Singularities of Cubic Vector Fields on the Plane

Abbas F. A. AL-shimmar - Electrical Engineering Department, College of Engineering -AL-Mustansiriya University

2007/4/ 2006 البحث قبول تاريخ 25 :البحث تقديم تاريخ /9/ 3

الخلاصة

الهدف الرئيسي من هذا البحث هو البرهنة على إن عائلة بعض الحقول المتجهية التكعيبية على المعرفة منطو ثنائي البعد ، جميع النقاط الانفرادية ذات المسارات المميزة يمكن تحديدها تبولوجيا" بمجموعة منتهية . بالإضافة إلى ذلك برهنا بأن نقطتين منفردتين (R2 ,0r,X) و (R2 ,0r,X) تكون متكافئتين من النمط C_0 بعد النفخ القطبي لهما , بشرط وجود مسارات مميزة .

ABSTRACT

The main aim of this paper is to prove that a families of some cubic vector fields on 2 – dimensional manifolds ,all singularities with characteristic orbits are topologically determined by a finite set ,moreover we prove that two singularities (R^2,0 v ,X) and (R^2,0 v ,X) are C_0 - equivalent after polar blowing up ,provided there is characteristic orbits.

Key words. Normal forms, bifurcations, blowing up, singularity of vector Fields.

INTRODUCTION

The qualitative analysis of vector fields on the plane has been a subject of major interest in pure and applied mathematics. In (1), Poincaré laid the foundations of these fields. A major contribution was then the work of Andronov In (2), we propose an overview of 2D vector field topology taken from the qualitative theory of second order dynamic system, the singularities may occurs in linear vector fields, An affine linear vector field X is described in the following way

$\dot{X} = A X$

Or

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \ldots$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \textbf{ where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

And is uniquely determined by its Jacobian (or gradient matrix) at the location of its possible zero. That is depending on the eigenvalues of the matrix A, integral curves of X may have different aspects over the plane, the classification is given in ((3),(9),(10)). In this paper, First, we introduce the basic notations required for the qualitative analysis of vector fields, Second, We focus on nonlinear cubic vector fields determined by a finite set called the normal form, third, gives the polar blowing up of some cubic polynomial vector fields. In the following, we consider a steady vector fields defined on the plane.

1 Basic Concept:

1.1 Definition: A" singularity of a C^k-vector field" is a triple (R^n, p, X) such that X is a C^k-vector field on R^n with the property X (p) = 0 and p is a singular point of the vector fields X.

1.2 Definition of Germ: Two vector fields X and Y on R^n with X (0) =Y (0) =0 are germ-equivalent in 0, if they coincide on some neighborhood of 0. The equivalence class for this equivalence relation are called germ of vector fields in 0.Let G_n denoted the set of germs of C^∞ vector fields on R^n.

1.3 Definition of k-jet: Let $\tilde{X}, \tilde{Y} \in G_n$,then $\tilde{X}$ and $\tilde{Y}$ are k-jet equivalent (with $0 \le k \le \infty$) if for some(and hence for all) representatives of X all partial derivative up to and including the order k of the component functions in 0 coincide with those of Y .

This equivalence classes are called k-jets, the k-jet of C^∞ vector field X on 0 being denoted by $J^k(X)(0)$.

1.4 Definition of C^0 -equivalence: n G Y ~ , X ~ ∈ **are topologically equivalent if for some (and hence for all) representatives X and Y of** Y ~ nd a X ~ **,there exist neighborhoods U and V of 0 in R^n ,and a homeomorphism h : U→ V ,mapping integral curves of X to integral curves of Y ,preserving the "sense" but not necessarily the parameterization, if we denote the flow of a vector field X as usual by**

$\Phi_X : D \subset R \times R^n \to R^n$ **,then the condition in this definition means that if** $p \in U$ and $\Phi_X(p,[0, t_1]) \subset U, t_1 > 0$ **,then there is some** t_2 **>0 such that**

