Purely Goldie Extending Modules

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Abstract

An R-module M is extending if every submodule of M is essential in a direct summand of M. Following Clark, an R-module M is purely extending if every submodule of M is essential in a pure submodule of M. It is clear purely extending is generalization of extending modules. Following Birkenmeier and Tercan, an R-module M is Goldie extending if, for each submodule X of M, there is a direct summand D of Msuch that $X\beta D$.

In this paper, we introduce and study class of modules which are proper generalization of both the purely extending modules and $\mathcal{G}$-extending modules. We call an R-module M is purely Goldie extending if, for each $X \leq M$, there is a pure submodule P of Msuch that $X\beta P$. Many characterizations and properties of purely Goldie extending modules are given. Also, we discuss when a direct sum of purely Goldie extending modules is purely Goldie extending and moreover we give a sufficient condition to make this property of purely Goldie extending modules is valid.

Key words: extending module, purely extending module, $\mathcal{G}$-extending module, purely Goldie extending.

Introduction

Throughout all rings are associative and R denotes a ring with identity and all modules are unitary R-modules. A submodule X of a module M is called essential if every non-zero submodule of M intersects X nontrivially (notionally, $X \leq^e M$). Also, a submodule X of M is closed in M, if it has no proper essential extension in M[1].

Recall that a module M is extending if every submodule of M is essential in a direct summand of M. Equivalently, every closed submodule of M is direct summand [1]. Many generalizations of extending modules are extensively studied. Following Fuchs [2] and Clark [3], an R-module M is purely extending if every submodule of M is essentialin a pure submodule of M (recall that a submodule N of an R-module M is pure if $IM \cap N = IN$ for every finitely generated ideal I of R). Also in [4], the following relations on the set of submodules of an R-module M are considered. (1) $X\alpha Y$ if and only if there exists a submodule A of M such that $X \leq^e A$ and $Y \leq^e A$; (ii) $X\beta Y$ if and only if $X \cap Y \leq^e X$ and $X \cap Y \leq^e Y$. Following [4], α is reflexive and symmetric, but it may not be transitive. Also, β is an equivalence relation. Moreover, an R-module M is extending if and only if for each submodule X of M, there exists a direct summand D of M such that $X\alpha D$ [4]. In 2009 Birkenmeier and Tercan [4], an R-module $M$is called Goldie extending (shortly,$\mathcal{G}$-extending) if, for each X submodule of M, there is a direct summand D of M such that $X\beta D$.

In section one, we introduce purely $\mathcal{G}$-extending modules. An R-module M is $\mathcal{G}$-extending if, for each $X \leq M$, there is a pure submodule P of Msuch that $X\beta P$. It is clear that every $\mathcal{G}$-extending (purely extending) module is purely $\mathcal{G}$-extending module and the converse is not true in general. Additional conditions are given to make the converse true. In fact we prove that: let M be a pure split. Then M is a purely $\mathcal{G}$ -extending module if and only if M is a $\mathcal{G}$–extending
module. Moreover, the hereditary property of purely $\mathcal{G}$ -extending modules is discussed. We call an R-module M is purely $\mathcal{G}^+$-extending if every direct summand of M is purely $\mathcal{G}$–extending. We do not know whether every purely $\mathcal{G}$–extending module is purely $\mathcal{G}^+$-extending. Indeed, we conclude that every purely extending module is purely $\mathcal{G}^+$-extending. Finally, we prove that an Z-module is extending if and only if M is a purely extending and M is a $\mathcal{G}$-extending.

In section two, various characterizations of purely $\mathcal{G}$ -extending modules are given. For example, we prove that an R-module M is purely $\mathcal{G}$-extending if and only if every direct summand A of the injective hull $E(M)$ of M, there exists a pure submodule P of M such that$(A \cap M)\ \beta P$. On other direction, the direct sum property of purely $\mathcal{G}$–extending modules is discussed. We prove that, if M_i is purely $\mathcal{G}$-extending module for each $i \in I$ and every closed submodule of $M=\bigoplus_{i\in I} M_i$ is fully invariant, then $M=\bigoplus_{i\in I} M_i$ is purely $\mathcal{G}$-extending module.

1. Purely Goldie Extending Modules.

Recall that an R-module M is $\mathcal{G}$-extending if, for each Xsubmodule of M, there is a direct summand D of Msuch that $X\beta D$. Equivalently , M is Goldie extending if and only if for each closed submodule C of M, there is a direct summand D of Msuch that $C\beta D$[4], Also, an R-module M is purely extending module if every submodule of M is essential in a pure submodule of M [3].

