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Strongly Uniform Extending Modules

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Abstract

In this paper, we introduced and studied the concept of strongly uniform extending modules, an R-module M is called strongly uniform extending (or M has (1-SC1) condition) if every uniform submodule of M is essential in a stable (fully invariant) direct summand of M. As a proper stronger than uniform extending modules and as a generalization of strongly extending modules and give some properties of such modules in analogy with properties for strongly extending modules. We have the following implications:
Strongly extending modules ⇒ strongly uniform extending modules ⇒ Uniform extending modules.

Keywords: strongly uniform extending, strongly uniform continuous and strongly uniform quasi-continuous.

الخلاصة

لتكن R حلقة و M مقاسا معرفا على R. يقال للمقاس M بانه توسع (منتظم) اذا كان كل مقاس جزئي (منتظم)من M يكون جوهري من مركبة جمع مباشر من M. في 2007 سعد الساعدي عرض و درس صنف من المقاسات كمفهوم اقوى من مقاسات التوسع. المقاس M انه مقاس التوسع بشده , اذا كان كل مقاس جزئي من Mيكون جوهري من مركبة جمع مباشر مستقر من M. في هذا البحث , تم عرض و دراسة مفهوم مقاسات التوسع المنتظم بشدة كمفهوم اقوى من مقاسات التوسع المنتظم وتعميم لمقاسات التوسع بشدة .نقول عن المقاس M انه مقاس توسع منتظم بشدة, او (1-SC1)اذا كان مقاس جزئي منتظم من M يكون جوهري من مركبة جمع مباشر مستقر من M.
تم الحصول على المؤديات الفعلية:
مقاسات التوسع بشدة ⇐ مقاسات التوسع المنتظم بشدة ⇐ مقاسات التوسع المنتظم
تم أعطاء العديد من التشخيصات ,النتائج و الخواص لمفهوم مقاسات التوسع المنتظم بشدة.

Introduction

Extending modules have been studied a lot in newly years and many generalizations have been considered by S.H.Mohamed and B.J.Muller [9], who gave several characterizations and properties of such modules. IN 2007 S.A.Al-Saadi [2] introduced a class of modules which is stronger properly than extending module. Following [6] an R-module is called uniform extending every uniform submodule in M is essential in a direct summand. A submodule N of an R-module M is said to be essential if every non-zero submodule of M has non-zero intersection with N [10]. Also, an R-module M is called uniform if $M \neq 0$ and any two non-zero submodules of M have a non-zero intersection[10]. A submodule V of an R-module M is called closed in M, if it has no proper essential extension in M [10] By Zorn's lemma, any submodule of M is contained in a maximal essential extension (a closed submodule) in M. In this paper definition, notations, examples, and fundamental results of strongly uniform extending modules are introduced.

Definition 1.1: M is strongly uniform-extending modules (or M has ($1\text{-}SC_1$) condition) if every uniform submodule of M is essential in a stable direct summand of M.

Notices and Examples 1.2:
(1) Every strongly uniform extending is uniform extending, but the invert is not true in

general. For example, the Z-module $M = Z \oplus Z_2$ is uniform extending [6]. But M is not strongly uniform extending. Since $Z \oplus 0 \cong Z$ and Z is uniform Z-module, then $Z \oplus 0$ is uniform direct summand of M. But $Z \oplus 0$ is not stable submodule of M. Let $f: Z \oplus 0 \longrightarrow M$ by $f(x, \bar{0}) = (0, \bar{x})$ for each $(x, \bar{0}) \in Z \oplus 0$. Clearly, f is Z-homomorphism. But $f(Z \oplus 0) \not\subset Z \oplus 0$.

(2) Every strongly extending is strongly uniform extending. But the inverse is not true in general.

(3) Every uniform module is strongly uniform extending. In particular, Z_8 as Z-module is strongly uniform extending.

(4) The inverse of (3) is not true in general. For example, Z_6 as Z-module is strongly uniform extending which it is not uniform.

(5) Every semisimple fully stable module is strongly uniform extending. In particular, Z_{10} as Z-module is fully stable and semisimple so is strongly uniform extending.

(6) The Z-module $Z_8 \oplus Z_2$ is not strongly uniform extending, since the submodule N={(0,0),(2,1), (4,0), (6,1)} is uniform closed but not stable direct summand.

(7) If M_x is a uniform submodule of M such that $M_x \cong M_y$. Then M_y is uniform.