$h(\Phi_X(p,[0,t_1])) = \Phi_X(h(p),[0,t_2])$ **.**

1.5 Definition of C^0 -conjugacy: Let X and Y be two (germs of) vector fields with X(0)=Y(0)=0 .We say that X and Y are "(locally) C^0- conjugated) " if there exists a homeomorphism h in a neighborhood V of 0 such that if p$\in$ V **and t**$\in$R **have the property** $\Phi_X(p, t)\in V$ **, then** $h(\Phi_X(p,t)) = \Phi_Y(h(p),t))$ **.**

1.6 Definition of characteristic orbit: We say that a vector field X on R^n ,

X (0) =0, has a" characteristic orbit" in 0, if for some V there exists an integral curve $t \to \Phi_X(y_0, t)$ **remaining in V for t** ≥ 0 **(or for t** ≤ 0**) and such that**

$\Phi_X(y_0 , t)$ **>0,**$\forall t \geq 0$ (resp.$\forall t \leq 0$ **),**

$\Phi_X(y_0 , t) \to 0$,**For** $t \to \infty$ (resp. $t \to -\infty$ **),**

2 Normal forms of cubic vector fields:

Normal form theory (see (4), (5)), is a technique for transforming the ordinary differential equations describing nonlinear dynamical systems into standard form. Using a particular class of coordinate transformations, One can remove the inessential part of higher order nonlinearities, the standard development of normal form theory involves several technical assumptions (often restricted to homogenous polynomials), Normal forms allows the use of restricted class of coordinates transformations (typically homogeneous polynomials) to put the bifurcations found in nonlinear dynamical system into a few standard forms.

Now let X is a C^k -vector field on R^n with X (0) =0 .it is possible to bring the i– jet of X (with i ≤ k) in a simple form by a C^∞- change of coordinates. Let X_1 be the vector field, whose component functions are linear, and such that $J^1(X_1)(0) = J^1(X)(0)$ **. Let** H^h **denote the vector space on R^n given by homogeneous polynomials of degree h .Let** h h

$[X_1, -]_h : H \to H$ **be the linear map which assign to each Y**
$\in H_h$ **the Lie product** $[X, Y]_1$ **. We now consider the splitting**
$H_h = B_h \oplus G_h$ **,where** B Im([X ,]) 1
$_h = -$ **and** G_h **is some complementary**
space.
Taken's normal form theorem (see (4)):
Let X, X_1**,** B_h **and** G_h **be as above, then for** $i < k < \infty$**, there is**
a C^∞ **- diffeomorphism** $\varphi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ **such that** $X' = \varphi^*(X)$ **is of**
the form

$X' = X_1 + g_2 + g_3 + \ldots + g_l + R_l$ **where** $g_i \in G_i$**, i=1,2,…,** l **and**
$J_l(R_l)(0)=0$ **,** X' **is called the normal form of the vector field X.**
Now we say about all families of vector fields X on $\mathbf{R}^n$ **with**
J1(X) J2(X) 0
v
= = **and** $J_3(X) \neq 0$ **are a cubic vector fields**
Theorem (2.1): Let X be a C^k **-cubic vector field (for** $k \geq 4$**) with X (0)**
=0 and its
3-jet at 0
v **is** $J_3(X)(0) =$ **(0, a** x^3 **), a**$\in$ **R\ {0}, then there is a**
diffeomorphism
h: ($\mathbf{R}^2$,0 v **)** $\to$**(**$\mathbf{R}^2$,0 v **) such that**
$h_* f =$ **(0, a** x^3 **)+ (**ΣΣ ΣΣ
= = = =
α − β −
r
i 6
2
k 0
r
i 6
2
k 0
k k i k
i
k k i k
i x y , x y **) +** R_r**, where**
, k R
i
k
$\alpha_i\ \beta \in$ **and** R_r **is a vector field with zero r-jet.**

3 Polar Blowing Up:

3.1. Construction

The blowing – up method,(see (5),(6),(7),(8)),is the technique we use to decompose a singularity into a simple singularities. In its different appearances, has already been use quite often in studying

singularities of vector fields. However, I would like to give here a rather extended description of this method.

