Semi – Bounded Modules

Adwia Jassim Abdul-Al-Kalik*

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Abstract:
Let R be a commutative ring with identity, and let M be a unity R-module. M is called a bounded R-module provided that there exists an element $x \in M$ such that $\text{ann}_R(M) = \text{ann}_R(x)$. As a generalization of this concept, a concept of semi-bounded module has been introduced as follows: M is called a semi-bounded if there exists an element $x \in M$ such that $\sqrt{\text{ann}_R(M)} = \sqrt{\text{ann}_R(x)}$. In this paper, some properties and characterizations of semi-bounded modules are given. Also, various basic results about semi-bounded modules are considered. Moreover, some relations between semi-bounded modules and other types of modules are considered.

Key words: Commutative ring, R-module, semi-bounded modules.

Introduction:
Let R be a commutative ring with unity, and let M be an R-module. An R-module M is called a semi-bounded module if there exists an element $x \in M$ such that $\sqrt{\text{ann}_R(M)} = \sqrt{\text{ann}_R(x)}$, where $\text{ann}_R(M) = \{r \in R : r \cdot m = 0, \forall m \in M\}$. Our concern in this paper is to study semi-bounded modules and to look for any relation between semi-bounded modules and certain types of well-known modules especially with bounded modules. This paper consists of two sections. In the first section, the definition of a semi-bounded module is recalled and we illustrate it by some examples, we also give some of the basic properties of semi-bounded modules. We end the section by studying the localization of semi-bounded modules, see (1.13).

In section two, we study the relation between semi-bounded modules and bounded modules. It is clear that every bounded module is semi-bounded module, but the converse is not true in general. We give in (2.1), a conditions under which the two concepts are equivalent. Next we investigate the relationships between semi-bounded, prime, quasi-Dedekind, cyclic and multiplication modules see (2.3), (2.8).

1. Semi-Bounded Modules
Following [1] an R-module M is said to be a bounded module if there exists an element $x \in M$ such that $\text{ann}_R(M) = \text{ann}_R(x)$, where $\text{ann}_R(M) = \{r \in R : r \cdot m = 0, \forall m \in M\}$. In this section the concept of semi-bounded module is introduced as a generalization of a bounded module and we give some properties and characterizations for this concept. We end this section by studying the behaviour of semi-bounded modules under localization.

Definition 1.1:
An R-module M is said to be semi-bounded module if there exists an element $x \in M$ such that $\sqrt{\text{ann}_R(M)} = \sqrt{\text{ann}_R(x)}$. We give some examples and remarks.
Remarks and Examples 1.2:
1. Every bounded R-module is a semi-bounded module. But the converse is not true in general. However we have no example.

*Ministry of Education – Vocational Education – Khalis Industrial School
2. Every simple R-module is a semi-bounded module. But the converse is not true in general, for example: The Z-module Z is a semi-bounded module but not simple.

3. Q as a Z-module is a semi-bounded module.

4. Consider, the Z-module M = Z ⊕ Z6. Then 
\[ \sqrt{\text{ann}_Z M} = \sqrt{\text{ann}_Z (1,0)}. \]
Therefore, M is a semi-bounded module.

5. Z_{p^n} is not a semi-bounded Z-module.

**Proof:** We know that every submodule of Z_{p^n} is of the form \( \frac{1}{p^n} + Z \), where \( n \) be a non-negative integer, so \( \sqrt{\text{ann}_Z \frac{1}{p^n} + Z} = \sqrt{p^nZ} = pZ \).

But \( \sqrt{\text{ann}_Z \frac{1}{p^n} + Z} \neq \sqrt{\text{ann}_Z \frac{1}{p^n} + Z^+} \), so Z_{p^n} is not a semi-bounded module.

6. Every cyclic R-module is a semi-bounded.

**Proof:** It follows directly by [2, Corollary 1.1.3, ch.1] and (1.2,1).

However, the converse is not true in general for example: The Z-module Q is a semi-bounded by (1.2,3), but not cyclic.

7. For each positive integer n, Z_n as a Z-module is a semi-bounded.

It is known that, if M is an R-module and I is an ideal of R which is contained in ann_R M then M is an R/I-module, by taking \((r+I) x = r x, \forall x \in M, r \in R\). Now, we can give the following result.

**Theorem 1.3** Let M be an R-module and let I be an ideal of R, which is contained in ann_R M. Then M is a semi-bounded R-module if and only if M is a semi-bounded R/I-module.

**Proof:** If M is a semi-bounded R-module. To prove M is a semi-bounded R/I-module, we must prove \( \sqrt{\text{ann}_{R/I} M} = \sqrt{\text{ann}_{R/I} (x)} \), for some \( x \in M \).

