Numerical Solution of Ordinary Differential Equations of Fractional Order Using Variable Step Size Method

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Abstract

The main objective from this paper is to introduce and derive three methods depending on fractional Euler's method and improve the results modifying the approach by using variable step size method. [DOI: 10.22401/ANJS.00.1.19]

Keywords: Generalized Taylor's formula; fractional Euler's method; fractional differential equation; Caputo fractional derivative.

1- Introduction

In this paper we will derive the sum formulas which we have improved for the numerical solution with Caputo derivatives. The approach is a generalization of the fractional Euler's method. For the numerical solution of initial value problems of the form:

$$D_*^\alpha y(t) = f(t, y(t)),\ \ y(0) = y_0,\ \ 0 < \alpha \le 1$$

where $D_*^\alpha y(t)$ denotes the Caputo fractional derivatives operator. As it is known from the usual methods of numerical analysis, the step size is fixed in that methods during the approach of solution, but still there are some methods may be used to decrease the topical truncation error [2,16], and among such methods is the variable step size method. The numerical solution of Fractional Differential Equations (FDE's) will be found using this methods when may be consider as a new approach in this topic, the derivation of the modified fractional Euler's method is motivated by a few classic and many recent applications of FDE's. Among the classic problems we mention areas like the modeling of the behave of viscoelastic items in mechanics, [12]. More recently fractional calculus has been affected to statistical mechanics and continuum for viscoelasticity problems, fractional diffusion-wave equations and Brownian motion and many physical phenomena, [1]. Most nonlinear FDE's don't have analytical solutions, so approximations and numerical techniques should be used. The decomposition method and the variational iteration method (VIM) [4,17] are comparatively new approaches to provides an analytic approximate solution to linear and nonlinear problems, and they are in particular valuable as tools for applied mathematicians and scientists [3], because they provide quick and visible symbolical terms of analytical solutions, as good as approximate numerical solutions to the linear and nonlinear differential equations [18] are relatively new approach to provides an analytical approximate solution to linear and nonlinear problems. The analytic solution of FDE's in most cases is so difficult to be evaluated [8]; therefore numerical methods for solving FDE's are more reliable than analytic methods [14], since such type of equations has some difficulties in their definition, which can't be dealing with it easily. Then we will study the numerical solutions of FDE's using fractional Euler's method and its related methods and thane connect these methods with variable step size method to produce more accurate results.

2- Preliminaries

With FDE's, the idea of fractional derivatives we will that are adopted using Caputo's definition that may be consider as a modification of the Riemann Liouville definition and has the benefit of dealing correctly with initial value problems in that the initial conditions are given in terms of the field variables and their integer order that is the case in most physical processes. For the completion purpose of the used concepts, the Riemann Liouville fractional order derivative and integral and the Caputo derivative, will be only introduced next.

Definition 1, [10]:

A real function f(x), x > 0, in the space C_μ, $\mu \in R$ if there are exists a real number p(> μ), such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and is said in the space C_m^μ if and only if $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2, [17]:

The Riemann Liouville fractional derivative of order q of a function f is defined to be:

$$D_{a,x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(y)}{(x-y)^{\alpha+1-n}} dy, \quad x \ge a$$

where α is a positive fractional number and n is a natural number, such that $n-1<\alpha \le n$.

Definition 3, [7]:

The Riemann Liouville fractional integral operator of order $\alpha \ge 0$, of a function $f \in C_\mu, \mu \ge -1$, is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, x > 0$$

$$J^0 f(x) = f(x)$$

where Γ is the classical gamma function $m-1 < \alpha \le m$, $m \in \mathbb{N}$ and $f \in C_\mu^m$, The Riemann Liouville derivative has certain abuses when trying to model real-world phenomena with fractional differential equations. Hence, a modified fractional differential will be introduced next which is operator proposed by M. Caputo in his search on the theory of viscoelasticity, [9]. Caputo's definition has the feature of dealing correctly with initial value problems, in that the initial conditions are given in stipulation of the field variables and their integer order that is the case in most physical processes, [6].

