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ELASTIC INSTABILITY OF FRAMES WITH NONLINEAR TAPERED MEMBERS

S. Z. Al-Sarraf

Building and Construction Department-University of Technology

Wisam Victor Yousif

Civil Engineering Department-University of Baghdad

Abstract

Modified stability functions for a nonlinear taper beam-columns having wide flange, box section, and other cross-sectional shapes are developed and tabulated for different values of axial load parameters, end depth ratios and modified shape factors. They made possible the rapid prediction of the elastic critical load of structure with non-prismatic members using a hand-computing method.

Notation:

A	: Shear stiffness factor
C	: Carry over factor
E	: Young's modulus
G	: Shear modulus
I	: Moment of Inertia
I_2	: Moment of inertia at end 2
I_1	: Moment of inertia at end 1
L	: Member length
M	: Bending Moment
M_1, M_2	: Bending moments at end 1 and 2 of the member
Q_e	: Euler load $(Q_e)_2 = \pi^2 EI_2/L^2$
Q	: Axial load
S	: Stiffness factor
$\bar{S}$	: Modified stiffness factor
S_1	: Modified stability function at end 1
S_2	: Modified stability function at end 2
SC	: Carrying factor of the modified stability function
V	: Shear factor
$f_1, f_2, \ldots f_6$	: Parameters function
k	: Stiffness of the strut
m	: Shape factor
$\overline{m}$	: Modified Shape factor
u	: Taper ratio
y	: Deflection
λ	: Non-linearity factor
θ_1, θ_2	: Rotational at end 1 and 2 of member
ρ	: Axial load parameter

Introduction

The modified slope-deflection equations [6, 10, 12] associated with an elastic prismatic strut shown in Figure (1) are: -

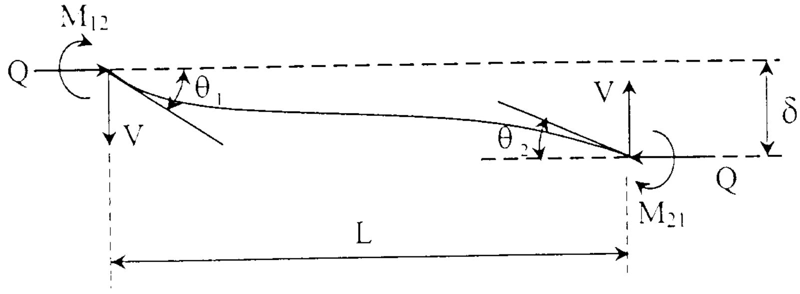


Figure (1): Prismatic strut

$$M_{12} = k\left(S\theta_1 + SC\theta_2 - S(1+C)\frac{\delta}{L}\right) \tag{1-a}$$

$$M_{21} = k\left(SC\theta_1 + S\theta_2 - S(1+C)\frac{\delta}{L}\right) \tag{1-b}$$

and

$$V = \frac{k}{L}\left(S(1+C)\theta_1 + S(1+C)\theta_2 - 2A\frac{\delta}{L}\right) \tag{1-c}$$

where

M_{12} : the clockwise moment in strut 12 at end 1

V : the end shear force

θ_1 : the clockwise rotation of end 1

δ : the clockwise displacement of end 2 perpendicular to strut 12

$k = EI/L$: the stiffness of the strut

E : Young's modulus

I : the moment of inertia

L : the length

S : the stiffness factor

SC : the moment carry-over factor

A : the shear stiffness factor

The stability functions S, SC and A are functions of the non-dimensional parameter $\rho = Q/Q_e$, where Q is the axial load and Q_e is the Euler load $\pi^2 EI/L^2$ and are extensively tabulated elsewhere[9].

For the non-prismatic strut shown in Figure (2), the stability functions of the two ends differ and the modified slope-deflection equations [3, 5] become: -

$$M_{12} = k_2\left(S_1\theta_1 + \overline{SC}\theta_2 - (S_1 + \overline{SC})\frac{\delta}{L}\right) \tag{2-a}$$

$$M_{21} = k_2\left(\overline{SC}\theta_1 + S_2\theta_2 - (S_2 + \overline{SC})\frac{\delta}{L}\right) \tag{2-b}$$

$$V = \frac{k_2}{L}\left((S_1 + \overline{SC})\theta_1 + (S_2 + \overline{SC})\theta_2 - 2\overline{A}\frac{\delta}{L}\right) \tag{2-c}$$

where

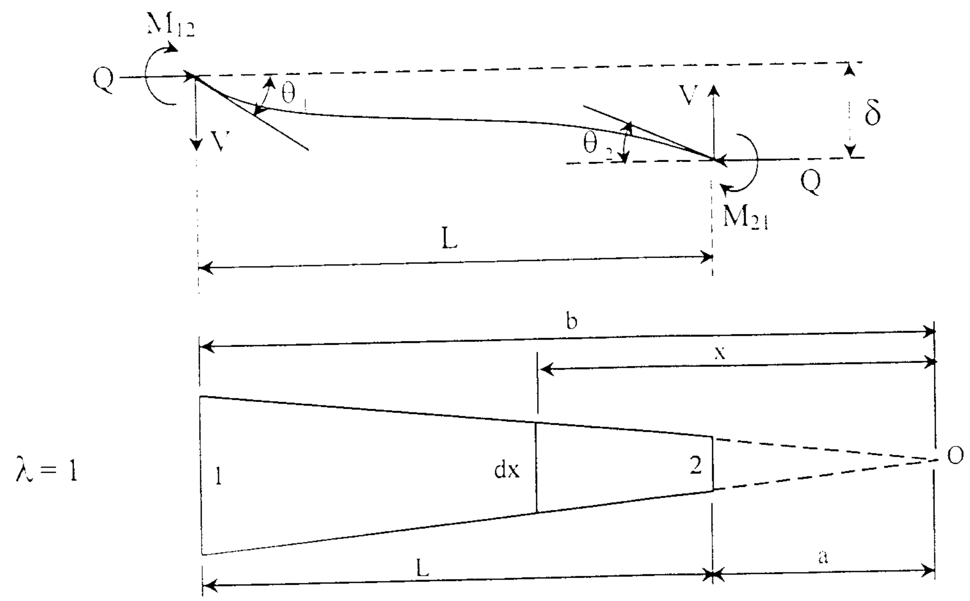
$$k_2 = EI_2 / L$$

S_1, S_2 : the modified stiffness factor of end 1, end 2 respectively

$\overline{SC}$: the modified carry-over factor for non-prismatic strut

$S_1 + \overline{SC}$: the modified sway moment factor of end 1

$2\overline{A}_2$: the modified shear stiffness factor $\left(= S_1 + S_2 + 2\overline{SC} - \pi^2 \rho_2\right)$



L: Length of non-prismatic member
a: the distance from end 2 to the origin O (zero depth)
b: the distance from end 1 to the origin O (zero depth)