Proposition 1.3: M is strongly uniform extending modules if and only if every uniform closed of M is a stable direct summand of M.

Now, we consider the properties of decompositions of strongly uniform extending module.

Theorem 1.4: M is strongly uniform extending modules if and only if for each uniform submodule N of M, there is a direct decomposition $M = M_1 \oplus M_2$ such that $N \subseteq M_1$, where M_1 is stable submodule of M and $N \oplus M_2$ is essential of M.

Proof: ($\Longrightarrow$) Suppose that M is strongly uniform extending R-module. Let N be a uniform submodule of M. Thus N is essential in a stable direct summand (say) X of M (i.e.) $M = X \oplus K_1$, where K_1 is a submodule of M. Also, since N is essential in X and K_1 is essential in K_1, thus $N \oplus K_1$ is essential in $X \oplus K_1 = M$. Hence $N \oplus K_1$ is essential submodule of M.

($\Longleftarrow$) Let N be a uniform submodule of M. By hypothesis, there is a direct decomposition $M = M_1 \oplus M_2$ such that $N \subseteq M_1$ where M_1 is a stable submodule of M and $N \oplus M_2$ is essential in M. We claim that N is essential in M_1. Let X be non-zero submodule of M_1, hence X is a submodule of M, so $(N \oplus M_2) \cap X \neq (0)$ (since $N \oplus M_2$ is essential in M). Let $x = n + m_2 (\neq 0)$, where $x \in X$, $n \in N$ and $m_2 \in M_2$, thus $m_2 = x - n$ which implies $m_2 \in M_1 \cap M_2 = (0)$, therefore, $0 \neq x = n \in X \cap N$, then $X \cap N \neq (0)$, hence N is essential in M_1. Thus M is strongly uniform extending R-module.

By the same argument in (Remark (2.1.11)[2]). Thus, we have the following characterization of strongly uniform extending modules.

Proposition 1.5: M is strongly uniform extending modules if and only if every uniform submodule of M is essential in a fully invariant direct summand of M.

Remark 1.6: For proposition (1.5) it can be restated that all results with "stable direct summand" being replaced by "fully invariant direct summand".

In the following result, we discuss when a submodule of strongly uniform extending is strongly uniform extending.

Proposition 1.7: A closed submodule (and hence direct summand) of strongly uniform extending module is strongly uniform extending.

Proof: Let X be a closed submodule of strongly uniform extending R-module M. Let Y be a uniform closed submodule of X. Since X is closed submodule of M, then Y is uniform closed submodule of M. Now, since M is strongly uniform extending, thus Y is a stable direct summand of M. Now, since $Y \subseteq X$, then Y is direct summand of X. Also, we claim that Y is a stable submodule of X. Let $f: Y \longrightarrow X$ be any R-homomorphism and consider the sequence $Y \xrightarrow{f} X \xrightarrow{i} M$, where i is the inclusion mapping. Then $(i \circ f): Y \longrightarrow M$ and since Y is stable of M then, $(i \circ f)(Y) \subseteq (Y)$. So $f(\mathrm{Y}) \subseteq Y$. Then Y is a stable direct summand of X.Therefore, X is strongly uniform extending.

Proposition 1.8: Every submodule V of a strongly uniform extending R-module M with property that the intersection of V with any stable direct summand of M is stable direct summand of V, is strongly uniform extending.

Proof: Let X be a uniform submodule of V. Since M is strongly uniform extending and X is a submodule of M, then there is a stable direct summand W of M such that X is essential in W. But $X \subseteq W \cap V \subseteq W$, thus X is essential in $W \cap V$ and by hypothesis, $W \cap V$ is a stable direct summand of V. Hence, V is strongly uniform extending.

A direct sum of strongly uniform extending need not be strongly uniform extending. In fact, Z_8 and Z_2 are strongly uniform extending Z-module (by Notices and Examples (1.2)(3) and (5)) but $Z_8 \oplus Z_2$ is not strongly uniform extending Z-module.

In the next results, we discuss when a direct sum of strongly uniform extending module is strongly uniform extending module.

Theorem 1.9: Let $M = M_1 \oplus M_2$ where M_1 and M_2 are both strongly uniform extending. Then M is strongly uniform extending if and only if every uniform closed submodule X of M with $X \cap M_1 = 0$ or $X \cap M_2 = 0$ is stable direct summand.