Let X: (R 2 ,0 v **) →(R**2 ,0 v **) be a vector field such that X (** 0) 0
v v
= **, and**
Φ :**R**2 →**R**2**be**
a diffeomorphism on R2 **such that** -1 X D X ~ or X) X ~ Φ* (= = Φ Φ **, Where**

∂
∂Φ
∂
∂Φ
Φ =
x
D x
2
1

∂
∂Φ
∂
∂Φ
y
y
2
1

If the vector field X has the property that j1(X) = J2 (X) = 0 **and**
j 0) X (3 ≠ **,then we divide** 2 r by X ~ **,Which makes sense .In the sequel, by "blowing up" we will always mean the construction leading to**
X ~ r
X 1k = **, where k is the largest integer such that** jk (X)(0) = 0 **.Write**
r
X 1(,r) r 2 (,r) ∂
∂
+ ζ α
∂α
∂
= ζ α **, where**
r
and
∂
∂
∂α
∂ **are such that**

) ((,r))
x
y
y
( )(,r) (x Φ α
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−
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α =
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Φ* ,

) ((,r)),
y
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x
(r)(,r) (x Φ α
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+
∂
∂
α =
∂α
∂
Φ*

3.2. Calculations Concerning Blowing Up

In this section we derive some formulas which we need later on. Let X be a C^{∞} - vector field on R^2 have the property j_1(X) = J_2 (X) = 0 and j $_3$(X) ≠ 0 , we blow up X in 0.The vector field X **induces a vector field.**

S^1 × **{0}:**X_0 **=** $_1$(,0)( ) ∂α
∂
ζ α **Where**S^1 = {(x, y) : x^2 + y^2 = 1} **, Looking at the Taylor expansion of X at 0 it is clear that**

ζ α = + α + α α + α
+
= Σ
k 1 i k 1-i
i
k 1
i
k 1
i 0
$_1$(,0) (b cos - a sin) cos sin

And)]

y
) (b x y)(
x
[(a x y)(k i k i
i
k
i 0
k i k i
i
k
k o i 0 ∂
∂
+
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∂ −
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=
∞
=
Σ Σ Σ **represent the** $\infty -$ jet **of X in 0. The singularities on** $S_1 \times$ **{0} are exactly the points where** $\zeta_1(\alpha,0) = 0$ **.**

Proposition. (See (6)) $\zeta_1(\alpha,0)$ **Satisfies one of the following conditions:**

I $\zeta_1(\alpha,0)$ **is nowhere zero.**

II $\zeta_1(\alpha,0)$**=0 has a finite number of solutions (at most 2k+4).**

III for every $\alpha \in [0,2\pi[$, $\zeta_1(\alpha,0)$ **=0.**

Whenα_0 **is a solution of** $\zeta_1(\alpha,0)$ **=0, it is clear that** $\alpha_0 + \pi$ **is:** $\zeta_1(\alpha_0 + \pi,0)$ **Furthermore, the orbits in the neighborhood of** $(\alpha_0 + \pi,0)$ **are analogous to the orbits of**X **in the neighborhood of** $(\alpha_0,0)$ **. The only possible difference is the sense .in order to have topology information about both. Two such singularities will be called equal up symmetry, and we will talk about a "symmetry pair of singularities."**

If

r
X 1(,r) r 2(,r) ∂
∂
+ ζ α
∂α
∂
= ζ α **is a** C^∞ **-vector field, we can consider its Taylor expansion in a singularity. We check in which way the** $j_k(X)(\alpha,0)$ **are related with the** $\infty -$ jet **of X in 0.**

We see that

$j_\infty(X)(\alpha,0) =$)
r
[(a (,0) r)( ) (b_n (,0) ir n i)(
i
n

i 0
n 1
n 1
n i
i
i 0 ∂
∂
+ α α
∂α
∂
α α −
=
−
∞
=
∞
=
Σ Σ Σ
With

(,0)
n! r
a (,0) 1 n i i 0
1
n
0
n
i α
∂ ∂α
∂ ξ
α = − **,**
(,0)
n! r
b (,0) (n i) n i 1 i 0
2
n 1
0
n
i α
∂ ∂α
− ∂ ξ
α = − +
−
.
Two remarks are pertinent:
(1) n. 0 b_{nn}
= ∀
(2) The n-jet of X **in a point of** S_1 × **{0} depends entirely with (k+n-1) –jet of X in 0.**
The formulas for the 1-jet of X **in a point** (α,0)**are**
α = ξ α = α + α α α