Figure (2): Linear non-prismatic

The modified stability functions $S_1, \overline{SC}, S_2$ and $2\overline{A}_2$ are functions of the non-dimensional parameters $\rho = Q/(Q_e)_2$, end depth ratio $u = d_1/d_2$ and shape factor m. where $(Q_e)_2 = \pi^2 EI_2 / L^2$ is the Euler load and d_1 the depth of end 1, respectively.
A similar approach had been adopted by Al-Sarraf [4] in the prediction of elastic critical load of frames having uniformly tapered struts with different values of u and m.
Krynicki and Mazurkiewicz [8] adopted a similar approach in the prediction of elastic critical load of frames having uniformly tapered struts with solid circular or square cross-sections which have a shape factor m = 4 only.
Gere and Carter [7] presented formulae and graphs for the determination of the elastic critical buckling load of uniformly tapered columns having different values of end depth ratio u and shape factors m for columns with pinned ends, fixed-free, fixed-pinned and fixed ends. In this paper the variation of the depth along the length of strut is taken as non-linear i.e. having convex tapered configuration

Modified stability functions:

The elastic stability of the non-prismatic struts, a member of a frame subjected to constant axial load Q shown in Figure (2) depends on the solution of the basic differential equation [5, 7, 8] :

$$EI(x)\frac{d^2y}{dx^2}+Qy=M_2\left(-1+\frac{(x-a)}{(b-a)}\right)+M_1\frac{(x-a)}{(b-a)}$$
$$=\frac{M_2}{L}(x-b)+\frac{M_1}{L}(x-a) \qquad (3)$$

where y represents lateral deflection at distance x along the strut. M_1 and M_2 are the applied end moments. a is the distance of end 2 from the origin O. point of zero depth and b = a − L. All the columns considered non-linear taper in either one or two directions. Therefore, the depth dx may be expressed by: -

$$d(x)=d_2\left(\frac{x}{a}\right)^{\lambda} \qquad (4)$$

for $\lambda = 1$, the taper is uniform [1], $\lambda > 1$, the taper is non-uniform i.e. concave configuration and for $\lambda < 1$, the taper is non-uniform i.e. convex configuration as shown in Figure (3) and is considered here.

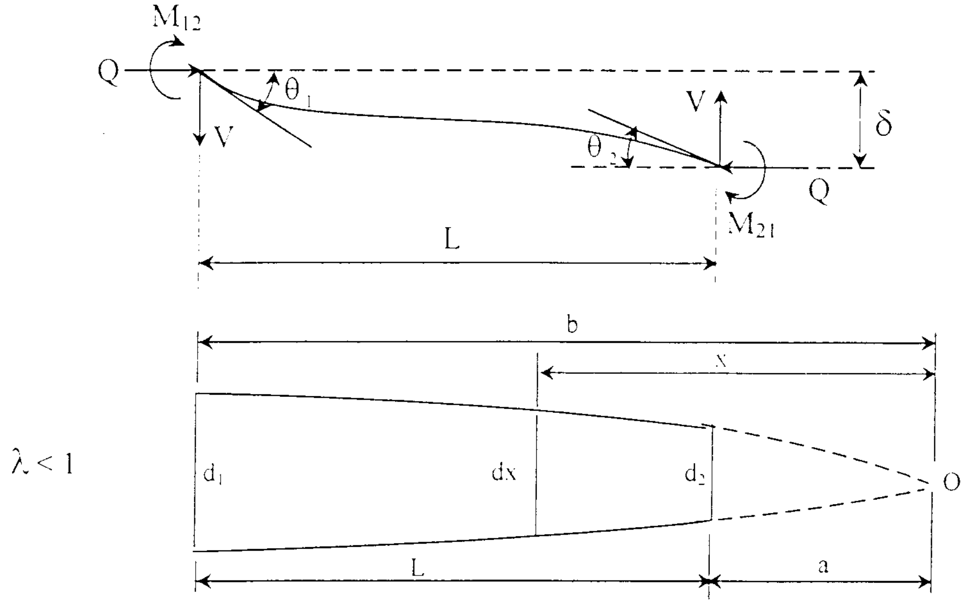


Figure (3): Non-linear (convex)

The depth at end 1 is: -

$$d_1=d_2\left(\frac{b}{a}\right)^{\lambda} \qquad (5)$$

$$\therefore \frac{d_1}{d_2}=u=\left(\frac{b}{a}\right)^{\lambda} \qquad (6)$$

where u is the depth ratio

The moment of inertia of the cross-sectional area of the strut about the axis of buckling may be expressed in the form: -

$$I(x)=I_2\left(\frac{x}{a}\right)^{\lambda m}=I_2\left(\frac{x}{a}\right)^{\overline{m}} \qquad (7)$$

where I(x) is the moment of inertia at distance x from the origin O and m is the shape factor that depends on the cross-sectional shape and dimensions of the strut. The shape factor m may be evaluated by observing that Equation (7) must give $I(x) = I_1$ when $x = b$. This condition yields the relation:

$$m = \log(I_1 / I_2)/\log u \tag{8}$$

where

$$u = \left(\frac{b}{a}\right)^{\lambda} = \frac{d_1}{d_2}$$

and

$\bar{m} = \lambda m$ [2]

Therefore, the shape factor [4] can be determined depending on the dimensions of the cross-sectional ends. The shape factor m is equal to either 1 or 3 depending on the axis about which buckling occurs for the rectangular cross-section. A strut having an open web, or an open box section consisting of equal areas at the corners of the cross-section has a value of 2 (approximately), and a tapered strut with solid circular or square cross-section has a value of m = 4. For struts of wide flange shape or closed box section, the shape factor will be between the limits m = 2.1 and 2.6 [3].

The basic differential equation of the beam subjected to constant axial force $\mp Q$ (–) tension or (+) compression is:

$$EI(x)\frac{d^2y}{dx^2} \mp Qy = \frac{M_1}{L}(x-a) + \frac{M_2}{L}(x-b) \tag{9}$$

Substituting equation (7) into equation (9) yields:

$$EI_2\left(\frac{x}{a}\right)^{\bar{m}}\frac{d^2y}{dx^2} \mp Qy = \frac{M_1}{L}(x-a) + \frac{M_2}{L}(x-b) \tag{10}$$

The right hand side of equation (9) can be reduced to zero by replacing "y" by "Z" when

$$Z = y - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b) \tag{11}$$

thus the differential equation becomes:

$$\frac{d^2Z}{dx^2} \mp \omega^2 x^{-\bar{m}} Z = 0 \tag{12}$$

where

$$\omega^2 = \mp Q a^{\bar{m}}/EI_2 \tag{13}$$

Equation (12) can be transformed into Bessel Equation of the form [4]

$$\frac{d^2Z}{dx^2} - \frac{(2\alpha-1)}{x}\cdot\frac{dZ}{dx} + \left(\beta^2\gamma^2x^{2\gamma-2} + \frac{\alpha^2 - n^2\gamma^2}{x^2}\right)Z = 0 \tag{14}$$

this equation has a general solution [11]

$$Z = x^{\alpha}\left[AJ_n(\beta x^{\gamma}) + BJ_{-n}(\beta x^{\gamma})\right] \tag{15-a}$$

or

$$Z = x^{\alpha}\left[AI_n(\beta x^{\gamma}) + BI_{-n}(\beta x^{\gamma})\right] \tag{15-b}$$

Depending on (n) is not an integer; here J_n and I_n are the Bessel and modified Bessel functions of the first kind of order (n); A and B are the constants of integration. Therefore the solution can at once be written down in terms of Bessel functions, by giving particular values to the constants α, β, γ and n comparing the two Equations (12) and (14) then the constants α, β, γ and n can be obtained:

$$\alpha = 0.5 \ , \ \beta = \pm\frac{2\omega}{2-\bar{m}} \ , \ \gamma = \frac{2-\bar{m}}{2} \ , \ n = \pm\frac{1}{2-\bar{m}}$$

hence the general solution of the fundamental Equation (9) is:

$$y = \sqrt{x}\left[AJ_n\left(\beta x^{\gamma}\right) + BJ_{-n}\left(\beta x^{\gamma}\right)\right] + \frac{M_1}{QL}(x-a) + \frac{M_2}{QL}(x-b) \qquad (16\text{-a})$$

or

$$y = \sqrt{x}\left[AI_n\left(\beta x^{\gamma}\right) + BI_{-n}\left(\beta x^{\gamma}\right)\right] - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b) \qquad (16\text{-b})$$

depending on whether Q is compression or tension.

