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Ising Chains Properties at Low Temperature Extended Analysis

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Abstract

The transfer matrix method is used to describe the properties of the random bond of a simple periodic frustrated Ising chain in a uniform magnetic field. Doman and Williams(1982) are found expressions for the thermodynamic functions of the system at low temperature. Their methods is extended to apply to a system of five-antiferromagnetic bond Ising chain terminated by a ferromagnetic bond at low temperatures. New expressions for the magnetization, energy, and entropy as a functions of magnetic field is found. All results are in agreement with the previous results.

Keywords:

Random bond , Ising chain , magnetic field

five-antiferromagnetic bond

1. Introduction

Ising model is one of the models that are used in statistical physics. The model is concerned with the physics of phase transitions which occur when the change in a parameters be small. The Ising model can solved the frustration problem and what happen to the system after this.

The existence of frustration as well as disorder, considered to be the main ingredients in spin glass models, and the random bond Ising chain in a uniform field, is the simplest model which displaying these features.

Numerical results for the random-bond model have been obtained by Fernandez (1977) and Puma and Fernandez (1978), and analytical expressions for various ground state properties of these models were first obtained by Derrida et al (1978). The results of Derrida were found by transfer matrix techniques which, due to the stochastic nature of the problem, led to a Markov chain whose limiting distribution was obtained in the low-temperature limit [J K Williams 1981].

In Edwards and Anderson model (EA), we note spins arranged on a d-dimensional lattice with only nearest neighbor interactions, resulting of new magnetic phase called the spin glass phase of the system. The Hamiltonian for this spin system is given by: $H= -\sum_{<ij>} J_{ij} s_i s_j$. In 1975 Sherrington and Kirkpatrick (SK) proposed the infinite ranged version of the EA Hamiltonian. They introduced an exactly solvable model of a spin glass. This model is an Ising model with ferro-antiferromagnetic couplings. In SK model the considered system is only two spins interactions, such that the range of each interaction can be potentially infinite. The Hamiltonian of this model is something similar to the EA model, and defined by:

$$H = \frac{-1}{\sqrt{n}} \sum_{<ij>} J_{ij}\, s_i s_j .$$

In Hamiltonian equation if all J>0, where J is the interaction energy constant for the nearest neighbour, then the interacting bond is ferromagnetic but when J<0 then the interacting bond is antiferromagnetic, and when J=0 the spins are non-interacting (diluted cases). See Peter Young 2008.

In 1981 Williams has shown how the properties of ground state may be obtained by considering the behavior of certain clusters of spins called superspins, at B=2J/r, where B is the external magnetic field, and r is an integer number.

The transfer matrix methods used to examine the ground state properties of the random bond Ising chain in a uniform magnetic field was extended to describe the system at low temperatures by Doman and Williams (1982). They examined a system of one-ferromagnetic bond and three-antiferromagnetic bonds, in this paper we will be using the idea to examine the changes as B passes through 2J/r at very low temperatures in Ising chains which consist of one-ferromagnetic bond and five-antiferromagnetic bonds.

2. Frustration effect

The Hamiltonian equation:

$$H = -\sum_{<ij>} J_{ij}\, s_i s_j - B\sum_i s_i \qquad (1)$$

Where B>0 represents the external magnetic field, and J_{ij} the interaction energy constant for the nearest neighbour, and $s_i=\pm1$.

When B>2J all the spins must point up in ground state such that $s_i=+1$, and all the antiferromagnetic bonds are frustrated. But if B reduce such that J<B<2J some of interior spins will point down if both its neighboring bonds are antiferromagnetic, and neighboring spins point up, so at each end the terminated bond is ferromagnetic.