Proof: The necessary condition is valid by Proposition (1.7). Conversely, let L be a uniform closed submodule of M. By Zorn's lemma there exists a complement submodule Y of L such that $L \cap M_2$ is essential in Y and also since L is a closed submodule of M, so Y is a closed submodule of M. Clearly, since $(L \cap M_2) \cap M_1$ is essential in $Y \cap M_1$ so $Y \cap M_1 = 0$. Then by hypothesis, $M = Y \oplus Y'$ for some submodule Y' of M and Y is a stable direct summand of M. Now, $L = L \cap M = L \cap (Y \oplus Y') = Y \oplus (L \cap Y')$. So $(L \cap Y')$ is closed in M since $(L \cap Y')$ is closed submodule of L. Moreover, $L \cap Y'$ is uniform. Also, $(L \cap Y') \cap M_2 = 0$. By hypothesis, $(L \cap Y')$is a stable direct summand of M and hence of Y' (since, $(L \cap Y') \subseteq Y'$). Thus, $Y' = (L \cap Y') \oplus X$, where X is a submodule of Y'. Now, $M = Y \oplus Y' = Y \oplus \left((L \cap Y') \oplus X\right) = (Y \oplus (L \cap Y')) \oplus X = L \oplus X$.Also since Y and $L \cap Y'$ are stable submodule of M and $L = Y \oplus (L \cap Y')$, then L is stable of M. So L is a stable direct summand of M. Therefore, M is strongly uniform extending.

Proposition 1.10: Let $M = \bigoplus_{i \in I} M_i$ be an R-module, where M_i is a submodule of M for each $i \in I$. Then, the following statements are equivalent:

(1) M is strongly uniform extending;

(2) Each M_i is strongly uniform extending and each uniform closed submodule of M is fully invariant;

(3) Each M_i is uniform extending and each uniform closed submodule of M is fully invariant.

Proof: (1) $\Longrightarrow$ (2) By using proposition (1.7) together with proposition (1.3).

(2) $\Longrightarrow$ (3) It is clear.

(3) $\Longrightarrow$ (1) Let U be a uniform closed submodule of M and $\pi_i : M \longrightarrow M_i$ be the natural projection mapping on M_i for each $i \in I$. Let $x \in U$, then $x = \sum_{i \in I} m_i$, where $m_i \in M_i$ and hence $\pi_i(x) = m_i$. Now, since U is uniform closed submodule of M, then by hypothesis, U is fully invariant and hence $\pi_i(U) \subseteq U \cap M_i$.So $\pi_i(x) = m_i \in U \cap M_i$ and hence $x \in \bigoplus_{i \in I} (U \cap M_i)$.Thus $U \subseteq \bigoplus_{i \in I} (U \cap M_i)$. Since always $\bigoplus_{i \in I} (U \cap M_i) \subseteq U$ Therefore $U = \bigoplus_{i \in I} (U \cap M_i)$. Now, $U \cap M_i$ is direct summand of U, then $U \cap M_i$ is closed in U. Since U is uniform, then $U \cap M_i$ is uniform. But U is uniform closed in M, thus $U \cap M_i$ is uniform closed in M. Since $U \cap M_i \subseteq M_i \subseteq M$, then $U \cap M_i$ uniform closed in M_i. By uniform extending property of M_i, $U \cap M_i$ is a direct summand of M_i. Thus $U = \bigoplus_{i \in I} (U \cap M_i)$ is a direct summand of $M = \bigoplus_{i \in I} M_i$ [8]. Thus, U is a fully invariant

direct summand of M. Therefore, by proposition (1.5), M is strongly uniform extending. □

An R-module M is uniform if and only if M is (strongly) extending and indecomposable ([2]) [9]. Also, we mentioned that, every uniform module is strongly uniform extending but the inverse is not true in general.

In verity, we do not know whether indecomposabilty property is sufficient to make strongly uniform extending module is uniform.

Here, we have the next result.

Proposition 1.11: Let M be an indecomposable R-module which contains a uniform submodule. If M is strongly uniform extending, then M is uniform.

Proof: Let V be a uniform submodule of M. By Zorn's lemma, there exists a closed submodule N of M such that V is essential in N. Since an essential extension of uniform is uniform by [5] so, N is a uniform submodule of M. Now, by strongly uniform extending property of M, so N is a stable direct summand of M. But, M is indecomposable and $N \neq (0)$ since it is uniform, so $N = M$. Hence M is uniform.