There are four unknowns A, B, M_1, and M_2, which have to be determined from the following conditions:

at $x = a$, $y = 0$ and $dy/dx = \theta_2$

and $x = b$, $y = 0$ and $dy/dx = \theta_1$

The solution of the basic differential equations of elastic curve for nonlinear non-prismatic members depends on the value of the shape factor m and non-linearity factor λ.
The non-linearity factor $\lambda < 1.0$ in convex taper member is considered here.

2.1 Convex Taper for m = 4 and λ = 0.8:

$\bar{m} = m\lambda = 3.2$

Case 1: Compressive Axial Force Q > 0: -

$$\alpha = 0.5 \ , \ \gamma = \frac{2-3.2}{2} = -0.6 \ , \ n = \frac{1}{2-3.2} = \pm 0.833 \ , \ \beta = \frac{2\omega}{2-3.2} = \pm 1.667\omega$$

The solution of equation (9) and the first derivation becomes:

$$y = x^{0.5}\left[AJ_{0.833}\left(\frac{5\omega}{3x^{0.6}}\right) + BJ_{-0.833}\left(\frac{5\omega}{3x^{0.6}}\right)\right] + \frac{M_1}{QL}(x-a) + \frac{M_2}{QL}(x-b) \qquad (17)$$

$$\frac{dy}{dx} = \frac{\omega}{x^{1.1}}AJ_{1.833}\left(\frac{5\omega}{3x^{0.6}}\right) - \frac{\omega}{x^{1.1}}J_{-1.833}\left(\frac{5\omega}{3x^{0.6}}\right) + \frac{M_1+M_2}{QL} \qquad (18)$$

The values of the constants A and B are obtained by substituting the boundary conditions in equation (17):

$$A = \frac{M_1 J_{-0.833}(\alpha)\sqrt{a} + M_2 J_{-0.833}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}\,ZQ} \qquad (19)$$

$$B = -\frac{M_1 J_{0.833}(\alpha)\sqrt{a} + M_2 J_{0.833}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}\,ZQ} \qquad (20)$$

where

$$Z = J_{0.833}(\alpha)J_{-0.833}(\beta) - J_{-0.833}(\alpha)J_{0.833}(\beta) \qquad (21)$$

$$\alpha = 1.667\frac{\omega}{a^{0.6}} \ , \ \beta = 1.667\frac{\omega}{b^{0.6}} \ , \ \rho_2 = \frac{QL^2}{EI_2\pi^2} \ , \ \omega = \left(\frac{a^{3.2}Q}{EI_2}\right)^{0.5}$$

solving equations (17) and (18) to obtain the modified stability functions:

$$S_1 = \left(\omega L f_4 + Za^{1.6}\right)\left(\frac{-LZQb^{1.6}}{\omega PEI_2}\right) \qquad (22)$$

$$\overline{SC} = \left(\omega L f_5 + Z a^{0.5} b^{1.1}\right)\left(\frac{LZQa^{1.1}b^{0.5}}{\omega PEI_2}\right) \quad (23)$$

$$S_2 = \left(\omega L f_3 + Z b^{1.6}\right)\left(\frac{-LZQa^{1.6}}{\omega PEI_2}\right) \quad (24)$$

where

$$P = Z\left[a^{1.1}\left(b^{0.5}f_5 - a^{0.5}f_3\right) - b^{1.1}\left(b^{0.5}f_4 - a^{0.5}f_6\right)\right] - \omega L f_1 f_2 \quad (25)$$

and

$$f_1 = J_{-1.833}(\alpha)J_{1.833}(\beta) - J_{1.833}(\alpha)J_{-1.833}(\beta)$$
$$f_2 = J_{-0.833}(\alpha)J_{0.833}(\beta) - J_{0.833}(\alpha)J_{-0.833}(\beta)$$
$$f_3 = J_{-0.833}(\alpha)J_{1.833}(\beta) + J_{0.833}(\alpha)J_{-1.833}(\beta)$$
$$f_4 = J_{-0.833}(\beta)J_{1.833}(\alpha) + J_{0.833}(\beta)J_{-1.833}(\alpha)$$
$$f_5 = J_{0.833}(\beta)J_{-1.833}(\beta) + J_{-0.833}(\beta)J_{1.833}(\beta)$$
$$f_6 = J_{-0.833}(\alpha)J_{1.833}(\alpha) + J_{0.833}(\alpha)J_{-1.833}(\alpha)$$

Case 2: Tensile Axial Force Q < 0: -

$$\alpha = 0.5 \ , \ \gamma = \frac{2-3.2}{2} = -0.6 \ , \ n = \frac{1}{2-3.2} = \pm 0.833 \ , \ \beta = \frac{2\omega}{2-3.2} = \pm 1.667\omega$$

The solution of equation (9) and the first derivation become:

$$y = x^{0.5}\left[AI_{0.833}\left(\frac{5\omega}{3x^{0.6}}\right) + BI_{-0.833}\left(\frac{5\omega}{3x^{0.6}}\right)\right] - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b) \quad (26)$$

$$\frac{dy}{dx} = \frac{\omega}{x^{1.1}}AI_{1.833}\left(\frac{5\omega}{3x^{0.6}}\right) - \frac{\omega}{x^{1.1}}I_{-1.833}\left(\frac{5\omega}{3x^{0.6}}\right) - \frac{M_1 + M_2}{QL} \quad (27)$$

The values of the constants A and B are obtained by substituting the boundary conditions in equation (26)

$$A = \frac{M_1 I_{-0.833}(\alpha)\sqrt{a} + M_2 I_{-0.833}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}\,ZQ} \quad (28)$$

$$B = -\frac{M_1 I_{0.833}(\alpha)\sqrt{a} + M_2 I_{0.833}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}\,ZQ} \quad (29)$$

where

$$Z = I_{-0.833}(\alpha)I_{0.833}(\beta) - I_{0.833}(\alpha)I_{-0.833}(\beta) \quad (30)$$

$$\alpha = 1.667\frac{\omega}{a^{0.6}} \ , \ \beta = 1.667\frac{\omega}{b^{0.6}} \ , \ \rho_2 = \frac{QL^2}{EI_2\pi^2} \ , \ \omega = \left(-\frac{a^{3.2}Q}{EI_2}\right)^{0.5}$$

using the same procedure to determine the modified stability functions

$$S_1 = \left(\omega L f_4 + Z a^{1.6}\right)\left(\frac{LZQb^{1.6}}{\omega PEI_2}\right) \quad (31)$$

