{\lambda_k\}$ in ₵, $|\lambda_k| = 1$ for all $k \ni x_k \to x$ and $T^{n_k}\lambda_k x_k \to y$. Since, $w$ is a neighborhood for zero in X, then there is $k \geq 0 \ni x_k - x \in w$ and $T^{n_k}(\lambda_k x_k) - y \in w$. Take $z = x_k,\ \alpha = \lambda_k, n = n_k$.

$(3) \Rightarrow (1)$ let $u, v$ be non empty open sets in X, let $x \in u, y \in v$ and let $B_1 = N(0,1)$ by (3) there are $z_1 \in$X$, n_1 \in N, \lambda_1 \in ₵, |\lambda_1| = 1 \ni z_1 - x \in B_1$ and $T^{n_1}\lambda_1 z_1 - y \in B_1$. Now, let $B_2 = $N$(0,1/2)$ by (3) there are $z_2 \in$X$, n_2 \in N, \lambda_2 \in ₵, |\lambda_2| = 1 \ni z_2 - x \in B_2$ and $T^{n_2}\lambda_2 z_2 - y \in B_2$ ... and so on. Then we get sequences $\{z_k\}$ in X, $\{n_k\}$ in $N$ and $\{\lambda_k\}$ in $₵, |\lambda_k| = 1$ for all $k \geq 1 \ni z_k - x \in B_k$ and $T^{n_k}\lambda_k z_k - y \in B_k$. Thus, ${}_q\|z_k - x\| < 1/k$ and ${}_q\|T^{n_k}(\lambda_k z_k) - y\| < 1/k$ for all $k$. Then we get $z_k \to x$ and $T^{n_k}\lambda_k z_k \to y$ as $k \to \infty$. Since $u, v$ are open sets containing $x, y$ respectively, then there are subsequences of $\{z_k\}$ in $u$, and $\{T^{n_k}\lambda_k z_k\}$ in $v$ say $\{s_k\}$, $\{T^{n_k}\lambda_k s_k\}$ respectively $\ni s_k \to x$ and $T^{n_k}\lambda_k s_k \to y$. Hence, there are $n \geq 0, \lambda \in ₵, |\lambda| = 1 \ni T^n(\lambda u) \cap v \neq \emptyset$. Thus, by theorem (2.1) T is quasi circle cyclic operator.

**Proposition 2.1:**

Let $x$ be a quasi circle cyclic vector for an operator T on quasi Banach space X, then

$inf\{{}_q\|T^n x\| | n \geq 0\} = 0$ and $Sup\{{}_q\|T^n x\| | n \geq 0\} = \infty$.

**Proof:**

Suppose that $inf\{{}_q\|T^n x\| | n \geq 0\} = t > 0$. Since $0 \in$ X, then (by theorem (2.2)-2) there are sequences $\{n_j\}$ in $N$ and $\{\lambda_j\}$ in ₵, $|\lambda_j| = 1$ for all $j \ni \lambda_j T^{n_j} x \to 0$, thus, ${}_q\|\lambda_j T^{n_j} x\| \to {}_q\|0\| = 0$ (see [8]), therefore, $\exists k \in N \ni {}_q\|\lambda_j T^{n_j} x\| < t$ for all $j > k$. Since $|\lambda_j| = 1$ for all $j$, then ${}_q\|\lambda T^{n_j} x\| = |\lambda_j|\ {}_q\|T^{n_j} x\| = {}_q\|T^{n_j} x\|$, thus ${}_q\|T^{n_j} x\| < t$ for all $j > k$ a contradiction. Now, assume that $sup\{{}_q\|T^n x\| | n \geq 0\} = t < \infty$. Let $y \in X \ni {}_q\|y\| > t$. Since, $x$ is a quasi circle cyclic vector for T, then (by theorem (2.2)-2) there exists sequences $\{m_j\}$ in $N$ and $\{\lambda_j\}$ in ₵,

$|\lambda_j| = 1$ for all $j \ni \lambda_j T^{m_j} x \to$y, thus $_q\|\lambda_j T^{m_j} x\| \to {}_q\|y\|$ (see[8]), therefore, $_q\|y\| \leq t$ which contradicts with $_q\|y\| > t$. Thus, $sup\{{}_q\|T^n x\| | n \geq 0\} = \infty$.

## 3. Quasi Disk Cyclicity

In this section, we introduce another concept of a quasi cyclic phenomena that stands midway between the concepts of quasi supercyclicity and quasi hypercyclicity.