Corollary 1.12: Let M be an R-module which contains a uniform submodule. Then M is uniform if and only if M is strongly uniform extending and indecomposable.

Remark 1.13: Note that the property of a module which contains a uniform submodule is not sufficient to make strongly uniform extending imply uniform property. For example, Z_6 as Z-module is strongly uniform extending which has uniform submodule $\{\bar{0}, \bar{2}, \bar{4}\}$, but it is not uniform.

Following [2], the concept of SS-modules introduced as a generalization of fully stable modules. An R-module M is called SS-module if every direct summand of M is stable.

We introduce the following concept which will play as a link between stronger uniform extending modules and uniform extending modules.

Definition 1.14: An R-module M is called U-SS-module if, every uniform direct summand of M is stable.

Notices and Examples 1.15:

(1) Every uniform module is U-SS-module, since the only uniform direct summand of a uniform R-module M is M.

(2) The inverse of (1) is not true in general. For example, the Z-module Z_{10} is U-SS-module but it is not uniform.

(3) If M uniform extending module, then the following conditions are equivalent:

(a) M is U-SS-module;

(b) Every uniform closed submodule of M is stable;

(c) Every uniform closed submodule of M is fully invariant.

(4) Every strongly uniform extending module is U-SS-module. Since in strongly extending R-module M every uniform closed is stable direct summand.

By using the concepts of U-SS-module, we give the next characterization of strongly uniform extending modules.

Lemma 1.16: M is strongly uniform extending module if and only if M is uniform extending and M is U-SS-module.

An R-module M is 1-quasi-continuous if M has uniform extending (has (1-C_1) condition) and the condition (1-C_3): If two uniform direct summands have zero intersection, then their direct sum is a direct summand [4]. Here, we shall use the term uniform quasi-continuous modules instead of 1-quasi-continuous modules.

Moreover, we call, an R-module M has the condition (1-C_2): Every uniform submodule of M which is isomorphic to a direct summand of M is a direct summand of M. Also, we insert the next concept as follows:

Definition 1.17: An R-module M is called uniform continuous (or 1-continuous), if it satisfies the conditions (1-C_1) and (1-C_2).

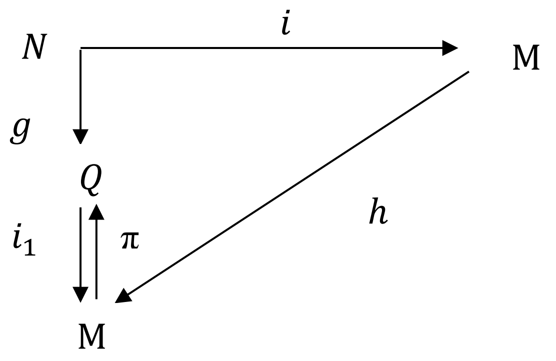
Notices and Examples 1.18:

(1) Every (quasi-)continuous module is uniform (quasi-)continuous module. But the

invert is not true in general. For example, let X be a division ring and V be a vector space over X of infinite dimension, S=End (V) and let $R = \begin{pmatrix} S & S \\ S & S \end{pmatrix}$. Then R is uniform continuous (uniform quasi-continuous) since R are uniform extending and C_2 but it is not (quasi-) continuous since R is not C_1 [10].
(2) If an R-module M has (1-C_2) condition, then M has (1-C_3) condition.
(3) As a consequence of (2), we have every uniform continuous module is uniform quasi-continuous. But the inverse is not true in general. For example, Z as Z-module.
In the next result, we assert that uniform continuous property inherited by direct summands.

Proposition 1.19: A direct summand of uniform continuous is uniform continuous.
Proof: Let P be a direct summand of uniform continuous R-module M. Then P has (1-C_1) condition since M has (1-C_1) condition [3]. Now, to prove that P has the condition (1-C_2). Let X be a uniform submodule of P such that P isomorphic to a direct summand Y of P. Put $P = Y \oplus B$ for some submodule B of P. Also, let $M = P \oplus P_1$ for some submodule $P_1$of M. Then, we have $M = P \oplus P_1 = (Y \oplus B) \oplus P_1 = Y \oplus (B \oplus P_1)$. Hence Y is direct summand of M. But $X \cong Y$ and M has (1-C_2) condition so X is direct summand of M. Since $X \subseteq P$, then X is direct summand of P. Then P has (1-C_2) condition. So P is uniform continuous.
In the next result, we get a characterization of modules has (1-C_2) condition.