The partition function is given by: [see Mansoor AZ. Habeeb 1984].

$$Z_n = \sum_{\{s\}_{i=1}^{n}} \exp(\beta J \sum_{i=1} s_i s_{i+1} + \beta B \sum_{i=1} s_i) \qquad (2)$$

Where $\beta = \frac{1}{kT}$, and k is Boltzmann's constant, so that we can write the partition function as:

$$Z_n = \sum_{\{s\}} L(s_1, s_2) \ldots L(s_i, s_{i+1}) \ldots L(s_n, s_1)$$

;where $L(s_i, s_{i+1}) = e^{\beta J s_i s_{i+1} + \beta B (s_i + s_{i+1})/2}$ (3)

With $s_{n+1}=s_1$. The chain is closed as a ring. Taking all the probabilities or cases of spin s_i, then:

$$L = \begin{pmatrix} e^{\beta(J+B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-B)} \end{pmatrix} \qquad (4)$$

Therefore:

$$Z_n = \sum_{s_1=\pm1} L^n(s_1, s_1) = \mathrm{tr}[L^n] = \lambda_+^n + \lambda_-^n \qquad (5)$$

Where λ_i are the eigenvalues of the matrix L, and can be found as:

$$\left(e^{\beta(J+B)} - \lambda\right)\left(e^{\beta(J-B)} - \lambda\right) - e^{-2\beta J} = 0$$

$$\lambda_\pm = e^{\beta J}\cosh\beta B \pm \sqrt{e^{2\beta J}\sinh^2\beta B + e^{-2\beta J}} \qquad (6)$$

The free energy per spin is given by: [see Mansoor AZ. Habeeb 1984].

$$F = -kT\ln\lambda_+$$

$$F = -\beta^{-1}\ln[e^{\beta J}\cosh\beta B + \sqrt{e^{2\beta J}\sinh^2\beta B + e^{-2\beta J}}] \quad (7)$$

All other thermodynamic functions, such as, the average energy E, the entropy S, the magnetization M, and the specific heat C, can be obtained by differentiating equation (7) as follows:

$$E = \frac{\partial \beta F}{\partial \beta}, \qquad S = \frac{-\partial F}{\partial T},$$

$$M = \frac{-\partial F}{\partial B}, \qquad C = \frac{\partial E}{\partial T} \qquad (8)$$

3. The extended periodic bond model

In 1982 Doman and Williams has examined the properties of a simple periodic chain in which one bond is ferromagnetic and the next three antiferromagnetic and this is repeated periodically. They had examined this model by using standard transfer matrix techniques as follows: [B.G.S Doman and J K Williams 1982]

$$T = \begin{pmatrix} e^{\beta(J+B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-B)} \end{pmatrix} \begin{pmatrix} e^{\beta(-J+B)} & e^{\beta J} \\ e^{\beta J} & e^{\beta(-J-B)} \end{pmatrix}^3$$

$$T = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \qquad (9)$$

For the Hamiltonian equation (1), assume that J_{ij} is –J with probability *x*, and J with probability 1-*x*. Hence for simple chain with one ferromagnetic bond of strength J and the next five antiferromagnetic bond of strength -J which is repeated periodically and using the transfer matrix:

$$T = \begin{pmatrix} e^{\beta(J+B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-B)} \end{pmatrix} \begin{pmatrix} e^{\beta(-J+B)} & e^{\beta J} \\ e^{\beta J} & e^{\beta(-J-B)} \end{pmatrix}^5$$

$$T = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \qquad (10)$$

The eigenvalues of T are:

$$\lambda_{\pm} = \frac{(x_{11}+x_{22}) \pm \sqrt{(x_{11}-x_{22})^2 + 4x_{12} * x_{21}}}{2} \qquad (11)$$

The thermodynamic functions per site, in such case, average energy E, entropy S, magnetization M, and specific heat C are given by: [See B.G.S Doman and J K Williams 1982].

$$E = \left(\frac{-1}{6}\right)\frac{\partial}{\partial\beta}\ln\lambda_+ \qquad (12)$$

$$S = \frac{1}{6}k\ln\lambda_+ - \frac{1}{6}k\beta\frac{\partial}{\partial\beta}\ln\lambda_+ \qquad (13)$$

$$M = \left(\frac{1}{6\beta}\right)\frac{\partial}{\partial B}\ln\lambda_+ \qquad (14)$$

$$C = \frac{1}{6}\beta^2\frac{\partial^2}{\partial\beta^2}\ln\lambda_+ \qquad (15)$$