**Definition 3.1:**

An operator T on a quasi-Banach space X is called quasi disk cyclic if there is a vector $x$ in $X$ such that the set $\{\alpha T^k x | k \in N, \alpha \in ₵ \ni |\alpha| \leq 1\}$ is quasi dense in X, and such a vector $x$ is said to be a quasi disk cyclic vector for $T$.

**Remark 3.1:**

Every quasi circle cyclic operator is quasi disk cyclic and every quasi disk cyclic operator is quasi supercyclic.

**Proposition 3.1:**

Let $x$ be a quasi disk cyclic vector for an operator $\mathrm{T}: \mathrm{X} \to \mathrm{X}$ on a quasi Banach space X, and let $\mathrm{S}: \mathrm{X} \to \mathrm{X}$ be an operator on $X$, such that $ST = TS$ and the range of $S$ is a quasi dense in $X$. Then $Sx$ is quasi disk cyclic vector for T. In particular if $x$ is quasi disk cyclic vector for T then $T^n x$ is a quasi disk cyclic vector for $T$ for all $n \geq 0$.

**Proof:**

Let $x$ be a quasi disk cyclic vector for T, then $\{\alpha T^n x | n \in N, \alpha \in ₵ \ni |\alpha| \leq 1\}$ is quasi dense in X, therefore,

$\overline{\{\alpha T^n (Sx) \ |n \in N, \alpha \in ₵ \ni |\alpha| \leq 1\}} =$

$\overline{\{S(\alpha T^n x) \ |n \in N, \alpha \in ₵ \ni |\alpha| \leq 1\}} =$

X

(by quasi density of the range of S). Then $Sx$ is quasi disk cyclic vector for T.

**Proposition 3.2:**

Let $T: X \to X$ be a quasi disk cyclic operator on the quasi Banach space X. Then the set of all quasi disk cyclic vectors for T is $\bigcap_k \left[\bigcup_{\substack{\alpha\in\mathbb{C}\\|\alpha|\geq 1}} \bigcup_n T^{-n}(\alpha u_k)\right]$, where $\{u_k\}_{k\in N}$ is a countable base for the quasi topology on X.

**Proof:**

The vector $x$ is a quasi disk cyclic vector for T if and only if $\{\alpha T^n x | n \in N,$
$\alpha \in \mathbb{C},\ 0 < |\alpha| \leq 1\}$ is quasi dense in X if and only if for each $k > 0$ there are $\lambda\alpha \in \mathbb{C}, 0 < |\lambda| \leq 1$ and $n \in N \ni \lambda T^n x \in u_k$ if and only if $x \in \bigcap_k \left[\bigcup_{\substack{\alpha\in\mathbb{C}\\|\alpha|\geq 1}} \bigcup_n T^{-n}(\alpha u_k)\right]$ where $\alpha = 1/\lambda$.

**Proposition 3.3:**

A non empty set of a quasi disk cyclic vectors is a quasi dense $G_\delta$-set in X.

**Proof:**

Let $x$ be a quasi disk cyclic vector for an operator T. Then (by proposition (3.1)) $\alpha T^n x$ is a quasi disk cyclic vector for $T$ for all $0 \neq \alpha \in \mathbb{C}, |\alpha| \leq 1$ for all $n \geq 0$. Thus, $[\{\alpha T^n x | n \in N, \alpha \in \mathbb{C}, |\alpha| \leq 1\} - \{0\}] \subset \{ x \in X | x$ is a quasi disk cyclic vector for T }. Hence, the set of all quasi disk cyclic vectors for T is quasi dense in X. By proposition (3.2) $\{x \in X | x \text{ is a quasi disk cyclic vectors for } T\} =$

$$\bigcap_k \left[\bigcup_{\substack{\alpha\in\mathbb{C}\\|\alpha|\geq 1}} \bigcup_n T^{-n}(\alpha u_k)\right]$$

where $\{u_k\}_{k\in N}$ is a countable base for the quasi topology on X, and since T is continuous and $\alpha u_k$ is an open set for all $k \geq 1$ and all $\alpha \in \mathbb{C}, |\alpha| \geq 1$. Therefore, $\bigcap_k \left[\bigcup_{\substack{\alpha\in\mathbb{C}\\|\alpha|\geq 1}} \bigcup_n T^{-n}(\alpha u_k)\right]$ is a quasi dense $G_\delta$-set, by theorem (1.3).

**Proposition 3.4:**

Let $T: X \to X$ be a quasi disk cyclic operator on the quasi Banach space $X$. Then the range of $T$ is quasi dense in $X$.