Definition 4, [11]:

The fractional derivative of f(x) in the Caputo meaning is defined as:

$$D_*^\alpha f(x) = J^{m-a} D^m f(x) = \frac{1}{\Gamma(m-a)} \int_0^x (m-t)^{m-a-1} f^m(t) dt$$

for $m-1 < \alpha \le m$, $m \in \mathbb{N}$ and $f \in C_\mu^m$

Definition 5 (Generalized Mean Value Theorem), [13]:

Assume that f(x) ∈ C[0,a] and $D_*^\alpha f(x) \in C(0,a]$, for $0 < \alpha \le 1$. Then:

$$f(x) = f(0+) + \frac{1}{\Gamma(\alpha)} (D_*^\alpha f)(\xi) x^\alpha$$

with $0 \le \xi \le x$, $\forall\, x \in (0, a]$.

Definition 6 (Generalized Taylor's formula), [16].

Assume that $D_*^{k\alpha} f(x) \in C(0,a]$ for k = 0,1,…, n + 1, where $0 < \alpha \le 1$ then:

$$f(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} (D_*^{i\alpha})(0+) + \frac{(D_*^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} x^{(n+1)\alpha} \quad \text{............}(1)$$

with $0 \le \xi \le x$, $\forall\, x \in (0, a]$.

3- Fractional Euler's Method

For convenience, we subdivide the interval [0, a] into k subintervals $[t_n, t_{n+1}]$ of equal step size h = a/k by using the nodes points $t_n = nh$, for n = 0, 1,..., k. Assume that y(t), $D_*^\alpha y(t)$ and $D_*^{2\alpha} y(t)$ are continuous on [0, a] and use the formula (1) to expand y(t) about $t = t_0 = 0$. For each value t, there is a value c_1 so that:

$$y(t) = y(t_0) + (D_*^\alpha y(t))(t_0) \frac{t^\alpha}{\Gamma(\alpha+1)} + (D_*^{2\alpha} y(t))(c_1) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad \text{..................}(2)$$

when $(D_*^\alpha y(t))(t_0) = f(t_0, y(t_0))$ and $h = t_1$ are substituted into equation (2), the result is an expression for $y(t_1)$:

$$y(t_1) = y(t_0) + f(t_0, y(t_0)) \frac{h^\alpha}{\Gamma(\alpha+1)} + (D_*^{2\alpha} y(t))(c_1) \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} \quad \text{.................}(3)$$

If the step size h is chosen sufficiently small, then we may omission the second-order term (involving $h^{2\alpha}$) and get:

$$y(t_1)=y(t_0)+\frac{h^\alpha}{\Gamma(\alpha+1)}f(t_0,y(t_0)) \qquad \text{...........}(4)$$

The process is repeated and will generates a sequence of points an approximation to the solution y(t) at a special node point. The general fractional Euler's method at $t_{n+1} = t_n + h$, is:

$$y(t_{n+1})=y(t_n)+\frac{h^\alpha}{\Gamma(\alpha+1)}f(t_n,y(t_n)) \qquad \text{........}(5)$$

for n = 0, 1, ..., k − 1. It is clear that if $\alpha = 1$, then the explicit fractional Euler's method (5) reduces to the classical Euler's method, [15].