$$\overline{SC} = \left(\omega L f_5 - Z a^{0.5} b^{1.1}\right)\left(\frac{LZQa^{1.1}b^{0.5}}{\omega PEI_2}\right) \quad (32)$$

$$S_2 = \left(\omega L f_3 + Z b^{1.6}\right)\left(\frac{LZQa^{1.6}}{\omega PEI_2}\right) \tag{33}$$

where

$$P = Z\left[a^{1.1}\left(b^{0.5}f_5 + a^{0.5}f_3\right) + b^{1.1}\left(b^{0.5}f_4 - a^{0.5}f_6\right)\right] - \omega L f_1 f_2 \tag{34}$$

and

$$f_1 = I_{-1.833}(\alpha)I_{1.833}(\beta) - I_{1.833}(\alpha)I_{-1.833}(\beta)$$
$$f_2 = I_{-0.833}(\alpha)I_{0.833}(\beta) - I_{0.833}(\alpha)I_{-0.833}(\beta)$$
$$f_3 = I_{-0.833}(\alpha)I_{1.833}(\beta) - I_{0.833}(\alpha)I_{-1.833}(\beta)$$
$$f_4 = I_{-0.833}(\beta)I_{1.833}(\alpha) - I_{0.833}(\beta)I_{-1.833}(\alpha)$$
$$f_5 = I_{0.833}(\beta)I_{-1.833}(\beta) - I_{-0.833}(\beta)I_{1.833}(\beta)$$
$$f_6 = I_{-0.833}(\alpha)I_{1.833}(\alpha) - I_{0.833}(\alpha)I_{-1.833}(\alpha)$$

2.2 Convex Taper for m = 4 and $\lambda = 0.6$:

$\overline{m} = m\lambda = 2.4$

Case 1: Compressive Axial Force Q > 0: -

$$\alpha = 0.5\ ,\ \gamma = \frac{2-2.4}{2} = -0.2\ ,\ n = \frac{1}{2-2.4} = \pm 2.5\ ,\ \beta = \frac{2\omega}{2-2.4} = \pm 5\omega$$

The solution of equation (9) and the first derivation become:

$$y = x^{0.5}\left[AJ_{2.5}\left(\frac{5\omega}{x^{0.2}}\right) + BJ_{-2.5}\left(\frac{5\omega}{x^{0.2}}\right)\right] + \frac{M_1}{QL}(x-a) + \frac{M_2}{QL}(x-b) \tag{35}$$

$$\frac{dy}{dx} = \frac{\omega}{x^{0.7}}AJ_{3.5}\left(\frac{5\omega}{x^{0.2}}\right) - \frac{\omega}{x^{0.7}}J_{-3.5}\left(\frac{5\omega}{x^{0.2}}\right) + \frac{M_1 + M_2}{QL} \tag{36}$$

using the same procedure to determine the modified stability functions

$$S_1 = \left(\omega L f_4 + Z a^{1.2}\right)\left(\frac{-LZQb^{1.2}}{\omega PEI_2}\right) \tag{37}$$

$$\overline{SC} = \left(\omega L f_5 + Z a^{0.5} b^{0.7}\right)\left(\frac{LZQa^{0.7}b^{0.5}}{\omega PEI_2}\right) \tag{38}$$

$$S_2 = \left(\omega L f_3 + Z b^{1.2}\right)\left(\frac{-LZQa^{1.2}}{\omega PEI_2}\right) \tag{39}$$

<u>Case 2: Tensile Axial Force Q < 0: -</u>

$$\alpha = 0.5\ ,\ \gamma = \frac{2-2.4}{2} = -0.2\ ,\ n = \frac{1}{2-2.4} = \pm 2.5\ ,\ \beta = \frac{2\omega}{2-2.4} = \pm 5\omega$$

The solution of equation (9) and the first derivation becomes:

$$y = x^{0.5}\left[AI_{2.5}\left(\frac{5\omega}{x^{0.2}}\right) + BI_{-2.5}\left(\frac{5\omega}{x^{0.2}}\right)\right] - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b) \tag{40}$$

$$\frac{dy}{dx} = \frac{\omega}{x^{0.7}}AI_{3.5}\left(\frac{5\omega}{x^{0.2}}\right) - \frac{\omega}{x^{0.7}}I_{-3.5}\left(\frac{5\omega}{x^{0.2}}\right) - \frac{M_1 + M_2}{QL} \tag{41}$$

using the same procedure to determine the modified stability functions

$$S_1 = \left(\omega L f_4 + Z a^{1.2}\right)\left(\frac{LZQb^{1.2}}{\omega PEI_2}\right) \tag{42}$$

$$\overline{SC} = \left(\omega L f_5 - Z a^{0.5} b^{0.7}\right)\left(\frac{LZQa^{0.7} b^{0.5}}{\omega PEI_2}\right) \tag{43}$$

$$S_2 = \left(\omega L f_3 + Z b^{1.2}\right)\left(\frac{LZQa^{1.2}}{\omega PEI_2}\right) \tag{44}$$

where the equations of f_1, f_2, f_3, f_4, f_5, f_6, Z, α, β, ω, P and the constant of integration are shown in Table 1.

2.3 Convex Taper for m = 4 and λ = 0.4:

$\overline{m} = m\lambda = 1.6$

Case 1: Compressive Axial Force Q > 0: -

$$\alpha = 0.5\ ,\ \gamma = \frac{2-1.6}{2} = 0.2\ ,\ n = \frac{1}{2-1.6} = \pm 2.5\ ,\ \beta = \frac{2\omega}{2-1.6} = \pm 5\omega$$

The solution of equation (9) and the first derivation becomes:

$$y = x^{0.5}\left[AJ_{2.5}\left(5\omega x^{0.2}\right) + BJ_{-2.5}\left(5\omega x^{0.2}\right)\right] + \frac{M_1}{QL}(x-a) + \frac{M_2}{QL}(x-b) \tag{45}$$

$$\frac{dy}{dx} = \frac{\omega}{x^{0.3}} AJ_{1.5}\left(5\omega x^{0.2}\right) - \frac{\omega}{x^{0.3}} J_{-1.5}\left(5\omega x^{0.2}\right) + \frac{M_1 + M_2}{QL} \tag{46}$$

using the same procedure to determine the modified stability functions

$$S_1 = \left(\omega L f_4 - Z a^{0.8}\right)\left(\frac{LZQb^{0.8}}{\omega PEI_2}\right) \tag{47}$$

$$\overline{SC} = \left(\omega L f_5 - Z a^{0.3} b^{0.5}\right)\left(\frac{-LZQa^{0.5} b^{0.3}}{\omega PEI_2}\right) \tag{48}$$

$$S_2 = \left(\omega L f_3 - Z b^{0.8}\right)\left(\frac{LZQa^{0.8}}{\omega PEI_2}\right) \tag{49}$$

Case 2: Tensile Axial Force Q < 0: -

$$\alpha = 0.5\ ,\ \gamma = \frac{2-1.6}{2} = 0.2\ ,\ n = \frac{1}{2-1.6} = \pm 2.5\ ,\ \beta = \frac{2\omega}{2-1.6} = \pm 5\omega$$