**Proof**

Let $R(T)$ be the range of $T$ and let $x$ be a quasi disk cyclic vector for $T$. Since, $₵orb(T,x)$-span $\{x\} \subseteq R(T)$, then it is enough to prove that $x \in \overline{R(T)}$. Let $B_1 = N(x,1)$, then there is in $B_1$ an element $\alpha_1 T^{n_1} x \neq \gamma x$ for all $\gamma \in ₵$ for some $\alpha_1 \in ₵$ and $n_1 \in N$, otherwise, there is a neighborhood $B \subseteq B_1 \ni B \cap ₵orb(T,x) = \emptyset$, which contradicts with the quasi density of the set $₵orb(T,x)$. Now, let $B_2 = N(x, 1/2)$, thus there is in $B_2$ an element $\alpha_2 T^{n_2} x$ which is different from $\alpha_1 T^{n_1} x$ and $\gamma x$ for all $\gamma \in ₵$ for some $\alpha_2 \in ₵, n_2 \in N$ ... ... and so on. Thus, we get sequences $\{\alpha_j\}$ in $₵$, $\{n_j\}$ in $N \ni \alpha_j T^{n_j} x \to x$. Therefore, $x \in \overline{R(T)}$.

The following theorem gives the characterization of the quasi disk cyclic operators.

**Theorem 3.1:**

Let $T: X \to X$ be an operator on a quasi Banach space $X$. Then the following statements are equivalent:

1) $T$ is a quasi disk cyclic operator.
2) For each non empty open sets $u, v$ there are $\alpha \in ₵, 0 < |\alpha| \leq 1$ and $n \in N \ni T^n(\alpha u) \cap v \neq \emptyset$.
3) For each $x, y \in X$ there are sequences $\{x_k\}$ in $X$, $\{n_k\}$ in $N$, $\{\alpha_k\}$ in $₵, 0 < |\alpha_k| \leq 1$ for all $k$ such that $x_k \to x$ and $T^{n_k} \alpha_k x_k \to y$.
4) For each $x, y \in X$ and each neighborhood $w$ for zero in $X$, there are $z \in X, n \in N, \alpha \in ₵, |\alpha| \leq 1$ such that $z - x \in w$ and $T^n \alpha z - y \in w$.

**Proof:**

$(1) \Rightarrow (2)$ let $\{w_k\}_{k \in N}$ be a countable base for the quasi topology on $X$. By proposition (3.2) $\{x \in X | x$ is a quasi disk cyclic vectors for T $\}$.

$= \bigcap_k \left[ \bigcup_{\substack{\alpha\in ₵ \\ |\alpha|\geq 1}} \bigcup_n T^{-n}(\alpha w_k) \right]$. Now, for all $k \geq 1$, put

$M_k = \left[ \bigcup_{\substack{\alpha\in ₵ \\ |\alpha|\geq 1}} \bigcup_n T^{-n}(\alpha w_k) \right]$, then (by proposition (3.3)) $M_k$ is quasi dense for all $k \geq 1$, since $u$ is open, then $u \cap M_k \neq \emptyset$ for all $k \geq 1$. Therefore, there are $n \in N$, $\alpha \in ₵, |\alpha| \geq 1 \ni u \cap T^{-n}(\alpha v) \neq \emptyset$ where $v = \cup\, w_i$ , $w_i \in \{w_k\}^{\infty}$. Hence, $T^n(\lambda)u \cap v \neq \emptyset$ where $\lambda = 1/\alpha$ (Note that $0 < |\lambda| \leq 1$).

$(2) \Rightarrow (3)$ let $x, y \in X$, let $B_1 = N(x,1)$, $B'_1 = N(y,1)$. By (2) there are $n_1 \in N_1, \alpha_1 \in ₵, 0 < |\alpha_1| \leq 1 \ni T^{n_1}\alpha_1 B_1 \cap B'_1 \neq \emptyset$. Thus $\exists\, x_1 \in B_1 \ni T^{n_1}\alpha_1 x_1 \in B'_1$. Now, let $B_2 = N(x, 1/2)$ and $B'_2 = N(y, 1/2)$. By (2) there are $n_2 \in N, \alpha_2 \in ₵, 0 < |\alpha_2| \leq 1 \ni T^{n_2}\alpha_2 B_2 \cap B'_2 \neq \emptyset$. Therefore, $\exists x_2 \in B_2 \ni T^{n_2}\alpha_2 x_2 \in B'_2$ ... ... and so on. Then we have get sequences $\{n_k\}$ in $N$, $\{\alpha_k\}$ in $₵, 0 < |\alpha_k| \leq 1$ for all $k \geq 1$ and $\{x_k\}$ in $X \ni x_k \in B_k$ for all $k \geq 1$ and $T^{n_k}\alpha_k x_k \in B'_k$ for all $k \geq 1$. Then ${}_q\|x_k - x\| < 1/k$ and ${}_q\|T^{n_k}\alpha_k x_k - y\| < 1/k$ for all $k \geq 1$. Thus, $x_k \to x$ and $T^{n_k}\alpha_k x_k \to y$ as $k \to \infty$.