We introduce and study the class of modules which is a generalization of both$\mathcal{G}$-extending modules and purely extending modules.

Definition (1.1)

An R-module M is called purely Goldie extending (shortly, purely $\mathcal{G}$-extending) if, for each $X \leq M$, there is a pure submodule P of Msuch that $X\beta P$.

Remarks and Examples (1.2)

1) Every purely extending module is a purely $\mathcal{G}$ -extending, but the converse is not true in general. For example, the Z-module $M = Z_p \oplus Q$ is a purely $\mathcal{G}$–extending since M is $\mathcal{G}$-extending [4]. But by [4, Example (3.20)] and proposition (1.14), $M = Z_p \oplus Q$ is not purely extending Z-module.
2) Every $\mathcal{G}$- extending module is purely $\mathcal{G}$-extending, but the converse is not true in general. For example, by [5, Example (3.4)], the Z-module $M = \oplus_{i\in I}$ Z is purely extending but it is not extending. So M is a purely $\mathcal{G}$–extending while, by proposition (1.14), M is not $\mathcal{G}$-extending .
3) Every uniform module is purely $\mathcal{G}$-extending, but the converse is not true in general. For example, Z_6 as Z-module is purely $\mathcal{G}$- extending but it is not uniform.

Recall that an R-module M is a pure-split if every pure submodule of M is a direct summand [6].The following proposition gives conditions under which the concepts of $\mathcal{G}$-extending modules and purely $\mathcal{G}$- extending modules are equivalent.

Proposition (1.3):

Let M is a pure split R-module. Then M is a purely $\mathcal{G}$ -extending if and only if M is a $\mathcal{G}$-extending. ■

Following [7], a non-zero$R$-module $M$is pure-simple if the only pure submodules of M are 0 and M itself.

Proposition (1.4)

Let M be a pure- simple R-module. Then M is a purely $\mathcal{G}$- extending if and only if M is a uniform module.

Proof:(⟹) Let X be a submodule of M. By assumption, there is a pure submodule P of Msuch that $X\beta P$. So, $X \cap P$ is essential in P.But M is a pure- simple then P=M, then X is essential in M. Thus, M is a uniform module.

(⟸) Let X be a submodule of M. Since M is a uniform module, then X is essential in M, but M is a pure submodule of M, then $X\beta M$. Hence, M is a purely $\mathcal{G}$- extending. ■

Corollary (1.5)

Let Mbe a pure- simple$R$-module. Then the following statements are equivalent.

(1) M is a purely extending module.
(2) M is a purely $\mathcal{G}$-extending module.
(3) M is uniform module.

Following [4], a submodule of $\mathcal{G}$-extending module need not to be $\mathcal{G}$-extending. Moreover, a submodule of purely extending module need not to be purely extending [5]. In fact, we do not know whether a submodule of a purely $\mathcal{G}$-extending module is purely $\mathcal{G}$-extending. Indeed, we have the following result.

Proposition (1.6)

Every submodule N of a purely $\mathcal{G}$-extending$R$-module M with the property that the intersection of N with any pure submodule of M is a pure submodule of N is purely $\mathcal{G}$-extending.

Proof : Let A be a submodule of N. Since M is a purely $\mathcal{G}$-extending, then there is a pure submodule P of M such that $A\beta P$. By assumption, $P \cap N$ is a pure submoduleof N.But, $(A \cap P) \leq^e P$ and $(A \cap P) \leq^e A$, so $(A \cap (P \cap N)) \leq^e (P \cap N)$ and $(A \cap (P \cap N)) \leq^e (A \cap N) = A$. Therefore, $A\beta(P \cap N)$. Thus, N is purely $\mathcal{G}$-extending module. ■

From [4], recall that M is $\mathcal{G}^+$-extending module if every direct summand of M is $\mathcal{G}$-extending.This lead us to introduce the following.

Definition (1.7):

An R-module $M$is called purely $\mathcal{G}^+$-extending if every direct summand of $M$is purely $\mathcal{G}$-extending.

In fact, we do not know whether, every purely $\mathcal{G}$-extending module is purely $\mathcal{G}^+$-extending. In fact, we have the following result.

Proposition (1.8):

Every purely extending module is purely $\mathcal{G}^+$-extending module.