The solution of equation (9) and the first derivation becomes:

$$y = x^{0.5}\left[AI_{2.5}\left(5\omega x^{0.2}\right) + BI_{-2.5}\left(5\omega x^{0.2}\right)\right] - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b) \tag{50}$$

$$\frac{dy}{dx} = -\frac{\omega}{x^{0.3}} AI_{1.5}\left(5x^{0.2}\right) - \frac{\omega}{x^{0.3}} I_{-1.5}\left(5x^{0.2}\right) - \frac{M_1 + M_2}{QL} \tag{51}$$

using the same procedure to determine the modified stability functions

$$S_1 = \left(\omega L f_4 + Z a^{0.8}\right)\left(\frac{-LZQb^{0.8}}{\omega PEI_2}\right) \tag{52}$$

$$\overline{SC} = \left(-\omega Lf_5 + Za^{0.5}b^{0.3}\right)\left(\frac{LZQa^{0.3}b^{0.5}}{\omega PEI_2}\right) \qquad (53)$$

$$S_2 = \left(\omega Lf_3 + Zb^{0.8}\right)\left(\frac{-LZQa^{0.8}}{\omega PEI_2}\right) \qquad (54)$$

where the equations of $f_1, f_2, f_3, f_4, f_5, f_6, Z, \alpha, \beta, \omega$, P and the constant of integration are shown in Table 1.

2.4 Convex Taper for m = 4 and $\lambda = 0.2$:

$\overline{m} = m\lambda = 0.8$

Case 1: Compressive Axial Force Q > 0: -

$$\alpha = 0.5\ ,\ \gamma = \frac{2-0.8}{2} = 0.6\ ,\ n = \frac{1}{2-0.8} = \pm 0.833\ ,\ \beta = \frac{2\omega}{2-0.8} = \pm 1.667\omega$$

The solution of equation (9) and the first derivation becomes:

$$y = x^{0.5}\left[AJ_{0.833}\left(\frac{5\omega x^{0.6}}{3}\right) + BJ_{-0.833}\left(\frac{5\omega x^{0.6}}{3}\right)\right] + \frac{M_1}{QL}(x-a) + \frac{M_2}{QL}(x-b) \qquad (55)$$

$$\frac{dy}{dx} = \omega x^{0.1}AJ_{-0.166}\left(\frac{5\omega x^{0.6}}{3}\right) - \omega x^{0.1}BJ_{0.166}\left(\frac{5\omega x^{0.6}}{3}\right) + \frac{M_1 + M_2}{QL} \qquad (56)$$

using the same procedure to determine the modified stability functions

$$S_1 = \left(\omega Lf_4 - Za^{0.4}\right)\left(\frac{LZQb^{0.4}}{\omega PEI_2}\right) \qquad (57)$$

$$\overline{SC} = \left(\omega Lf_5 - Za^{-0.1}b^{0.5}\right)\left(\frac{-LZQa^{0.5}b^{-0.1}}{\omega PEI_2}\right) \qquad (58)$$

$$S_2 = \left(\omega Lf_3 - Zb^{0.4}\right)\left(\frac{-LZQa^{0.4}}{\omega PEI_2}\right) \qquad (59)$$

Case 2: Tensile Axial Force Q < 0: -

$$\alpha = 0.5\ ,\ \gamma = \frac{2-0.8}{2} = 0.6\ ,\ n = \frac{1}{2-0.8} = \pm 0.833\ ,\ \beta = \frac{2\omega}{2-0.8} = \pm 1.667\omega$$

The solution of equation (9) and the first derivation becomes:

$$y = x^{0.5}\left[AI_{0.833}\left(\frac{5\omega x^{0.6}}{3}\right) + BI_{-0.833}\left(\frac{5\omega x^{0.6}}{3}\right)\right] - \frac{M_1}{QL}(x-a) - \frac{M_2}{QL}(x-b) \qquad (60)$$

$$\frac{dy}{dx} = \omega x^{0.1}AI_{-0.166}\left(\frac{5\omega x^{0.6}}{3}\right) - \omega x^{0.1}BI_{0.166}\left(\frac{5\omega x^{0.6}}{3}\right) - \frac{M_1 + M_2}{QL} \qquad (61)$$

using the same procedure to determine the modified stability functions

$$S_1 = \left(\omega Lf_4 + Za^{0.4}\right)\left(\frac{-LZQb^{0.4}}{\omega PEI_2}\right) \qquad (62)$$

$$\overline{SC} = \left(-\omega Lf_5 + Za^{0.5}b^{-0.1}\right)\left(\frac{-LZQa^{-0.1}b^{0.5}}{\omega PEI_2}\right) \qquad (63)$$

$$S_2 = \left(\omega L f_3 + Z b^{0.4}\right)\left(\frac{-LZQa^{0.4}}{\omega P E I_2}\right) \quad (64)$$

where the equations of $f_1, f_2, f_3, f_4, f_5, f_6, Z, \alpha, \beta, \omega$, P and the constant of integration are shown in Table 1.

3. Modified Stability Functions at No Axial Load:

The basic differential equation for a tapered beam is:

$$EI_2\left(\frac{x}{a}\right)^{\overline{m}}\frac{d^2y}{dx^2} = \frac{M_1}{L}(x-a) + \frac{M_2}{L}(x-b) \quad (65)$$

and may be written in the form

$$EI_2\frac{d^2y}{dx^2} = \frac{a^{\overline{m}}}{L}\left[(M_1+M_2)x^{1-\overline{m}} - a(M_1+uM_2)x^{-\overline{m}}\right] \quad (66)$$

and integrating with respect to x

$$EI_2\frac{dy}{dx} = \frac{a^{\overline{m}}}{L}\left[(M_1+M_2)\frac{x^{2-\overline{m}}}{2-\overline{m}} - a(M_1+uM_2)\frac{x^{1-\overline{m}}}{1-\overline{m}}\right] \quad (67)$$

and

$$EI_2 y = \frac{a^{\overline{m}}}{L}\left[(M_1+M_2)\frac{x^{3-\overline{m}}}{(2-\overline{m})(3-\overline{m})} - a(M_1+uM_2)\frac{x^{2-\overline{m}}}{(1-\overline{m})(2-\overline{m})}\right] + Ax + B \quad (68)$$

It may be noted that this solution is used when the value of $\overline{m} \neq 1,2,3$.

There are four unknowns A, B, M_1, and M_2, which have to be determined from the boundary conditions at the ends.
Then, the modified stability functions are:

$$S_1 = \frac{(2-\overline{m})L}{P}\left[\frac{2b}{a^{2\overline{m}-2}}\left(3-4\overline{m}+\overline{m}^2\right) + \frac{3\overline{m}-2-\overline{m}^2}{a^{2\overline{m}-3}} + \frac{2}{a^{\overline{m}}b^{\overline{m}-3}} - \frac{b^2}{a^{2\overline{m}-1}}\left(6-5\overline{m}+\overline{m}^2\right)\right] \quad (69)$$

$$\overline{SC} = \frac{(\overline{m}-2)L}{P}\left[\frac{b(\overline{m}-3)}{a^{2\overline{m}-2}} + (1-\overline{m})\left(a^{3-2\overline{m}} - \frac{b^{3-\overline{m}}}{a^{\overline{m}}}\right) + \frac{a^{1-\overline{m}}}{b^{\overline{m}-2}}(3-\overline{m})\right] \quad (70)$$

$$S_2 = \frac{(2-\overline{m})L}{P}\left[\frac{2-3\overline{m}+\overline{m}^2}{a^{\overline{m}}b^{\overline{m}-3}} + \frac{8\overline{m}-2\overline{m}^2-6}{a^{\overline{m}-1}b^{\overline{m}-2}} - \frac{2}{b^{2\overline{m}-3}} + \frac{a^{1-\overline{m}}}{b^{\overline{m}-2}}(3-\overline{m})\right] \quad (71)$$

where

$$P = \frac{2b^{2-\overline{m}}}{a^{\overline{m}-2}}\left(3-4\overline{m}+\overline{m}^2\right) + \left(\frac{b^{1-\overline{m}}}{a^{\overline{m}-3}} + \frac{b^{3-\overline{m}}}{a^{\overline{m}-1}}\right)\left(4-4\overline{m}+\overline{m}^2\right) + b^{4-2\overline{m}} + a^{4-2\overline{m}} \quad (72)$$

