

Power Series Method For Solving Nonlinear Volterra Integro-Differential Equations of The Second Kind

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Abstract

In this work, we present the power series method for solving special types of the first order nonlinear Volterra integro-differential equations of the second kind. To show the efficiency of this method, we solve some numerical examples.

Keywords: Integro-differential, power series.

طريقة متسلسلات القوى لحل معادلات فولتيرا التكاملية - التفاضلية اللاخطية ذات الرتبة الاولى ومن النوع الثاني

الخلاصة

في هذا العمل قمنا بتقديم طريقة متسلسلات القوى لحل أنواع خاصة من معادلات فولتيرا التكاملية-التفاضلية اللاخطية ذات الرتبة الأولى ومن النوع الثاني. ولإثبات كفاءة هذه الطريقة قمنا بحل بعض الأمثلة العددية.

1. Introduction

It is known that the integro-differential equations arise in a great many branches of sciences, for example, in potential theory, acoustics, elasticity, fluid mechanics, theory of population, [4], [3].

Many researchers studied the integro-differential equations, see [4] discussed the existence of the solutions for special types of integro-differential equations, [5], devoted some analytic methods for solving linear Volterra integro-differential equations, [6], [1] gave some numerical methods for solving linear and nonlinear Volterra integro-differential equations, [5] used some approximated methods for solving linear Volterra integro-differential equations.

The power series method is one of the important methods that can

be used to solve the initial value problem of the linear Volterra integro-differential equations of the second kind, [2].

In [7], the power series method is used to solve the nonlinear Volterra integral equations of the second kind of the form:

$$u(x) = f(x) + I \int_0^x k(x,t)[u(t)]^p dt, p \in \mathbb{N}$$

where f and k are known functions, λ is a scalar parameter and u is the unknown function that must be determined.

Here we use the same method to solve the initial value problem that consists of the first order non-linear Volterra integro-differential equations of the second kind of the form:

$$u'(x) = f(x) + I \int_0^x k(x,t)[u(t)]^p dt,$$

$$p \in \mathbb{N} \quad \dots(1.a)$$

together with the initial condition:

$$u(0) = \alpha \quad \dots (1.b)$$

where f and k are known functions, α is a known constant, λ is a scalar parameter and u is the unknown function that must be determined.

2. Power Series Method for Solving Equations (1):

Consider the initial value problem given by equations (1). Assumed the solution of equations (1) takes the form:

$$u(x) \cong e_0 + e_1x + e_2x^2 \dots (2)$$

Then by setting $x=0$ into equation (2) one can get:

$$u(0) \cong e_0.$$

By using the initial condition given by equation (1.b), one can get:

$$e_0 = \alpha.$$

Then by differentiating equation (2) with respect to x and setting $x=0$ in the resulting equation one can have:

$$u'(0) = e_1.$$

On the other hand, from equation (1.a), one can have:

$$u'(0) = f(0).$$

Therefore

$$e_1 = f(0).$$

Thus the approximated solution takes the form:

$$u(x) \cong \alpha + f(0)x + e_2x^2 \dots (3)$$

where e_2 is the unknown parameter that must be determined. To do this, we expand $k(x,y)$ and $f(x)$ as a power series. That is,

$$k(x,t) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} k_{ij} x^i t^j \dots (4)$$

and

$$f(x) = \sum_{i=0}^{\infty} f_i x^i \dots (5)$$

By substituting equations (3)-(5) into equation (1.a) one can get:

$$f(0) + 2e_2x = \sum_{i=0}^{\infty} f_i x^i + \int_0^x \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} k_{ij} x^i t^j [a + f(0)t + e_2t^2]^p dt \dots (6)$$

But

$$\begin{aligned}
 [a + f(0)t + e_2t^2]^p &= \binom{p}{1} a t^{p-1} \sum_{l=0}^{p-1} \binom{p-1}{l} [f(0)]^l \\
 \sum_{k=0}^p \binom{p}{k} a^k [f(0)t + e_2t^2]^{p-k} &= [e_2t]^{p-1-1} + \binom{p}{2} a^2 t^{p-1} \\
 \sum_{k=0}^p \binom{p}{k} a^k t^{p-k} [f(0) + e_2t]^{p-k} &= \sum_{l=0}^{p-1} \binom{p-2}{l} [f(0)]^l [e_2t]^{p-2-1} + \mathbf{L} \\
 \sum_{k=0}^p \binom{p}{k} a^k t^{p-k} \sum_{l=0}^{p-k} \binom{p-k}{l} & \binom{p}{p} a^p t^{p-p} \sum_{l=0}^{p-p} \binom{p-p}{l} \\
 [f(0)]^l [e_2t]^{p-k-1} & [f(0)]^l [e_2t]^{p-p-1} dt
 \end{aligned}$$