It is clear that \( \sqrt{\text{ann}_{R/I} M} \subseteq \sqrt{\text{ann}_{R/I} (x)} \). Let \( r \) and \( n \) such that \( r^n x = 0 \), then \( r^n x = (r^n + I)x = (r+I)^n x = 0 \), hence \( r^n x = 0 \). Therefore, \( r^n x = 0 \) and so are \( r^n \in \text{ann}_{R/I} M \). Hence, M is a semi-bounded R/I-module. The following result is an immediate consequence of theorem (1.3).
Corollary 1.4:
Let M be an R-module, then M is a semi-bounded R-module if and only if M is a semi-bounded R\ann R M-module.
Recall that an R-module M is called the internal direct sum of two R-modules M1 and M2 of M, written as M=M1\oplus M2 if and only if M=M1+M2 and M1\cap M2={0}.
Now, the following result has been stated and proved.
Proposition 1.5:
Let M1 and M2 be two semi-bounded R-modules, then M1\oplus M2 is a semi-bounded R-module.
Proof: There exists x\in M1 such that \sqrt{\ann R M_1}=\sqrt{\ann R (x)} . Also there exists y\in M2 such that \sqrt{\ann R M_2}=\sqrt{\ann R (y)} . So (x,y)\in M1\oplus M2. We claim that \sqrt{\ann R (M_1 \oplus M_2)}=\sqrt{\ann R ((x,y))} .
Let r\in \sqrt{\ann R ((x,y))} , then r^n(x,y)=(0,0) for some n\in \mathbb{Z}_+, and so (r^n x,r^n y)=(0,0). It follows that r^n x=0 and r^n y=0, that is r^n\in \ann R(x) and r^n\in \ann R(y) and so r\in \sqrt{\ann R (x)} =\sqrt{\ann R M_1} and r\in \sqrt{\ann R (y)} =\sqrt{\ann R M_2} . Now, if (m,m')\in M1\oplus M2 then r^n(m,m')=(r^n m,r^n m')=(0,0) for some n\in \mathbb{Z}_+, implies that r^n\in \ann R(M_1 \oplus M_2) and so r\in \sqrt{\ann R (M_1 \oplus M_2)} .
Therefore \sqrt{\ann R (M_1 \oplus M_2)}=\sqrt{\ann R ((x,y))} .
Recall that a submodule B of an R-module M is called a direct summand of M if and only if there exists a submodule C of M such that M=B\oplus C, [3,p.31].
Note that a direct summand of a semi-bounded module need not be semi-bounded in general for example:
\[ M=\mathbb{Z}\oplus \mathbb{Z}_{p^\infty} \] as a Z-module, M is semi-bounded since \sqrt{\ann R M}=\sqrt{\ann R ((1,0))}, but \mathbb{Z}_{p^\infty} is not semi-bounded \mathbb{Z}-module, by (1.2,(5)).
By proposition (1.5) and by mathematical induction we have the following:
Corollary 1.6:
A finite direct sum of semi-bounded R-modules is semi-bounded.
Remark 1.7:
If M/N is a semi-bounded R-module, then it is not necessary that M is a semi-bounded R-module as the following example shows.
Let M=\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p as a Z-module and p>2.
N=\bigoplus_{p^2} \mathbb{Z}_p is a submodule of M, where p is a prime number, so M/N=\mathbb{Z}_2 is a semi-bounded Z-module. But M is not a semi-bounded Z-module.
Recall that a submodule N of an R-module M is said to be pure if I M \cap N=I N for every ideal I of R. In case R is a principal ideal domain (PID) or M is cyclic, then N is pure if and only if rM \cap N=rN, \forall r\in R, [4].
By using this concept, we have the following:
Proposition 1.8:
Let N be a pure submodule of an R-module M such that M/N is a semi-bounded R-module and \ann R M=[N : M]. Then M is a semi-bounded R-module, where [N : M]={r\in R: rM \subseteq N}.
Proof: Since M/N is a semi-bounded R-module, then there exists x\in M/N such that \sqrt{\ann R (M/N)}=\sqrt{\ann R (x)} .
But \[ [N : M]=\ann R M \] by hypothesis.
And on the other hand \[ \ann R M/N=[N : M] \], hence
\[ \sqrt{\text{ann}_R M} = \sqrt{\text{ann}_R (x)} \]

...(1)

No we can show that
\[ \sqrt{\text{ann}_R (x)} = \sqrt{\text{ann}_R (m)} \]
for some \( m \in M \). By \([2, \text{proposition (1.1.21), ch.1}]\), we have \( \text{ann}_R (x) = \text{ann}_R (m) \)
for some \( m \in M \). Therefore
\[ \sqrt{\text{ann}_R (x)} = \sqrt{\text{ann}_R (m)} \]
...(2)

Thus by (1) and (2),
\[ \sqrt{\text{ann}_R M} = \sqrt{\text{ann}_R (m)} \]
for some \( m \in M \) and \( M \) is a semi-bounded \( R \)-module by definition (1.1).