Proposition 1.20: For any R-module M, the following statements are equivalent:
(1) M has (1-C_2).
(2) For any uniform submodule N of M which is isomorphic to a direct summand of M, each R-homomorphism $f: N \to M$ can be extended to an R-endomorphism of M.
Proof:(1) $\Longrightarrow$ (2) Directly, by using the known fact every homomorphism of direct summand of M in to M can be extended to endomorphism of M.
(2) $\Longrightarrow$ (1) Let N be an uniform submodule of M which isomorphic to a direct summand Q of M. We can consider the following homomorphisms, $i: N \to M$ be the inclusion mapping $g: N \to Q$ be an isomorphism, $\pi: M \to Q$ the natural projection of M onto Q and, $i_1: Q \to M$ be the inclusion mapping. So, we have $\pi \circ i_1: Q \to Q$ and $(\pi \circ i_1)(q) = \pi(i_1(q)) = \pi(q) = q$ for each$q$ in Q, (i.e.) $\pi \circ i_1 = I_Q$ the identity of Q.



By hypothesis, the homomorphism$(i_1 \circ g)$ can be extend to an R-endomorphism $h: M \to M$, hence $(h \circ i)(n) = h(i(n)) = h(n) = (i_1 \circ g)(n)$ for n in N. Hence $i_1 \circ g = h \circ i$ .So $(g^{-1} \circ \pi \circ h \circ i)(n) = (g^{-1} \circ \pi)(h \circ i)(n) = g^{-1} \circ (\pi \circ i_1) \circ g)(n) = (g^{-1} \circ g)(n) = n$ for each n in N, (i.e.) $g^{-1} \circ \pi \circ h \circ i = I_N$. Take $\alpha = g^{-1} \circ \pi \circ h$, then $\alpha \circ i = I_N$ (by [7]) we get that i is a split monomorphism, hence $Im(i) = N$ is a direct summand of M, therefore, M has (1-C_2) condition.

Corollary 1.21: Let M be an R-module, then the following statements are equivalent:
(1) M is uniform continuous
(2) M has (1-C_1) condition and for each uniform submodule N of M which isomorphic to a direct summand of M, each R-homomorphism $f: N \longrightarrow M$ extends to an R-endomorphism of M.

Now, we are ready to consider the following conditions.

(1-SC$_1$): Every uniform submodule of M is essential in a stable direct summand of M.

(1-SC$_2$): Every uniform submodule of M which is isomorphic to a direct summand of M is a stable direct summand of M.

(1-SC$_3$): If two uniform direct summand of M have zero intersection, then their sum is a stable direct summand of M.

As stronger concepts of uniform (quasi-) continuous modules and as a popularization of strongly (quasi-) continuous modules we insert the following concepts:

Definition 1.22: An R-module M is called strongly uniform continuous, if it satisfies the conditions (1-SC$_1$) and (1-SC$_2$).

Definition 1.23: An R-module M is called strongly uniform quasi-continuous, if it satisfies the conditions (1-SC$_1$) and (1-SC$_3$).

Notices and Examples 1.24:

(1) Every strongly (quasi-)continuous module is strongly uniform (quasi-) continuous.

(2) Every strongly uniform (quasi-) continuous module is uniform (quasi-) continuous. But invert is not true in general. For example, consider $M = Z_{p^\infty} \oplus Z_{p^\infty}$ as Z-module is uniform (quasi)continuous. But, M is not strongly uniform (quasi-)continuous since M does not satisfy (1-SC$_1$) condition.

(3) Every uniform module is strongly uniform quasi-continuous.

(4) The inverse of (3) is not true in general. For example, Z_6 as Z-module is strongly uniform quasi-continuous which is not uniform.

(5) If an R-module M has (1-SC$_2$) condition, then M has (1-SC$_3$) condition (the proof is essentially the same as that the proof (1-C$_2$) $\Longrightarrow$ (1-C$_3$) in Lemma (1.18)).

(6) By using (5), we have every strongly uniform continuous module is strongly uniform quasi-continuous but the inverse is not true in general (see (7)).