4- Modified Fractional Euler's Method

The modification the Fractional Euler's method may be introduced in this section. And will be addressed as the modified Fractional Euler's method from equation (3),

$$y(t_{n+1})=y(t_n)+\frac{h^\alpha}{\Gamma(\alpha+1)}y'(t_n)+\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}y''(c_1) \qquad \text{.............................}(6)$$

when the flowing abbreviation is used, $f(t_j,y(t_j))=y'(t_n)$ and $(D_*^{2\alpha}y(t))(c_1)=y''(c_1)$ derive equation (6) with integer order, yields:

$$y'(t_{n+1})=y'(t_n)+\frac{h^\alpha}{\Gamma(\alpha+1)}y''(t_n)+\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}y'''(c_1)$$

$$=y'(t_{n+1})-\frac{h^\alpha}{\Gamma(\alpha+1)}y''(t_n)-\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}y'''(c_1)$$

$$\text{...........}(7)$$

by reparation equation (7) in equation (6)

$$y(t_{n+1})=y(t_n)+\frac{h^\alpha}{\Gamma(\alpha+1)}[y'(t_{n+1})-\frac{h^\alpha}{\Gamma(\alpha+1)}y''(t_n)-\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}y'''(c_1)]+\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}y''(c_1)$$

$$=y(t_n)+\frac{h^\alpha}{\Gamma(\alpha+1)}y'(t_{n+1})-\frac{h^{2\alpha}}{(\Gamma(\alpha+1))^2}y''(t_n)-\frac{h^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}y'''(c_1)+\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}y''(c_1)$$

If the step size h is chosen small enough, then we may neglect the second-order term (involving $h^{2\alpha}$ and $h^{3\alpha}$) and get the implicit fractional Euler's method.

$$y(t_{n+1})=y(t_n)+\frac{h^\alpha}{\Gamma(\alpha+1)}y'(t_{n+1})$$

$$=y(t_n)+\frac{h^\alpha}{\Gamma(\alpha+1)}f(t_{n+1},y(t_{n+1})) \qquad \text{....}(8)$$

When $f(t_{n+1},y(t_{n+1}))=y'(t_{n+1})$ and hence, by adding equation (5) and (8) we get the implicit fractional Trapezoidal rule:

$$y(t_{n+1})=y(t_n)+\frac{h^\alpha}{2\Gamma(\alpha+1)}f(t_n,y(t_n))f(t_{n+1},y(t_{n+1}))$$

$$\text{............}(9)$$

by subtraction equation (5) from (8), we get:

$$f(t_n,y(t_n))=f(t_{n+1},y(t_{n+1})) \qquad \text{...................}(10)$$

by compensation equation (10) and (9) the explicit fractional trapezoidal rule is obtained:

$$y(t_{n+1})=y(t_n)+\frac{h^\alpha}{2\Gamma(\alpha+1)}f(t_{n-1},y(t_{n-1}))f(t_n,y(t_n))$$

$$\text{..........}(11)$$

5- Variable Step Size Method for Solving FDE's

In this section, the variable step size methods for solving FDE's will be derived that may be considered as a new approach for solving FDE's. In all fixed step-size methods, the local truncation error will depends on step size h and on the numerical method used. But, in variable step-size methods, we shall find the numerical solution y_t for the FDE,s given in equation (5), (8), (9) and (11), with $y_{t_0}=y_0$ that is accurate to within a pre-specified tolerance ε. Therefore, it turns out for acceptable effective estimates of the step-size, it is required to attain a custom local truncated error (tolerance) ε. The variable step-size method which will be consider here, is based upon comparing to between the estimates of the one and two steps of the numerical value of y_t at some time obtained by the numerical method with local truncation error term that is of the form Ch^p, where C is unknown constant and p is the order of the method. Assume that we started with the initial condition y_0 with step-size h using certain to find the solution $y^{(1)}_{t_0+h}$ and $y^{(2)}_{t_0+h}$ using the step-size h and $\frac{h}{2}$, respectively.

Let $E_{est} = \left| y^{(1)}_{t_{0+h}} - y^{(2)}_{t_{0+h}} \right|$. Hence if $E_{est.} \le \varepsilon$, then there is no problem and one may consider $y^{(2)}_{t_0+h}$ as the solution at $t_0 + h$. Otherwise if $E_{est.} > \varepsilon$, then one can to find another estimation of the step-size say h_{new} . If this approximation was accepted then this value of h_{new} will be used as the new value of h in the next step; if not, then it will be used as an old h and repeat similarly as above, [5].