Example:

To find the elastic critical load for non-sway mode unti-symmetrical joint rotation model of the frame shown in Figure (4), which has prismatic beam and convex columns shape of $\lambda = 0.8$ and the stiffnesses of beam and column shown in the Figure (4).

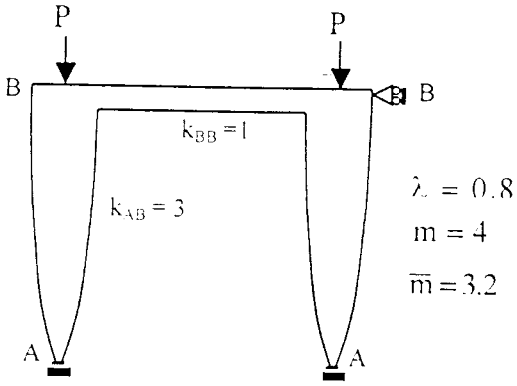


Figure (4-a): Example Frame

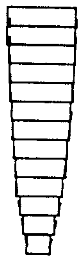


Figure (4-b): Equivalent Strut

$$\begin{aligned}\sum M_B &= k_{BB}[S\theta_B + SC\theta_B] + k_{AB}[S_1\theta_B] \\ &= 1(4+2)\theta_B + 3S_1\theta_B \\ &= 3(2+S_1)\theta_B\end{aligned}$$

where S, SC are 4 and 2 [11] respectively at axial load = 0

At elastic critical load, the stiffness is vanish where

$k = 2 + S_1$

$S_1 = -2$

By substituting ρ value from $\rho = 0$ up to ρ which verify $S_1 = -12$ by using equation (22) are get $\rho = 6.7196$.

The other solution of the stability function can be obtained depending on the stability function of prismatic member by dividing the convex beam-column member into equivalent prismatic segments.

The approximation method gives an approximate stability functions of beam-column member and this functions values will be more closed to the exact method when dividing element into a large number of segments as shown in Figures (5), (6) and (7), which represent the exact and approximate stability functions $(S_1, SC \text{ and } S_2)$ for different member of segments 5, 10, 15 and 20 subjected to tensile and compressive axial force.

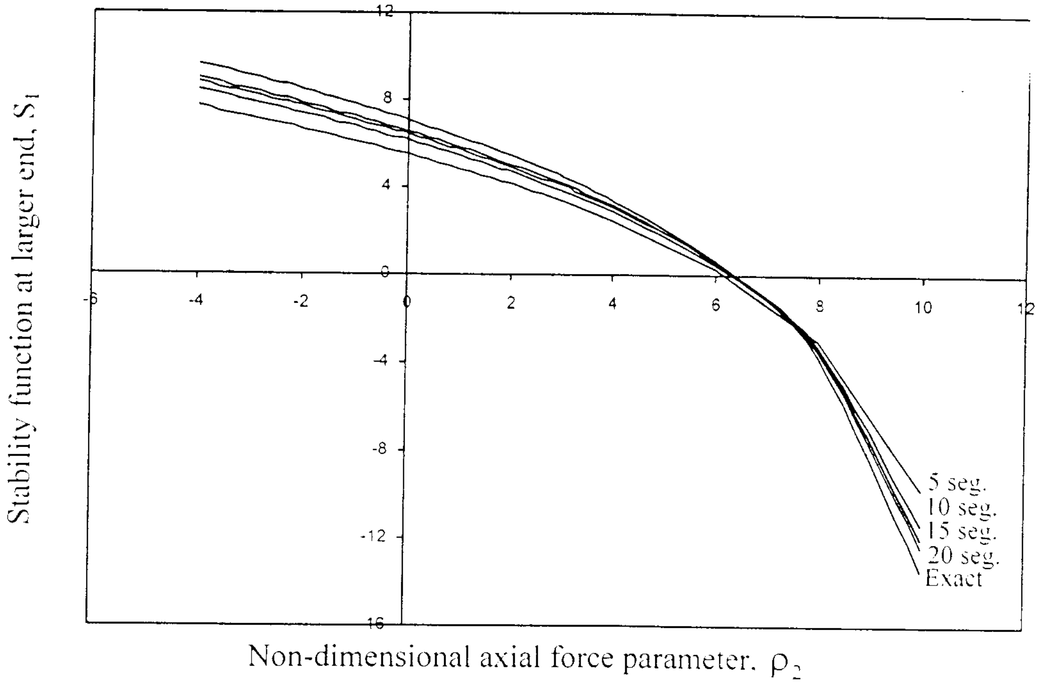


Figure (5): Stability function (S_1) for approximate and exact solution for tensile and compressive axial force for the considered example

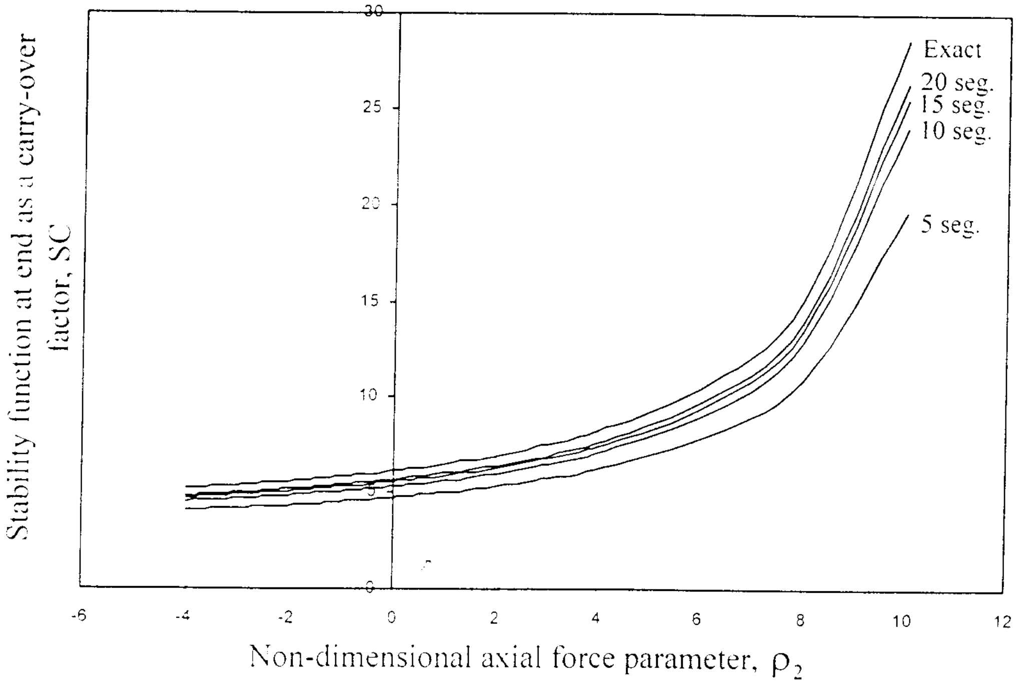


Figure (6): Stability function (SC) for approximate and exact solution for tensile and compressive axial force for the considered example