Therefore equation (6) becomes

$$\begin{aligned}
 f(0) + 2e_2x &= \sum_{i=0}^{\infty} f_i x^i + \\
 \int_0^x \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} k_{ij} x^i t^j \sum_{k=0}^p \binom{p}{k} a^k t^{p-k} & \\
 \sum_{l=0}^{p-k} \binom{p-k}{l} [f(0)]^l [e_2t]^{p-k-1} dt &= \\
 f_0 + f_1x + f_2x^2 + \mathbf{L} + & \\
 \int_0^x (k_{00} + k_{10}x + k_{01}t + \mathbf{L}) & \\
 \left[\binom{p}{0} t^p \sum_{l=0}^p \binom{p}{l} [f(0)]^l [e_2t]^{p-1} + \right. &
 \end{aligned}$$

$$\begin{aligned}
 & f_0 + f_1x + f_2x^2 + \mathbf{L} + \\
 & \int_0^x (k_{00} + k_{10}x + k_{01}t + \mathbf{L}) [t^p \{(e_2t)^p + \\
 & \binom{p}{1} f(0)(e_2t)^{p-1} + \\
 & \binom{p}{2} [f(0)]^2 (e_2t)^{p-2} + \mathbf{L} + \\
 & [f(0)]^p \} + pat^{p-1} \{(e_2t)^{p-1} + \\
 & \binom{p-1}{1} f(0)(e_2t)^{p-1} + \\
 & \binom{p-1}{2} [f(0)]^2 (e_2t)^{p-3} + \mathbf{L} + \\
 & [f(0)]^{p-1} \} + \frac{p(p-1)}{2} a^2 t^{p-2} \\
 & \{(e_2t)^{p-2} + \binom{p-2}{1} f(0)(e_2t)^{p-3} + \\
 & \binom{p-2}{2} [f(0)]^2 (e_2t)^{p-4} + \mathbf{L} + \\
 & [f(0)]^{p-2} \} + a^p] dt
 \end{aligned}$$

$$\begin{aligned}
 Q(x^2) = & f_2x^2 + f_3x^3 + \mathbf{L} + \\
 & \int_0^x (k_{00} + k_{10}x + k_{01}t + \mathbf{L}) [t^p \{(e_2t)^p + \\
 & \binom{p}{1} f(0)(e_2t)^{p-1} + \\
 & \binom{p}{2} [f(0)]^2 (e_2t)^{p-2} + \mathbf{L} + \\
 & [f(0)]^p \} + pat^{p-1} \{(e_2t)^{p-1} + \\
 & \binom{p-1}{1} f(0)(e_2t)^{p-1} + \\
 & \binom{p-1}{2} [f(0)]^2 (e_2t)^{p-3} + \mathbf{L} + \\
 & [f(0)]^{p-1} \} + \frac{p(p-1)}{2} a^2 t^{p-2} \\
 & \{(e_2t)^{p-2} + \binom{p-2}{1} f(0)(e_2t)^{p-3} + \\
 & \binom{p-2}{2} [f(0)]^2 (e_2t)^{p-4} + \mathbf{L} + \\
 & [f(0)]^{p-2} \}
 \end{aligned}$$

It is clear that $f(0) = f_0$ and $Q(x^2)$ is a polynomial of degree greater than or equal two. By neglecting $Q(x^2)$ and solving the equation $e_2 = \frac{f_1 + k_{00}\alpha^p}{2}$, the unknown parameter e_2 is determined and therefore the coefficient of x^2 in equation (3) is obtained.

By repeating the above procedure $m-1$ iterations, a power series of the following form derives:

$$y(x) = \sum_{i=0}^m e_i x^i \quad \dots\dots (7)$$

Equation (7) is an approximated solution of the initial value problem given by equations (1).

3. Numerical Examples:

In this section we present two examples that are solved by using power series method. These examples shows the efficiency of this method.