$(3) \Rightarrow (4)$ let $x, y \in$X and let $w$ be a neighborhood for zero in X. Then by (3) there are sequences $\{x_k\}$ in X, $\{n_k\}$ in $N$, $\{\alpha_k\}$ in $₵, 0 < |\alpha_k| \leq 1 \ni x_k \to x$ and $T^{n_k}\alpha_k x_k \to y$. Then $\exists\, k \in N \ni x_k - x \in w$ and $T^{n_k}\alpha_k x_k - y \in w$. Thus, $z - x \in w$ and $T^n \alpha z - y \in w$, by take $z = x_k, n = n_k \,, = \alpha_k$.

$(4) \Rightarrow (3)$ let $x, y \in$X, and $B_1 = N(0,1)$. Then by (4) there are $z_1 \in$X$, n_1 \in N, \alpha_1 \in ₵, 0 < |\alpha_1| \leq 1 \ni z_1 - x \in B_1$ and $T^{n_1}\alpha_1 z_1 - y \in B_1$. Let $B_2 = N(0,1/2)$ then by (4) there are $z_2 \in$X$, n_2 \in N, \alpha_2 \in ₵, 0 < |\alpha_2| \leq 1 \ni z_2 - x \in B_2$ and $T^{n_2}\alpha_2 z_2 - y \in B_2$ ... ... and so on. Therefore, we get sequences $\{z_k\}$ in X, $\{n_k\}$ in $N$, $\{\alpha_k\}$ in $₵, 0 < |\alpha_k| \leq 1$ for all $k \geq 1 \ni z_k - x \in B_k$ and

$T^{n_k}\alpha_k z_k - y \in B_k$ for all $k \geq 1$. Thus $_q\|z_k - x\| < 1/k$ and $_q\|T^{n_k}\alpha_k z_k - y\| < 1/k$ for all $k \geq 1$. Therefore, $x_k \to x$ and $T^{n_k}\alpha_k x_k - y$ as $k \to \infty$ by taking $x_k = z_k$.

$(3) \Rightarrow (1)$, since X is separable quasi-Banach space, then there is a countable quasi dense set say $\{x_j\}_{j\in N}$. Set $F(j,k) = N(x_j, 1/k)$ for some $j \in N$, $k \geq 1$. We will show that the collection of all $F(j,k)$ is a base for the quasi topology on X. Let $u$ be an open set in X, let $y \in u$, thus $\exists \in < 0 \ni N(y, \in) \subseteq u$. Let $k \geq 1, k > 2 \notin$. Since $\{x_j\}_{j\in N}$ is quasi dense, $\exists j \in N \ni y \in N(x_j, 1/k) \subseteq N(y, \in) \subseteq u$. Then (by proposition (3.2)) $\{x \in$X$|x$ is quasi disk cyclic vector for T$\}$ $= \bigcap_j \bigcap_k \left[\bigcup_{\substack{\alpha\in\mathbb{C} \\ |\alpha|\geq 1}} \bigcup_n T^{-n}(\lambda F(j,k))\right]$. Therefore, (by theorem (1.3)), it is enough to prove that $\bigcup_{\substack{\alpha\in\mathbb{C} \\ |\alpha|\geq 1}} \bigcup_n T^{-n}(\lambda F(j,k))$ is quasi dense in X for all $j \in N$ and all $k \geq 1$. For a fixed $j$ and $k$, let $y \in$X, by (3) there are sequences $\{x_\ell\}$ in X, $\{\alpha_\ell\}$ in $\mathbb{C}$, $0 < |\alpha_\ell| \leq 1$ for all $\ell$ and $\{n_\ell\}$ in $N \ni x_\ell \to y$ and $T^{n_\ell}\alpha_\ell x_\ell \to x_j$. Then $\exists\, t > 0 \ni {}_q\|T^{n_\ell}\alpha_\ell x_\ell - x_j\| < 1/k$ for all $\ell > t$. Thus $T^{n_\ell}\alpha_\ell x_\ell \in F(j,k)$ for all $\ell > t$, hence, $x_\ell \in T^{-n_\ell}(1/\alpha_\ell)F(j,k)$ for all $\ell > t$. Then $x_\ell \in \bigcup_{\substack{\lambda\in\mathbb{C} \\ |\lambda|\geq 1}} \bigcup_n T^{-n}(\lambda F(j,k))$ for all $\ell > t$. Therefore, $\exists$ susequence $\{x'_\ell\}$ of $\{x_\ell\} \ni x'_\ell \in \bigcup_{\substack{\lambda\in\mathbb{C} \\ |\lambda|\geq 1}} \bigcup_n T^{-n}(\lambda F(j,k))$ and $x'_\ell \to y$. Thus, $\bigcup_{\substack{\lambda\in\mathbb{C} \\ |\lambda|\geq 1}} \bigcup_n T^{-n}(\lambda F(j,k))$ is quasi dense in $X$.