Proof : Let Nbe a direct summand of a purely extending module M. By [5], N is purely extending module. Hence N is purely $\mathcal{G}$-extending module. Thus, M is a purely $\mathcal{G}^+$-extending. ■

But the converse of proposition (1.8) is not true in general, for example, the Z-module $M = Z_p \oplus Q$ (for any prime number p) is not purely extending by (1.2), but M is purely $\mathcal{G}^+$-extending, since the only direct summands of M, $(Z_p \oplus 0)$, $(0 \oplus Q)$, $(0 \oplus 0)$ and M, which are purely $\mathcal{G}$-extending.

Recall that an R-module M has the pure intersection property (PIP) if the intersection of any two pure submodule of M is pure [8].

Proposition (1.9) :

Let M be a purely $\mathcal{G}$-extending and M has the PIP. Then M is a purely $\mathcal{G}^+$-extending.

Proof : Let N be a direct summand of M and A be a submodule of N. Since $M$is a purely $\mathcal{G}$-extending, then there is a pure submodule P of M such that $A\beta P$. But Msatisfies PIP, then $P \cap N$ is a pure submodule of M. But $P \cap N \subseteq N$, hence $P \cap N$ is a pure submodule of N.Therefore, $A = (A \cap N)\beta(P \cap N)$ by [9], and so M is a purely $\mathcal{G}^+$-extending. ■

Corollary (1.10) :

Let M be a prime module over a Bezout domain. If $M$is a purely $\mathcal{G}$-extending module, then M is a purely $\mathcal{G}^+$-extending. ■

Recall that an R-module $M$is a multiplication if for each submodule A of M, there exists an ideal $I$of R such that $A = IM$ [10]. Since every multiplication module has the PIP[8]. Thus, we have the next corollary.

Corollary (1.11):

Let M be a multiplication purely $\mathcal{G}$-extending module. Then M is a purely $\mathcal{G}^+$-extending. ■

Corollary (1.12) :

Let $M$is cyclic module over a commutative ring R. If M is a purely $\mathcal{G}$-extending, then N is purely $\mathcal{G}$-extending. ■

Corollary (1.13) :

Let R be a purely $\mathcal{G}$-extending commutative ring, then R is a purely $\mathcal{G}^+$-extending. ■

The following result gives a characterization of extending abelian groups.

Proposition (1.14):

A Z-module M is extending module if and only if M is a purely extending and M is a $\mathcal{G}$-extending as Z-module.

Proof : (⟹) it is clear that .

(⟸) Let N be a closed submodule of M. Since M is a purely extending, then N is a pure submodule of M by [5]. Also, since M is a $\mathcal{G}$-extending as Z-module by [4], then N is a direct summand of M.Therefore, $M$is extending module. ■

2. Characterizations of Purely Goldie Extending Modules

It is known that M is a purely extending module if and only if every closed submodule in M is a pure in M [5]. Also from [4], M is $\mathcal{G}$-extending module if and only if for every closed submodule C of M, there is a direct summand D of M such that $C\beta D$.

Here, we give analogous characterization of purely $\mathcal{G}$-extending modules.

Proposition (2.1):

An R-module M is purely $\mathcal{G}$-extending if and only if for every closed submodule C of M, there is a pure submodule P of M such that $C\beta P$.

Proof : (⟹) it is clear .

(⟸) Let Abe a submodule of M.By Zorn's lemma, there exists a closed submodule C of M such that A is essential in C . So, we have $A\beta C$ .By assumption, there exists a pure submodule P of M such that $C\beta P$. Since β is transitive relation, then $A\,\beta\,P$.Therefore, $M$is purely $\mathcal{G}$-extending module. ■

Proposition (2.2):

An R-module M is purely $\mathcal{G}$-extending if and only if every direct summand A of the injective hull $E(M)$, there exists a pure submodule P of M such that$(A \cap M)\,\beta P$.

Proof : (⟹) Let A be a direct summand of the injective hull $E(M)$ of M, then $(A \cap M)$ is a submodule of M, since $M$is purely$\mathcal{G}$-extending,then there exists a pure submodule P of M such that $(A \cap M)\beta P$.