Example (1):

Consider the first order nonlinear integro-differential equation of the second kind:

$$u'(x) = -e^{-x} + \frac{1}{3} \left(x^2 + x + \frac{1}{3} \right) e^{-3x} - \frac{1}{3} x^2 - \frac{1}{9} + \int_0^x (x^2 + t) [u(t)]^3 dt \dots(8.a)$$

together with the initial condition:

$$u(0)=1 \dots(8.b)$$

Here

$$f(x) = -e^{-x} + \frac{1}{3} \left(x^2 + x + \frac{1}{3} \right) e^{-3x} - \frac{1}{3} x^2 - \frac{1}{9}, p=3$$

and

$$k(x,t) = x^2 + t.$$

We solve this example by using the power series method. To do this, let $e_0 = u(0)$ and $e_1 = u'(0)$. Therefore $e_0 = 1$ and $e_1 = f(0) = -1$. Assume the solution of the above initial value problem takes the form:

$$u(x) \cong e_0 + e_1 x + e_2 x^2.$$

Hence

$$u(x) \cong 1 - x + e_2 x^2.$$

But

$$e^{-x} = \sum_{i=0}^{\infty} \frac{(-x)^i}{i!}$$

and

$$e^{-3x} = \sum_{i=0}^{\infty} \frac{(-3x)^i}{i!}.$$

Therefore

$$\begin{aligned} f(x) &= -\sum_{i=0}^{\infty} \frac{(-x)^i}{i!} + \frac{1}{3} \left(x^2 + x + \frac{1}{3} \right) \sum_{i=0}^{\infty} \frac{(-3x)^i}{i!} - \frac{1}{3} x^2 - \frac{1}{9} \\ &= -\left(1 - x + \frac{x^2}{2!} - \mathbf{L} \right) + \frac{1}{3} \left(x^2 + x + \frac{1}{3} \right) \left(1 - 3x + \frac{9x^2}{2!} - \mathbf{L} \right) - \frac{1}{3} x^2 - \frac{1}{9} \\ &= \left(-1 + \frac{1}{9} - \frac{1}{9} \right) + \left(1 + \frac{1}{3} - \frac{1}{3} \right) x - \sum_{i=2}^{\infty} \frac{(-x)^i}{i!} + \frac{1}{3} \left(x^2 + x + \frac{1}{3} \right) \sum_{i=2}^{\infty} \frac{(-3x)^i}{i!} - x^3 - x^2 \\ &= -1 + x - \sum_{i=2}^{\infty} \frac{(-x)^i}{i!} + \frac{1}{3} \left(x^2 + x + \frac{1}{3} \right) \sum_{i=2}^{\infty} \frac{(-3x)^i}{i!} - x^3 - x^2 \end{aligned}$$

Hence $f_1 = 1$. On the other hand

$$k_{ij} = \begin{cases} 1 & \text{for } (i, j) = (0,1) \text{ and} \\ & (i, j) = (0,2) \\ 0 & \text{e.w} \end{cases}$$

Thus $k_{00} = 0$. Therefore

$$e_2 = \frac{f_1 + k_{00}\alpha^p}{2} = \frac{1}{2}.$$

In this case

$$\begin{aligned} Q(x^2) &= \frac{1}{8}(e_2)^3 x^8 - \frac{1}{7}(e_2)^3 x^9 - \\ &\frac{6}{35}(e_2)^2 x^7 + \frac{1}{2}(e_2)^2 x^8 - \frac{1}{2}(e_2)^2 x^6 + \\ &e_2 x^6 + \frac{1}{5}e_2 x^5 - \frac{4}{5}x^5 - \frac{3}{5}e_2 x^7 - \\ &\frac{3}{4}e_2 x^4 + \frac{3}{4}x^4 + \frac{1}{4}x^6 - \frac{1}{6}x^2 + 1 + \\ &\frac{x^2}{2!} - \frac{x^3}{3!} + \mathbf{L} - \\ &\frac{1}{3}x^2 \left[1 - 3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \mathbf{L} \right] - \\ &\frac{1}{3}x \left[-3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \mathbf{L} \right] - \\ &\frac{1}{9} \left[\frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \mathbf{L} \right] + \frac{1}{3}x^2. \end{aligned}$$