∂
∂ ξ + + +
+
= Σ
k 1 i k 1-i
i
k 1
i
k 1
i 0
2
2)(,0) (,0) (cos . a sin . b)cos .sin
r
(r
This will be called the "radial eigenvalue" of X **in**(α,0) **.**
+ + − α × α α
α = α + α α + − α α
∂
∂ξ
+ − −
+ + +
+
= Σ
k 1 3 i 1 k i
i
k 1 2
i
k 1 2
i
k 1 3
i
k 1
i 0
1
(k 1 i)b cos] cos .sin
)(,0) [i a sin - (i 1) b) sin cos - a (k 2 i)sin cos
r
(
This will be called the "tangential eigenvalue" of X **in**(α,0) **.**
Furthermore)(,0) 0
r
(r 2 α =
∂
∂ ξ **,and**
α = α α α α
∂
∂ξ + + + −
+
= Σ
k 2 i k 2 i
i
k 2
i
k 1

i 0
1)(,0) [b cos - a sin] cos .sin
r
(
Taken's theorem (see (4)): there is a C ∞ -vector field X ~ **on** $S_1 \times \mathbf{R}_2$ **such that**)) q ((X)) q (X ~ Φ_* (= Φ **for all q**$\in$ $S_1 \times \mathbf{R}_2$ **,**
r
r) r , (X ~
∂
∂
+
∂α
∂
α = α
• •
Where f
r
1
2 = α
•
and f = ⟨ X , (-y,x) ⟩
g
r
r = 1
•
and g = ⟨ X, (x, y) ⟩
Because f ,g are in general position ,X $_{S_1\times\{0\}}$ **is a Morse-smale system and in each point** (ϕ ,0)$\in S_1 \times\{0\}$ o **where** X **is zero ,** X **has a hyperbolic singularity ,if Y has the same k-jet as X and is at least** $\mathbf{C_{k+1}}$**,and if Y and** Y **are defined analogous to X and** X **,then all the above remarks concerning** X **also hold for** Y **.this means that we can make a homeomorphism h of neighborhood** $\mathbf{U_1}$**of** $S_1 \times\{0\}$**in** $S_1 \times\{r \in R, r \ge 0\}$ **Onto another such neighborhood** $\mathbf{U_2}$ **such that h maps integral curves of** X **to integral curves of** Y **,i.e. if**

$p \in U_1$ and $D_X(p,[0, t_1])$, $t_1 > 0$ **,is contained in** $\mathbf{U_1}$ **, then there is a t** $_2$**> 0 such that** $h(D_X(p,[0, t_1])) = D_Y(h(p),[0, t_2])$ **.**
Using h, we construct a $\mathbf{C_0}$ **-equivalence between the two germs by taking:** $\phi : \Phi(U_1) \to \Phi(U_2)$ **defined by** $\phi(0) = 0$ and $\phi(p) = \Phi h \Phi_{-1}(p)$ **for p**$\neq 0$, where $\Phi_{-1}(p)$ **has to be chosen so that it's r-coordinate is positive .The fact that** ϕ **is a homeomorphism which sends X integral curves to Y integral curves follows immediately.**

Theorem (3.1):
Let X be a cubic vector field on $\mathbf{R}_2$ **, with** $J_1(X)$ $J_2(X)$ 0
v
= = **,such that**

it's J (X) (x3 xy2 ,3y3 - x2y)
3 = + **,then for k=2 , there is**
Φ : S1 ×**R**2 →**R**2 **such that the polar blowing up** X **of the vector field X has the following form:**
X =
r
2 cos sin (sin2 - cos2) r (cos4 3 sin4)
∂
∂
+ α + α
∂α
∂
α α α α

4 Proof of the results:

Proof of theorem (2.1)

We define a suitable vector space Hi−2 **to determine a basis for the space G**i **.**