4. Results and Discussions

At low temperature, the thermodynamic functions are obtained simply by taking the $\beta\to\infty$ limit of the above expressions. Now we consider these quantities for various ranges of the magnetic field as follows:

B>2J

$$\lambda_+ = e^{\beta(6B-4J)} \qquad (16)$$

Then

$$E = \frac{-1}{6}(6B - 4J), \quad S = 0, \quad M = 1 \qquad (17)$$

B=2J

$$\lambda_+ = e^{\beta(6B-4J)} + 2e^{\beta(2B+4J)} \qquad (18)$$

Then

$$E = -\frac{4}{3}J, \quad S = \frac{1}{6}k\ln 3, \quad M = \frac{5}{9} \qquad (19)$$

J<B<2J

$$\lambda_+ = 2e^{\beta(2B+4J)} \qquad (20)$$

Then

$$E = \frac{-1}{3}(B + 2J), \quad S = \frac{1}{6}K\ln 2, \quad M = \frac{1}{3} \qquad (21)$$

When B~J

$$\lambda_+ = \frac{1}{2}\left[3e^{\beta(4J+2B)} + \sqrt{3e^{2\beta(4J+2B)} + 4e^{12\beta J}}\right] \qquad (22)$$

Then

$$E = -J, \quad S = \frac{1}{6}K\ln\frac{3+\sqrt{13}}{2}, \quad M = \frac{1}{\sqrt{13}} \qquad (23)$$

Let a pair of spins connected by a ferromagnetic bond define a new 'superspin' and we regard the chain as being built up of 'superspin' connected by ferromagnetic or antiferromagnetic 'superbond';

where an antiferromagnetic superbond consist of an odd sequence of antiferromagnetic bonds, and a ferromagnetic superbond consists of either a ferromagnetic bond or an even sequence of antiferromagnetic bonds terminated by a ferromagnetic bond. [J K Williams 1981]

Then when B=J a superspin (which are linked together with antiferromagnetic superbonds) can point down if its neighbouring superspins point up, without changing the ground state energy. Any frustrated superbond can be in one of two configurations, figure 1b,1c, depending on which of its four frustratable antiferromagnetic bonds is frustrated. [B.G.S Doman and J K Williams 1982].

The spin behavior is represented by the thermodynamic functions. When B>2J all the spins align with the field in the ground state, leaving the five antiferromagnetic bonds per cell frustrated and the ferromagnetic bond per cell satisfied. While in case B=2J there are three allowed configurations per cell, the first, all the ferromagnetically coupled spins align with the field in the ground state, so in second and third configurations, a spin that is coupled antiferromagnetically to both its neighboring spins can flip down if these neighboring spins point up, as shown in figure 1.

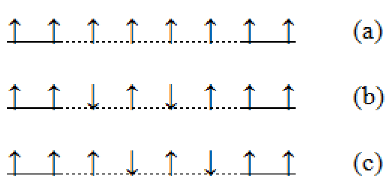

Figure 1: The three allowed configurations. Where antiferromagnetic bonds represented by broken lines and ferromagnetic bonds represented by full lines.

When J<B<2J the ferromagnetically coupled spins align with the field in the ground state, but two of the four spins coupled antiferromagnetically to both its neighboring spins points down, leaving one frustrated antiferromagnetic bond per cell. there are now two allowed configurations per cell, as shown in figure 1(b) and 1(c).

5. Conclusions

In general in low temperature is noted that when the external magnetic field be small, the magnetization of system is lowest, while the entropy and the energy be higher. Now we can extended this results to apply to a system of seven, nine,…,2n+1 antiferromagnetic bonds with one ferromagnetic bond Ising chain.

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