(⟸) Let A is a submodule of M and let B be a relative complement of Asuch that $A\oplus B$ is essential in M [11]. Since M is essential in $E(M)$, then $A\oplus B$ is essential in $E(M)$. Thus, $E(A)\oplus E(B) = E(A\oplus B)$=$E(M)$[10]. By hypothesis, there exists a pure submodule P of M such that $(E(A) \cap M)\beta P$. But A is essential in $E(A)$. Therefore, $A = (A \cap M)\leq^{e}(E(A) \cap M)$. But $(A \cap M)$=$(A \cap M) \cap (E(A) \cap M)\leq^{e}(E(A) \cap M)$ and $(A \cap M)$=$(A \cap M) \cap (E(A) \cap M)\leq^{e}(A \cap M)$. So, $A = (A \cap M)\beta(E(A) \cap M)$. Since β is transitive, then $A = (A \cap M)\beta P$. So $M$is purely $\mathcal{G}$-extending. ■

Proposition (2.3):

The following statements are equivalent for an an R- module M:

(1) M is purely $\mathcal{G}$ – extending module.

(2) For each Y is a submodule of M, there exists X a submodule of M and a pure submodule P of M, such that $X \leq^{e} Y$ and $X \leq^{e} P$.

Proof: (1)⟹(2) Let Y be a submodule of M. Then there exists a pure submodule P of M such that $Y\beta P$, so $Y \cap P \leq^{e} P$ and $Y \cap P \leq^{e} Y$. The proof is complete put $= Y \cap P$.

(2)⟹(1) Let Y be a submodule of M. By (2), there exists a submodule X of M and a pure submodule P of M such that $X \leq^{e} Y$ and $X \leq^{e} P$. Now, since $X \leq Y \cap P \leq Y$ and $X \leq Y \cap P \leq P$ then $Y \cap P \leq^{e} Y$ and $\cap P \leq^{e} P$. So $Y\beta P$ and so M is purely $\mathcal{G}$ – extending module. ■

Following [4], a direct sum of $\mathcal{G}$-extending modules need not be $\mathcal{G}$-extending module. Also, a direct sum of purely extending modules need not be purely extending module [5]. Here, we discuss when a direct sum of purely $\mathcal{G}$-extending modules is a purely $\mathcal{G}$-extending.

Recall that a submodule N of an R-module M is fully invariant if $f(N) \subseteq N$ for each R-endomorphism f of M [12]. M is called Duo if every submodule of M is fully invariant [13].

Proposition (2.4)

Let M_i is purely $\mathcal{G}$-extending R-module for each $i \in I$ such that every closed submodule of $M = \bigoplus_{i\in I} M_i$ is fully invariant, then $M = \bigoplus_{i\in I} M_i$ is purely $\mathcal{G}$-extending module.

Proof : Let K be a closed submodule of M and let $\pi_i : M \longrightarrow M_i$ be the natural projection on M_i for each $i \in I$. Let $x \in K$, so $x = \sum_{i\in I} m_i$, where $m_i \in M_i$ and hence $\pi_i(x) = m_i$. Now, since K is closed submodule of, then by hypothesis , K is fully invariant and hence $\pi_i(K) \subseteq K \cap M_i$. So $\pi_i(x) = m_i \in K \cap M_i$ and hence $x \in \bigoplus_{i\in I} (K \cap M_i)$. Thus $K \subseteq \bigoplus_{i\in I} (K \cap M_i)$. Also, $\bigoplus_{i\in I} (K \cap M_i) \subseteq K$ and so $\bigoplus_{i\in I} (K \cap M_i) = K$. Since $(K \cap M_i) \subseteq M_i$ and by purely $\mathcal{G}$-extending property of M_i, then there is a pure submodule P_i of M_i such that $(K \cap M_i)\beta(P_i)$, $\forall i \in I$.

Now, since P_i is a pure submodule of $M_i, \forall i \in I$, then $\bigoplus_{i\in I} P_i$ is a pure submodule in $M = \bigoplus_{i\in I} M_i$ [8].So, $K = \bigoplus_{i\in I} (K \cap M_i)\beta(\bigoplus_{i\in I} P_i)$ [9].Thus, M is purely $\mathcal{G}$ -extending module. ■

Corollary (2.5) :

Let $M = M_1 \oplus M_2$ be a duo module such that M_1 and M_2 are purely $\mathcal{G}$-extending modules. Then M is a purely $\mathcal{G}$-extending. ■

By the same argument of the proof proposition (2.4), one can get the following result. Firstly, recall that an R- module M is distributive if for all submodules K, L and N of M, $K \cap (L + N) = (K \cap L) + (K \cap N)$[14].