## 4. Quasi Codisk Cyclicity

In this section, we introduce another new concept of a quasi cyclic phenomena that also stands midway between the concepts of quasi supercyclicity and quasi hypercyclicity.

**Definition 4.1:**

An operator T o a quasi Banach space X is said to be quasi codisk cyclic if there is a vector $x$ in X such that the set $\{\alpha T^k x | k \in N, \alpha \in ₵ \ni |\alpha| \geq 1\}$ is quasi dense in X, and such a vector $x$ is said to be quasi codisk cyclic vector for T.

**Remark 4.1:**

Every quasi circle cyclic operator is quasi codisk cyclic. And every quasi codisk cyclic operator is quasi supercyclic.

**Proposition 4.1:**

Let T:X→X be an operator on a quasi-Banach space X. And let $x$ be a quasi-codisk cyclic vector for T. Let S:X→X be an operator on X such that $ST = TS$ and the range of S is quasi dense in X. Then $Sx$ is a quasi codisk cyclic vector for T. In particular if $x$ is a quasi codisk cyclic vector for T then so is $T^n x$ for all $n \geq 0$.

**Proof:** see proposition (3.1)

**Proposition 4.2:**

Let T:X→X be an operator on a quasi-Banach space X. If T is a quasi codisk cyclic operator then the set of all quasi codisk cyclic vectors for T is $\bigcap_k \left[ \bigcup_{\substack{\alpha \in ₵ \\ 0<|\alpha|\leq 1}} \bigcup_n T^{-n}(\alpha u_k) \right]$, where $\{u_k\}_{k \in N}$ is a countable base for the quasi topology on X.

**Proof:**

The vector $x$ is quasi codisk cyclic if and only if $\{\alpha T^n x | n \in N, \alpha \in ₵ \ni |\alpha| \geq 1\}$ is quasi dense in X if and only if for each $k > 0$ there are $\lambda \in ₵, |\lambda| \geq 1$ and $n \in N$ $\ni \lambda T^n x \in u_k$ if and only if $x \in \bigcap_k \left[ \bigcup_{\substack{\alpha \in ₵ \\ 0<|\alpha|\leq 1}} \bigcup_n T^{-n}(\alpha u_k) \right]$, where $\alpha = 1/\lambda$.

**Proposition 4.3:**

A non empty set of quasi codisk cyclic vectors is a quasi dense $G_\delta$-set in X.

**Proof:**

See the proof of proposition (3.3).

By the same way of the proof of proposition (3.4) we can prove the following proposition.

**Proposition 4.4:**

The range of a quasi codisk cyclic operator on a quasi-Banach space is quasi dense.

In the following theorem we give a characterization of a quasi codisk cyclic operator.

**Theorem 4.1**

Let $T: X \rightarrow X$ be an operator on a quasi Banach space X. Then the following statements are equivalents:

1) T is a quasi codisk cyclic operator.
2) $\forall$ non empty open sets $u, v$ of X there exist $|\lambda| \geq 1$ and $n \in N \ni T^n(\lambda u) \cap v \neq \emptyset$.
3) $\forall x, y \in X$, there are sequences $\{x_k\}$ in $X$, $\{n_k\}$ in N, $\{\lambda_k\}$ in $\mathbb{C}, |\lambda_k| \geq 1$ for all $k \ni x_k \rightarrow x$ and $T^{n_k}\lambda_k x_k \rightarrow y$.
4) $\forall x, y \in X$, and each neighborhood $w$ for zero in X, there are $z \in$X, $n \in N, \lambda \in \mathbb{C}, |\lambda| \geq 1 \ni z - x \in w$ and $T^n \alpha z \rightarrow y \in w$.

**Proof**

See the proof of theorem (2.1).

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