Let Y (0, xi k 2yk) , k 0,1,2,...,i - 2
k = = − −
n 1
Z (x y ,A x y) ,n 0,1,2,...,i - 3 ,A m n
i n 3 n 1
n
i n 2 n
n +
= − − − − + = =
Z (yi 2 ,0)
i 1
−
− =
The set i 2
n n 0
i 2
{Yk}k 0 {Z } −
=
−
= ∪ **form a standard basis for the vector space H**i−2 **,to determine the space B**i **, we compute the following calculations:**

[(0, a x3 **), Y**k **] = (0, a k x**i−k+1 yk−1)
[(0, a x3 **), Z**n **] = (a n x**i−n+1yn−1**,0)**
[(0, a x3 **), Z**i−2 **] = (a (i-2) x**3 yi−3 ,−3ax2yi−2 **)**

The space H i **generated by B**i **and some complementary set, hence we choose G**i **such that**
Gi={X : X (x y , x y)} ,where , k R
i
k
i
k k i k

i
2
k 0
k k i k
i
2
k 0
= α β − α β ∈
=
−
=
Σ Σ **,and by**
Taken's theorem [4] ,there is a C ∞ **- homeomorphism function h :**
(R2 ,0 ∨ **) →(R**2 ,0 ∨ **) such that h**∗ **f = (0,a** x3 **) +**

(Σ Σ Σ Σ
= = = =
α − β −
r
i 6
2
k 0
r
i 6
2
k 0
k k i k
i
k k i k
i x y , x y **) + R**r **,where** , R k
i
k
αi β ∈ **and R**r **is a vector field with zero r-jet.**

Proof of theorem (3.1)

Let X be a C4 **-vector field on R**2 **,with**X (0) X (0) 0 1 2 = = **, and**
J (X) (x3 xy2 ,3y3 - x2y)
3 = + **, consider the following calculation:**
1 k 3 3 (,r) X , (y, x)
r
(, r) 1 + α ζ α = ⟨ − ⟩ **, (the symbol** ⟨ , ⟩ **represents the inner product)**
= (,r)
3 2 3 2
4 (x xy ,3y x y),(y, x)
r
1
α ⟨ __________+ + − ⟩
= { x y xy 3xy x y}
r
1 3 3 3 3

4− − + −
= {2y 2x y}
r
1 3 3
4−
=2 cosα sin α(sin2 α − cos2 α)
ζ α = ⟨(x + xy ,3y − x y), (x, y)⟩
r
(, r) 1 3 2 3 2
2 4
= {x x y 3y x y }
r
1 4 2 2 4 2 2
4+ + −
=cos4 α + 3sin4 α
So the polar blowing up X **is**
r
X 1(,r) r 2 (,r) ∂
∂
+ ζ α
∂α
∂
= ζ α
r
(x xy ,3y x y),(x, y)
r
(x xy ,3y x y) , (-y,x) 1
r
1 3 2 3 2
3
3 2 3 2
4∂
∂
+ ⟨ + − ⟩
∂α
∂
= ⟨ + − ⟩
=
r
(x x y 3y - x y)
r
(- x y x y 3xy - x y) 1
r
1 4 2 2 4 2 2
3
3 3 3 3
4∂
∂
+ + +
∂α

∂
− +
=
r
(2cos sin (sin2 - cos 2)) r (cos4 3sin4)
∂
∂
+ α + α
∂α
∂
α α α α
The singularities on $S_1 \times \{0\}$ **are exactly the points where** $\zeta_1(\alpha,r) = 0$
So the points where $2\cos\alpha \sin\alpha(\sin^2\alpha - \cos^2\alpha)$ **= 0 are** ,0),
2
,0),(
4
(0,0),(
π π

,0)
2
,0),(3
4
(,0),(5
π π
π **, to define the linear part of the vector field**
$X(X_1, X_2) =$ **at the above singular points, we calculate the following matrix**
(,r)
2 2
1 1
r
X X
r
X X
α
∂
∂

$\partial\alpha$
∂
∂
∂
$\partial\alpha$
∂
=
(,r)
0 cos 4 3sin4
2 cos 4 0
α
α + α
− α

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