Thus

$$u(x) \cong 1 - x + \frac{1}{2}x^2.$$

By repeating the above argument for the approximated solution:

$$u(x) \cong 1 - x + \frac{1}{2}x^2 + e_3 x^3$$

one can get:

$$\left(3e_3 - \frac{1}{6} - \frac{1}{3} + 1 \right) x^2 + Q(x^3) = 0$$

....(9)

where

$$\begin{aligned} Q(x^3) &= \frac{12}{35}x^7 + \frac{5}{8}x^6 + \frac{7}{64}x^8 - \\ &\frac{1}{56}x^9 + \frac{3}{8}x^4 - \frac{7}{10}x^5 - \\ &\frac{1}{11}(e_3)^3 x^{11} - \frac{2}{21}(e_3)^2 x^9 + \frac{29}{84}e_3 x^9 - \\ &\frac{1}{6}(e_3)^2 x^{11} - \frac{1}{10}(e_3)^3 x^{12} + \\ &\frac{9}{40}(e_3)^2 x^{10} - \frac{5}{8}e_3 x^8 - \frac{3}{8}(e_3)^2 x^8 - \\ &\frac{3}{32}e_3 x^{10} + \frac{12}{35}e_3 x^7 - \frac{3}{5}e_3 x^5 + \\ &\frac{1}{4}e_3 x^6 + \left[-\frac{x^2}{3!} + \frac{x^4}{4!} - \mathbf{L} \right] - \\ &\frac{1}{3}x^2 \left[-3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \mathbf{L} \right] - \\ &\frac{1}{3}x \left[\frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \mathbf{L} \right] - \\ &\frac{1}{9} \left[-\frac{27}{3!}x^3 + \frac{81}{4!}x^4 - \mathbf{L} \right]. \end{aligned}$$

By neglecting $Q(x^3)$ then equation (9) becomes

$$\left(3e_3 - \frac{1}{3} + 1 - \frac{1}{2} + \frac{1}{3} \right) x^2 = 0$$

and hence $e_3 = -\frac{1}{3!}$. Thus

$$u(x) \cong 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3.$$

By repeating the above argument for the approximated solution:

$$u(x) \cong 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + e_4 x^4$$

one can get:

$$\left(4e_4 - \frac{1}{6} + 1 - \frac{3}{2} + \frac{1}{2}\right)x^3 + Q(x^4) = 0 \quad \dots (10)$$

Where

$$\begin{aligned} Q(x^4) = & \frac{59}{765}x^9 + \frac{7}{320}x^{10} + \\ & \frac{5}{1188}x^{11} + \frac{13}{64}x^8 + \frac{3}{8}x^4 - \\ & \frac{1}{2160}x^{12} - \frac{2}{5}x^7 - \frac{1}{24}(e_4)^2x^{14} + \\ & \frac{7}{12}x^6 - \frac{3}{5}x^5 - \frac{1}{14}(e_4)^3x^{14} - \\ & \frac{26}{63}e_4x^9 + \frac{9}{35}e_4x^7 - \frac{1}{13}(e_4)^3x^{15} - \\ & \frac{14}{143}(e_4)^2x^{13} - \frac{1}{2}e_4x^6 - \frac{1}{4}e_4x^8 - \\ & \frac{7}{40}(e_4)^2x^{12} + \frac{31}{720}e_4x^{12} - \\ & \frac{1}{132}e_4x^{13} - \frac{2}{33}(e_4)^2x^{11} - \\ & \frac{59}{396}e_4x^{11} - \frac{3}{10}(e_4)^2x^{10} + \\ & \frac{13}{40}e_4x^{10} + \left[\frac{x^4}{4!} - \frac{x^5}{5!}\mathbf{L}\right] - \\ & \frac{1}{3}x^2\left[\frac{9}{2!}x^2 - \frac{27}{3!}x^3 + \mathbf{L}\right] - \\ & \frac{1}{3}x\left[\frac{27}{3!}x^3 + \frac{81}{4!}x^4 - \mathbf{L}\right] - \\ & \frac{1}{9}\left[\frac{81}{4!}x^4 + \frac{243}{5!}x^5 - \mathbf{L}\right]. \end{aligned}$$

By neglecting $Q(x^4)$ then equation (10) becomes

$$\left(4e_4 - \frac{1}{6} + 1 - \frac{3}{2} + \frac{1}{2}\right)x^3 = 0$$

and hence $e_4 = \frac{1}{4!}$. Thus

$$u(x) \cong 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4.$$

By continuing in this manner, one can get:

$$\begin{aligned} u(x) \cong & 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \\ & \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \dots = e^{-x} \end{aligned}$$

Note that this approximated solution is the exact solution of the initial value problem given by equations (8).