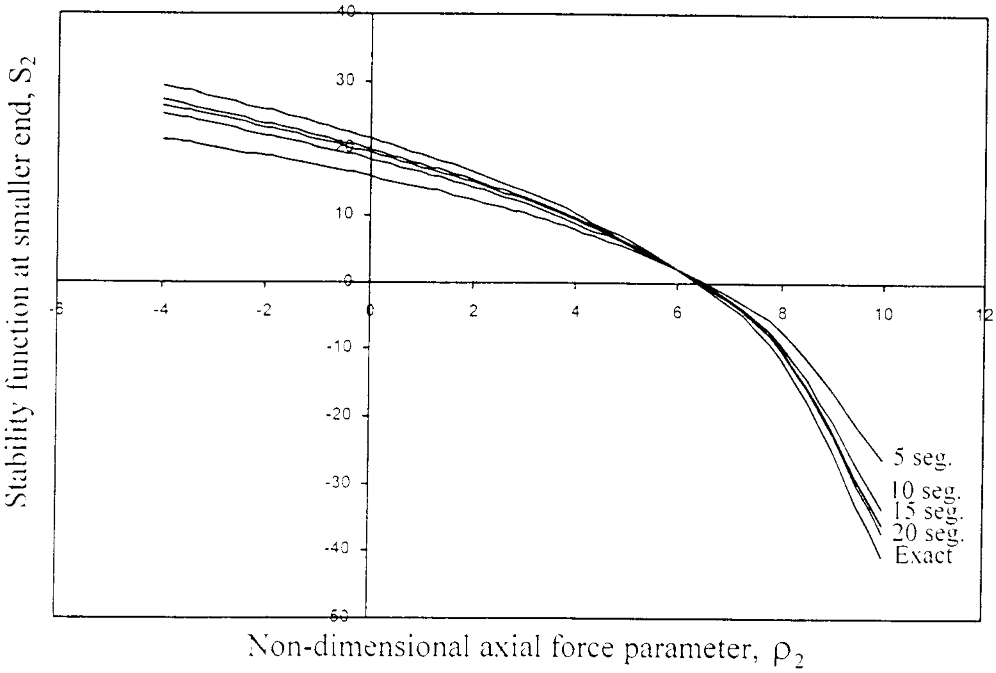


Figure (7): Stability function (S_2) for approximate and exact solution for tensile and compressive axial force for the considered example

The different percents between the exact and approximate elastic critical load of frame example with respect to exact and approximate stability functions at 5, 10, 15 and 20 segments are shown in Table (1) to estimate the validity of increasing number of segments.

Table (1): Elastic critical force for example frame

ρ	u		
	2	3	4
Exact	6.71866	12.9622	21.1797
Approximate (5 segments)	7.4830	14.1860	23.065
Difference %	11.3	9.4	20
Approximate (10 segments)	7.4492	14.347	23.0221
Difference %	10.8	10.6	8.67
Approximate (15 segments)	7.4184	14.2874	23.0113
Difference %	10.4	10.2	8.6
Approximate (20 segments)	7.400	14.243	22.9541
Difference %	10.1	9.8	8.4

The different tapering ratio represented in Table (1) comes from dimensions in Table (2).

Table (2): Dimensions of beam-column with respect of taper ratio

u	a	L	b
2	5.0	5.0	10.0
3	2.5	5.0	7.50
4	1.66	5.0	6.66

Conclusion

The non-linearity λ factor equals to 0.2, 0.4, 0.6 and 0.8 are used in derivation formulas for a member with convex configuration shape having a square or circular solid cross-sectional area, subjected to tensile, compressive or no axial force, which may used for any depth ratio at member ends.

These are obtained by solving the basic differential equation using Bessel functions and modified Bessel functions of the first and second kinds.

The present study introduces the exact formula of the modified stability function with respect to the non-dimensional axial force parameter ρ.