Theorem (1):

Suppose the $y^{(1)}_{t_0+h}$ and $y^{(2)}_{t_0+h}$ are the numerical solution obtained y the fractional Euler's method given in equation (1) with step sizes h^{α} and $\frac{h^{\alpha}}{2}$, respectively. If ε is the tolerance and $E_{est} = \left| y^{(1)}_{t_{0+h}} - y^{(2)}_{t_{0+h}} \right|$, then the new value of the step size is giving:

$$h_{new} = \left[\frac{\varepsilon/2}{E_{est}} \right]^{\frac{1}{\alpha}} h_{old} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Proof:

Suppose Y is the actual solution at t_0+h, then

$$E_{est} = \left| y^{(1)}_{t_{0+h}} - y^{(2)}_{t_{0+h}} \right| = ch^{\alpha} - c\left(\frac{h^{\alpha}}{2}\right) = c\left(\frac{h^{\alpha}}{2}\right)$$

this given the estimate $c = \left[\frac{E_{est}}{\left(h^{\alpha}_{old}/2\right)} \right]$ since,

$$\varepsilon = ch^{\alpha}_{new} = \left[\frac{E_{est}}{h^{\alpha}_{old}/2} \right] h^{\alpha}_{new}$$

and so $h_{new} = \left[\frac{\varepsilon/2}{E_{est}} \right]^{\frac{1}{\alpha}} h_{old}$, where h_{old} refers to the old value of the step size. Similar the variable step size (12) may be obtained for methods (5), (8), (9) and (11).

6- Numerical Examples

In this section, two examples will be presented for comparison purpose between the different proposed numerical methods.

Example 1:

Our first example deal with the homogeneous linear (FDE,s)

$$D^{\alpha}_{*} y(t) = -y(t), \quad y(0) = 1,\ t > 0,\ 0 < \alpha \le 1 \ \ldots\ldots (13)$$

the exact solution of equation (13) is given by

$$y(t) = E_{\alpha}(-t^{\alpha})$$

where E_{α} is the Mittag-Leffler function defend by:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+1)}$$

while upon using the Variable Step Size Method in connection with the methods, explicit fractional Euler's, the implicit fractional Euler's, the implicit fractional Trapezoidal and the explicit fractional trapezoidal which are presented in Table (1) and Table (2).

Table (1)

Numerical values for Example 1 when $\alpha = 0.5$ and $h = 0.01$.

t	*Exact Solution*	*Absolute Errors explicit fractional Euler's*	*Absolute Errors implicit fractional Euler's*	*Absolute Errors implicit fractional Trapezoidal*	*Absolute Errors explicit fractional Trapezoidal*
0	1.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.723578	0.167213	0.155995	0.130745	0.137244
0.2	0.643788	0.147135	0.134112	0.117126	0.119564
0.3	0.592018	0.102264	0.100434	0.091129	0.092389
0.4	0.553606	0.089545	0.082064	0.074409	0.077334
0.5	0.523157	0. 079655	0.070409	0.064279	0.656235
0.6	0.498025	0.067576	0.063498	0.051468	0.052886
0.7	0.476703	0.047554	0.046608	0.033885	0.032418
0.8	0.458246	0.031198	0.027358	0.013057	0.018186
0.9	0.442021	0.020405	0.016427	0.008685	0.009423
1.0	0.427584	0.016023	0.010825	0.001385	0.005612

Table (2)
Numerical values for Example (1) when $\alpha = 0.5$ with h = 0.1 and $\varepsilon = 0.5$.