Example (2):

Consider the first order nonlinear integro-differential equation of the second kind:

$$\begin{aligned} u'(x) = & 3x^2 - \frac{1}{8}x^8 \sin x + \\ & \int_0^x t \sin x [u(t)]^2 dt \dots(11.a) \end{aligned}$$

together with the initial condition:

$$u(0)=0 \quad \dots(11.b)$$

$$\text{Here } f(x) = 3x^2 - \frac{1}{8}x^8 \sin x,$$

$$p=2 \text{ and } k(x, t) = t \sin x.$$

We solve this example by using the power series method. To do this, let $e_0 = u(0)$ and $e_1 = u'(0)$. Therefore $e_0 = 0$ and $e_1 = f(0) = 0$. Assume the solution of the above initial value problem takes the form:

$$u(x) \cong e_0 + e_1 x + e_2 x^2$$

Hence

$$u(x) \cong e_2 x^2.$$

But

$$\sin x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}.$$

Therefore

$$\begin{aligned} f(x) &= 3x^2 - \frac{1}{8} x^8 \sin x \\ &= 3x^2 - \frac{1}{8} x^8 \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}. \end{aligned}$$

and

$$k(x, t) = t \sin x = t \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}.$$

Hence $f_1 = 0$ and $k_{00} = 0$ and this implies that

$$e_2 = \frac{f_1 + k_{00} \alpha^p}{2} = 0.$$

In this case

$$\begin{aligned} Q(x^2) &= -\frac{1}{6} (e_2)^2 x^6 \left[\sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} \right] - \\ & 3x^2 + \frac{1}{8} x^8 \left[\sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} \right] \end{aligned}$$

Thus

$$u(x) \cong 0.$$

By repeating the above argument for the approximated solution:

$$u(x) \cong e_3 x^3$$

one can get:

$$\begin{aligned} (3e_3 - 3)x^2 + Q(x^3) &= 0 \\ \dots (12) \end{aligned}$$

where

$$\begin{aligned} Q(x^3) &= -\frac{1}{8} (e_3)^2 x^8 \left[\sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} \right] + \\ & \frac{1}{8} x^8 \left[\sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} \right] \end{aligned}$$

By neglecting $Q(x^3)$ then equation (12) becomes

$$(3e_3 - 3)x^2 = 0$$

and hence $e_3 = 1$. Thus

$$u(x) \cong x^3.$$

By repeating the above argument for the approximated solution:

$$\begin{aligned} u(x) &\cong x^3 + e_4 x^4 \\ \text{one can have:} \\ (4e_4)x^3 + Q(x^4) &= 0 \quad \dots(13) \end{aligned}$$

where

$$\begin{aligned} Q(x^4) &= -\left[\frac{1}{10} (e_4)^2 x^{10} + \frac{2}{9} e_4 x^9 \right] \\ & \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} \end{aligned}$$

By neglecting $Q(x^4)$ then equation (13) becomes

$$(4e_4)x^3 = 0$$

and hence $e_4 = 0$. Thus

$$u(x) \cong x^3.$$

By continuing in this manner, one can get:

$$u(x) \cong x^3 + 0x^4 + 0x^5 + \dots = x^3$$

Note that this approximated solution is the exact solution of the initial value problem given by equations (11).

Remark(1):

The power series method can be also used to solve the initial value problem that consists of the first order nonlinear Volterra integro-differential equation of the second kind:

$$u'(x) = f(x) + \int_a^x k(x, t)[u(t)]^p dt \quad \dots(14.a)$$

together with the initial condition:

$$u(a) = \alpha \quad \dots(14.b)$$

To do this let $z = t - a$ then equation (14.a) becomes

$$u'(x) = f(x) + \int_0^{x-a} k(x, z+a)[u(z+a)]^p dz$$

Then by setting $s = x - a$ in the above equation one can have:

$$y'(s) = f(s+a) + \int_0^s k(s+a, z+a)[y(z)]^p dz \quad \dots(15.a)$$

where $y(s) = u(s+a)$. Thus

$$y(0) = u(a) = \alpha \quad \dots(15.b)$$

Therefore the initial value problem given by equations (14) reduces to the initial value problem given by equations (15).

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