Proposition (2.6)

Let $M = M_1 \oplus M_2$ be a distributive module such that M_1 and M_2 are purely $\mathcal{G}$-extending modules. Then M is a purely $\mathcal{G}$-extending.

Proof: Let A is a submodule of $M = M_1 \oplus M_2$ since M is a distributive module so $A = (A \cap M) = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$. But M_1 and M_2 are purely $\mathcal{G}$- extending, then there are a pure submodule P_1 of M_1 such that $(A \cap M_1)\beta P_1$ and pure submodule P_2 of M_2 such that $(A \cap M_2)\beta P_2$. So, $A = ((A \cap M_1) \oplus (A \cap M_2))\ \beta\ (P_1 \oplus P_2)$ by [9] and by [8] $(P_1 \oplus P_2)$ is a pure submodule of $M = M_1 \oplus M_2$. Thus, M is a purely $\mathcal{G}$-extending. ■

Proposition (2.7):

Let M and N be purely $\mathcal{G}$-extending R-modules such that $ann(M) + ann(N) = R$. Then $M \oplus N$ is a purely $\mathcal{G}$-extending module.

Proof : Let $A(\neq 0)$ be a submodule of $M \oplus N$. Since $ann(M) + ann(N) = R$, then $A = C \oplus D$, where C is a submodule of M and D is a submodule of N[15]. Since $A(\neq 0)$ then $C(\neq 0)$ or $D(\neq 0)$.If $C \neq 0$ and $D = 0$, then $A = C$ is a submodule of M. But M is purely $\mathcal{G}$-extending and hence there is a pure submodule H of M such that $A\beta H$. Since M is a direct summand of $M \oplus N$, then M is a pure submodule of $M \oplus N$, (by [16]), then H pure submodule of $M \oplus N$.Thus $M \oplus N$ is a purely $\mathcal{G}$-extending module. By the similar way if $C = 0$ and $D \neq 0$, then $M \oplus N$ is a purely $\mathcal{G}$-extending module. If $C(\neq 0)$ and $D(\neq 0)$, since M and N are purely $\mathcal{G}$-extending modules, then there is a pure submodule H of M such that $C\beta H$, and there is a pure submodule P of N such that $D\beta P$. But$(H \oplus P)$ is a pure submodule of $M \oplus N$ [8] and by [9], $(C \oplus D)\beta(H \oplus P)$. Therefore, $M \oplus N$ is a purely $\mathcal{G}$-extending module. ■

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مقاسات التوسع النقية من النمط -$\mathcal{G}$

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الخلاصة

لتكن R حلقة و M مقاساً معرفاً على R. يقال للمقاس M بأنه توسع إذا كان كل مقاس جزئي من M يكون جوهرياً من مركبة جمع مباشرمن M. تبعاً كلارك، يقال للمقاس M بأنه توسع نقي إذا كان كل مقاس جزئي من M يكون جوهرياً من مقاس جزئي نقي من M. من جهة اخرى، بركانمير و تيركان عرضا مفهوم مقاسات التوسع من النمط-$\mathcal{G}$. يقال للمقاس M بأنه توسع من النمط-$\mathcal{G}$ أذا كان لكل مقاس جزئي X من M يوجد مركبة جمع مباشرD من M بحيث $X\beta D$.

في هذا البحث، تم عرض و دراسة صنف من المقاسات كتعميم فعلي لكل من صنف مقاسات التوسع النقية ومقاسات التوسع من النمط-$\mathcal{G}$. نقول عن المقاس M بأنه توسع نقي من النمط-$\mathcal{G}$ إذا كان لكل مقاس جزئي X من M يوجد مقاس جزئي نقي P من M بحيث $X\beta P$. تم أعطاء العديد من التشخيصات و النتائج و الخواص لمقاسات التوسع النقية من النمط-$\mathcal{G}$. وكذلك تم مناقشة متى تكون مركبة الجمع المباشرلمقاسات التوسع النقية من النمط –$\mathcal{G}$ مقاس توسع نقي من النمط-$\mathcal{G}$. أكثر من ذلك، تم تقديم شروط كافية لجعل هذه الخاصية متحققة لمقاسات التوسع النقية من النمط –$\mathcal{G}$.

الكلمات المفتاحية: مقاسات التوسع، مقاسات التوسع النقية، مقاسات التوسع من النمط-$\mathcal{G}$ ، مقاسات التوسع النقية من النمط –$\mathcal{G}$.