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Table 3

	Symbol	$\overline{m}=2.4 \quad \lambda=0.6$	$\overline{m}=1.6 \quad \lambda=0.4$	$\overline{m}=0.8 \quad \lambda=0.2$
Compressive Axial Force, Q > 0	Z	$J_{2.5}(\alpha)J_{-2.5}(\beta)-J_{-2.5}(\alpha)J_{2.5}(\beta)$	$J_{-2.5}(\alpha)J_{0.25}(\beta)-J_{2.5}(\alpha)J_{-2.5}(\beta)$	$J_{-0.166}(\alpha)J_{0.166}(\beta)-J_{0.166}(\alpha)J_{-0.166}(\beta)$
	ω	$\left(a^{2.4}Q/EI_2\right)^{0.5}$	$\left(a^{1.6}Q/EI_2\right)^{0.5}$	$\left(a^{0.8}Q/EI_2\right)^{0.5}$
	α	$5\omega a^{-0.2}$	$5\omega a^{0.2}$	$1.667\omega a^{0.6}$
	β	$5\omega b^{-0.2}$	$5\omega b^{0.2}$	$1.667\omega b^{0.6}$
	A	$\frac{M_1J_{-2.5}(\alpha)\sqrt{a}+M_2J_{-2.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$\frac{M_1J_{-2.5}(\alpha)\sqrt{a}+M_2J_{-2.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$-\frac{M_1J_{-0.167}(\alpha)\sqrt{a}+M_2J_{-0.167}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$
	B	$-\frac{M_1J_{2.5}(\alpha)\sqrt{a}+M_2J_{2.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$\frac{M_1J_{2.5}(\alpha)\sqrt{a}+M_2J_{2.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$\frac{M_1J_{0.167}(\alpha)\sqrt{a}+M_2J_{0.167}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$
	f_1	$J_{-3.5}(\alpha)J_{3.5}(\beta)-J_{3.5}(\alpha)J_{-3.5}(\beta)$	$J_{-3.5}(\alpha)J_{3.5}(\beta)-J_{3.5}(\alpha)J_{-3.5}(\beta)$	$J_{-1.166}(\alpha)J_{1.166}(\beta)-J_{1.166}(\alpha)J_{-1.166}(\beta)$
	f_2	$J_{-2.5}(\alpha)J_{2.5}(\beta)-J_{2.5}(\alpha)J_{-2.5}(\beta)$	$J_{-2.5}(\alpha)J_{2.5}(\beta)-J_{2.5}(\alpha)J_{-2.5}(\beta)$	$J_{-0.166}(\alpha)J_{0.166}(\beta)-J_{0.166}(\alpha)J_{-0.166}(\beta)$
	f_3	$J_{-2.5}(\alpha)J_{3.5}(\beta)+J_{2.5}(\alpha)J_{-3.5}(\beta)$	$J_{-2.5}(\alpha)J_{3.5}(\beta)+J_{2.5}(\alpha)J_{-3.5}(\beta)$	$J_{-0.166}(\alpha)J_{1.166}(\beta)+J_{0.166}(\alpha)J_{-1.166}(\beta)$
	f_4	$J_{-2.5}(\beta)J_{3.5}(\alpha)+J_{2.5}(\beta)J_{-3.5}(\alpha)$	$J_{-2.5}(\beta)J_{3.5}(\alpha)+J_{2.5}(\beta)J_{-3.5}(\alpha)$	$J_{-0.166}(\beta)J_{1.166}(\alpha)+J_{0.166}(\beta)J_{-1.166}(\alpha)$
	f_5	$J_{2.5}(\beta)J_{-3.5}(\beta)+J_{-2.5}(\beta)J_{3.5}(\beta)$	$J_{2.5}(\beta)J_{-3.5}(\beta)+J_{-2.5}(\beta)J_{3.5}(\beta)$	$J_{0.166}(\beta)J_{-1.166}(\beta)+J_{-0.166}(\beta)J_{1.166}(\beta)$
	f_6	$J_{-2.5}(\alpha)J_{3.5}(\alpha)+J_{2.5}(\alpha)J_{-3.5}(\alpha)$	$J_{-2.5}(\alpha)J_{3.5}(\alpha)+J_{2.5}(\alpha)J_{-3.5}(\alpha)$	$J_{-0.166}(\alpha)J_{1.166}(\alpha)+J_{0.166}(\alpha)J_{-1.166}(\alpha)$
	P1	$(f_5b^{0.5}-f_3a^{0.5})$	$(f_5b^{0.5}-f_3a^{0.5})$	$(f_5b^{0.5}-f_3a^{0.5})$
	P2	$(f_4b^{0.5}-f_6a^{0.5})$	$(f_4b^{0.5}-f_6a^{0.5})$	$(f_4b^{0.5}-f_6a^{0.5})$
	P	$Z\left[a^{0.7}P_1-b^{0.7}P_2\right]-\omega Lf_1f_2$	$Z\left[b^{0.3}P_2-a^{0.3}P_1\right]-\omega Lf_1f_2$	$Z\left[-a^{-0.1}P_1-b^{-0.1}P_2\right]-\omega Lf_1f_2$
Tensile Axial Force, Q < 0	Z	$I_{-2.5}(\alpha)I_{2.5}(\beta)-I_{2.5}(\alpha)I_{-2.5}(\beta)$	$I_{2.5}(\alpha)I_{-2.5}(\beta)-I_{-2.5}(\alpha)I_{0.25}(\beta)$	$I_{-0.167}(\alpha)I_{0.167}(\beta)-I_{0.167}(\alpha)I_{-0.167}(\beta)$
	ω	$\left(-a^{2.4}Q/EI_2\right)^{0.5}$	$\left(-a^{1.6}Q/EI_2\right)^{0.5}$	$\left(-a^{0.8}Q/EI_2\right)^{0.5}$
	α	$5\omega a^{-0.2}$	$5\omega a^{0.2}$	$1.667\omega a^{0.6}$
	β	$5\omega b^{-0.2}$	$5\omega b^{0.2}$	$1.667\omega b^{0.6}$
	A	$\frac{M_1I_{-2.5}(\alpha)\sqrt{a}+M_2I_{-2.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$\frac{M_1I_{-2.5}(\alpha)\sqrt{a}+M_2I_{-2.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$-\frac{M_1I_{-0.167}(\alpha)\sqrt{a}+M_2I_{-0.167}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$
	B	$-\frac{M_1I_{2.5}(\alpha)\sqrt{a}+M_2I_{2.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$\frac{M_1I_{2.5}(\alpha)\sqrt{a}+M_2I_{2.5}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$	$\frac{M_1I_{0.167}(\alpha)\sqrt{a}+M_2I_{0.167}(\beta)\sqrt{b}}{\sqrt{a}\sqrt{b}ZQ}$
	f_1	$I_{-3.5}(\alpha)I_{3.5}(\beta)-I_{3.5}(\alpha)I_{-3.5}(\beta)$	$I_{-3.5}(\alpha)I_{3.5}(\beta)-I_{3.5}(\alpha)I_{-3.5}(\beta)$	$I_{-1.167}(\alpha)I_{1.167}(\beta)-I_{1.167}(\alpha)I_{-1.167}(\beta)$
	f_2	$I_{-2.5}(\alpha)I_{2.5}(\beta)-I_{2.5}(\alpha)I_{-2.5}(\beta)$	$I_{-2.5}(\alpha)I_{2.5}(\beta)-I_{2.5}(\alpha)I_{-2.5}(\beta)$	$I_{-0.167}(\alpha)I_{0.167}(\beta)-I_{0.167}(\alpha)I_{-0.167}(\beta)$
	f_3	$I_{-2.5}(\alpha)I_{3.5}(\beta)-I_{2.5}(\alpha)I_{-3.5}(\beta)$	$I_{-2.5}(\alpha)I_{3.5}(\beta)-I_{2.5}(\alpha)I_{-3.5}(\beta)$	$I_{-0.167}(\alpha)I_{1.167}(\beta)-I_{0.167}(\alpha)I_{-1.167}(\beta)$
	f_4	$I_{-2.5}(\beta)I_{3.5}(\alpha)-I_{2.5}(\beta)I_{-3.5}(\alpha)$	$I_{-2.5}(\beta)I_{3.5}(\alpha)-I_{2.5}(\beta)I_{-3.5}(\alpha)$	$I_{-0.167}(\beta)I_{1.167}(\alpha)-I_{0.167}(\beta)I_{-1.167}(\alpha)$
	f_5	$I_{2.5}(\beta)I_{-3.5}(\beta)-I_{-2.5}(\beta)I_{3.5}(\beta)$	$I_{2.5}(\beta)I_{-3.5}(\beta)-I_{-2.5}(\beta)I_{3.5}(\beta)$	$I_{0.167}(\beta)I_{-0.167}(\beta)-I_{-0.167}(\beta)I_{0.167}(\beta)$
	f_6	$I_{-2.5}(\alpha)I_{3.5}(\alpha)-I_{2.5}(\alpha)I_{-3.5}(\alpha)$	$I_{-2.5}(\alpha)I_{3.5}(\alpha)-I_{2.5}(\alpha)I_{-3.5}(\alpha)$	$I_{-0.167}(\alpha)I_{1.167}(\alpha)-I_{0.167}(\alpha)I_{-1.167}(\alpha)$
	P_1	$(f_5b^{0.5}+f_3a^{0.5})$	$(f_5b^{0.5}+f_3a^{0.5})$	$(f_5b^{0.5}+f_3a^{0.5})$
	P_2	$(f_4b^{0.5}-f_6a^{0.5})$	$(f_4b^{0.5}-f_6a^{0.5})$	$(f_4b^{0.5}-f_6a^{0.5})$
	P	$P=Z\left[a^{0.7}P_1+b^{0.7}P_2\right]-\omega Lf_1f_2$	$P=\omega Lf_1f_2-Z\left[a^{0.3}P_1+b^{0.3}P_2\right]$	$P=\omega Lf_1f_2-Z\left[a^{-0.1}p_1+b^{-0.1}p_2\right]$