t	*Exact Solution*	*Absolute Errors explicit fractional Euler's*	*Absolute Errors implicit fractional Euler's*	*Absolute Errors implicit fractional Trapezoidal*	*Absolute Errors explicit fractional Trapezoidal*
0	1.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.723578	0.038724	0.033786	0.029024	0.029987
0.2	0.643788	0.025795	0.021898	0.019893	0.019979
0.3	0.592018	0.013606	0.012463	0.009667	0.009857
0.4	0.553606	0.010369	0.00969	0.007996	0.008215
0.5	0.523157	0.009424	0.008609	0.006298	0.006758
0.6	0.498025	0.008798	0.008184	0.005876	0.005956
0.7	0.476703	0.007209	0.006546	0.005054	0.005286
0.8	0.458246	0.00678	0.005323	0.004984	0.005054
0.9	0.442021	0.00596	0.005046	0.003243	0.003554
1.0	0.427584	0.00408	0.003928	0.002847	0.002998

Example 2:

Consider the nonlinear equation:

$$D_*^{\alpha} y(t) = y^2(t) - \frac{2}{(t-1)^2},\ y(0) = -2 \quad \text{..........}(14)$$

where $0 < \alpha \leq 1$.

The exact solution of equation (14) in case of $\alpha = 1$ is given by:

$$y(t) = -\frac{2}{(t+1)}$$

While upon using the Variable Step Size Method in connection with the methods the explicit fractional Euler's, the implicit fractional Euler's, the implicit fractional Trapezoidal and the explicit fractional Trapezoidal which are presented in Table (3) and Table (4).

Table (3)
Numerical values for Example (2) when $\alpha = 0.5$ and h = 0.01.

t	*Exact Solution*	*Absolute Errors explicit fractional Euler's*	*Absolute Errors implicit fractional Euler's*	*Absolute Errors implicit fractional Trapezoidal*	*Absolute Errors explicit fractional Trapezoidal*
0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	–0.090000	0.112765	0.105022	0.091132	0.09314
0.2	–0.160000	0.110098	0.100212	0.090122	0.092176
0.3	–0.210000	0.102565	0.100115	0.087662	0.088565
0.4	–0.240000	0.095425	0.092214	0.072029	0.073245
0.5	–0.250000	0. 096522	0.090423	0.070279	0.071001
0.6	–0.240000	0.095769	0.088363	0.061433	0.062234
0.7	–0.210000	0.08654	0.080811	0.058542	0.059461
0.8	–0.160000	0.08023	0.076352	0.043644	0.044766
0.9	–0.090000	0.07470	0.067672	0.033256	0.034121
1.0	0.000000	0.05022	0.04812	0.012520	0.013224

Table (4)
Numerical values for Example (1) when $\alpha = 0.5$ with $h = 0.1$ and $\varepsilon = 0.5$.

t	*Exact Solution*	*Absolute Errors explicit fractional Euler's*	*Absolute Errors implicit fractional Euler's*	*Absolute Errors implicit fractional Trapezoidal*	*Absolute Errors explicit fractional Trapezoidal*
0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	-0.090000	0.030651	0.029077	0.011022	0.012873
0.2	-0.160000	0.023218	0.022120	0.010542	0.011644
0.3	-0.210000	0.011678	0.010981	0.008732	0.008843
0.4	-0.240000	0.010505	0.009643	0.007632	0.077454
0.5	-0.250000	0. 009843	0.009122	0.007071	0.007112
0.6	-0.240000	0.009277	0.008921	0.006621	0.006822
0.7	-0.210000	0.008905	0.008010	0.006011	0.06243
0.8	-0.160000	0.008021	0.007263	0.005121	0.005322
0.9	-0.090000	0.007704	0.006971	0.004290	0.044322
1.0	0.000000	0.006023	0.005511	0.003522	0.003722

7. Conclusions

The fundamental objective of this work has been to construct a numerical scheme to the numerical solution of the linear and nonlinear FDE's. Those objective has been obtained by using the submitted Modified Fractional Euler's method. From the results or the Table (1) to (4) we can see the accuracy of the optioned used approaches and there are step size methods, in which the efficiency of the results is increased, and more precisely. The Variable Step Size Method approximate solution in this case is in high agreement with the exact solution.

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