(7) Z as Z-module is uniform and hence it is strongly uniform quasi-continuous, while it is not strongly uniform continuous, since Z_Z does not satisfy (1-SC$_2$) condition. In fact, $2Z$ is uniform submodule of Z such that $2Z \cong Z$, but $2Z$ is not stable direct summand of Z as Z-module.

In the next result, we obtain a characterization of a module has (1-SC$_2$) condition.

Lemma 1.25: An R-module M has (1-SC$_2$) condition if and only if M has (1-C$_2$) condition and M is U-SS-module.

By using the Lemma (1.25) and Lemma (1.16), we have the following characterization of strongly uniform continuous module.

Proposition 1.26: An R-module M is strongly uniform continuous if and only if M is uniform continuous and U-SS-module.

Lemma 1.27: Let M be U-SS-module which has (1-C$_3$) condition. Then M has (1-SC$_3$) condition. By using Lemma (1.16), Lemma (1.25) and [1], we have the following:

Proposition 1.28: M is strongly uniform continuous module if and only if M satisfies the condition (1-SC$_1$) and (1-C$_2$).

Proposition 1.29: M is strongly uniform continuous module if and only if M satisfies the condition (1-C$_1$) and (1-SC$_2$).

Proposition 1.30: M is strongly uniform quasi-continuous module if and only if M satisfies the condition (1-SC$_1$) and (1-C$_3$).

Proposition 1.31: M is strongly uniform quasi-continuous module if and only if M satisfies the condition (1-C$_1$) and (1-SC$_3$).

The next result asserts that U-SS-modules property inherited by direct summands.

Proposition 1.32: Every direct summand of U-SS-module is U-SS-module.

Proof: Let M be U-SS-module and X be direct summand of M. Let Y be a uniform direct summand of X. Since X is direct summand of M, then Y is uniform direct summand of M. By U-SS-module, we have Y is a stable submodule of M. We claim that Y is a stable of X. Let $f: Y \longrightarrow X$ be any homomorphism and

$i: X \longrightarrow M$ is the inclusion mapping. Then $(i \circ f): Y \longrightarrow M$ and so $(i \circ f)(Y) \subseteq Y$ (i.e.) $f(Y) \subseteq Y$. Thus Y is a stable submodule of X. Hence X is U-SS-module.

Corollary 1.33: A direct summand of strongly uniform continuous module is strongly uniform continuous.

Proposition 1.34: A direct summand of strongly uniform quasi- continuous is strongly uniform quasi-continuous.

Proof: Let U be a direct summand of strongly uniform quasi-continuous R-module M. Then U has (1-SC_1) condition since M has (1-SC_1) condition [7]. Now, to prove that U has the condition (1-SC_3). Let X and Y are uniform direct summand of U such that $X \cap Y = 0$. Put $U = X \oplus A_1$ and $U = Y \oplus A_2$ for some submodules A_1 and A_2 of U. Since U is a direct summand of M, then $M = U \oplus N_1$ for some submodule N_1 of M. Hence, $M = U \oplus N_1 = X \oplus A_1 \oplus N_1$ and $M = U \oplus N_1 = Y \oplus A_2 \oplus N_1$. So, we have X and Y are direct summand of M and $X \cap Y = 0$. Thus by hypothesis since M has strongly uniform quasi-continuous then $X \oplus Y$ is stable direct summand of M (i.e.), $M = (X \oplus Y) \oplus K$ for some submodule K of M and $X \oplus Y$ is direct summand of M. Let $M = (X \oplus Y) \oplus K$ for some submodule K of M and since $X \oplus Y \subseteq U$, then $X \oplus Y$ is direct summand of U. So $X \oplus Y$ is direct summand of U. Now, to prove $(X \oplus Y)$ is stable of U. Let $f: (X \oplus Y) \rightarrow U$ be any R-homomorphism and consider the sequence $(X \oplus Y) \xrightarrow{f} U \xrightarrow{i} M$, where i is the inclusion mapping. Then $(i \circ f): (X \oplus Y) \longrightarrow M$ and since $(X \oplus Y)$ is stable of M, then $(i \circ f)(X \oplus Y) \subseteq (X \oplus Y)$. So $f(X \oplus Y) \subseteq (X \oplus Y)$. Then $(X \oplus Y)$ is stable of U. Hence $(X \oplus Y)$ is a stable direct summand of U. Thus, U has (1-SC_3). Therefore, U is strongly uniform quasi-continuous.

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