Recall that an \( R \)-module \( M \) is called \( F \)-regular if every submodule of \( M \) is pure, \([4]\).
The following result follows immediately from proposition (1.8).

Corollary 1.9:
Let \( M \) be \( F \)-regular \( R \)-module and \( N \) be a submodule of \( M \) such that \( M/N \) is a semi-bounded and \( \text{ann}_R M = [N : M] \).

Then \( M \) is a semi-bounded \( R \)-module.

Corollary 1.10:
Let \( N \) be a submodule of an \( R \)-module \( M \), if every finitely generated submodule of \( N \), is pure in \( M \) such that \( M/N \) is a semi-bounded and \( \text{ann}_R M = [N : M] \). Then \( M \) is a semi-bounded.

**Proof:** Since every finitely generated submodule of \( N \) is pure in \( M \), implies that \( N \) is pure submodule by \([4, \text{corollary 2}]\). Therefore \( M \) is a semi-bounded by proposition (1.8).

Hence we have another consequence of proposition (1.8).

Corollary 1.11:
Let \( R \) be a \((\text{PID})\), \( M \) is an \( R \)-module, \( N \) is a divisible submodule of \( M \) such that \( M/N \) is a semi-bounded \( R \)-module and \( \text{ann}_R M = [N : M] \), then \( M \) is a semi-bounded.

**Proof:** It is enough to show \( N \) is pure in \( M \). Since \( N \) is a divisible submodule of \( M \), then \( rN = N \) for every \( 0 \neq r \in R \) and so \( rM \cap N = rM \cap rN = rN \). Thus \( N \) is pure. Therefore \( M \) is a semi-bounded \( R \)-module by proposition (1.8). Recall that a subset \( S \) of a ring \( R \) is called multiplicatively closed if \( 1 \in S \) and \( ab \in S \) for every \( a, b \in S \), we know that a proper ideal \( P \) in \( R \) is prime if and only if \( R \setminus P \) is multiplicatively closed, \([5, \text{p.42}]\).

Now, let \( M \) be an \( R \)-module and \( S \) be a multiplicatively closed subset of \( R \) and let \( R_S \) be the set of all fractional \( r/s \) where \( r \in R \) and \( s \in S \) and \( M_S \) be the set of all fractional \( x/s \) where \( x \in M \) and \( s \in S \). For \( x_1, x_2 \in M \) and \( s_1, s_2 \in S \), \( x_1/s_1 = x_2/s_2 \) if and only if there exists \( t \in S \) such that \( t(s_1x_2 – s_2x_1) = 0 \). So, we can make \( M_S \) in to \( R_S \)-module by setting \( x/s + y/t = (tx + sy)/st \) and \( r/t \cdot x/s = rx/ts \) for every \( x, y \in M \) and \( s, t \in S \), \( r \in R \). \( M_S \) is the module of fractions. If \( S = R \setminus P \) where \( P \) is a prime ideal we write \( M_p \) instead of \( M_S \) and \( R_p \) instead of \( R_S \). \( R_p \) is often called the localization of \( R \) at \( P \), and \( M_p \) is the localization of \( M \) at \( P \). In order, we investigate the behaviour of a semi-bounded module under localization. But first we state and prove the following lemma.

Lemma 1.12:
Let \( M \) be an \( R \)-module, let \( S \) be a multiplicatively closed subset of \( R \). If \( (\text{ann}_R (x))_S = \text{ann}_R (x)_S \), then
\[ \left( \sqrt{\text{ann}_R (x)} \right)_S = \sqrt{\text{ann}_R (x)_S} . \]

**Proof:** Let \( m \in \left( \sqrt{\text{ann}_R (x)} \right)_S \), then \( m = r/s \) for some \( r \in \sqrt{\text{ann}_R (x)} \) and \( s \in S \) and hence \( m^n = (r/s)^n \in (\text{ann}_R (x))_S = \text{ann}_R (x)_S \) for some \( n \in \mathbb{Z}^+ \). Thus \( m = r/s \in \sqrt{\text{ann}_R (x)_S} \) and so
\( \left( \sqrt{\text{ann}_R(x)} \right)_S \subseteq \sqrt{\text{ann}_{R_S}(x)}_S \)

...(1)

Now, let \( m=r/s \in \sqrt{\text{ann}_{R_S}(x)}_S \) then
\( m^n=(r/s)^n=r^n/s^n \in \text{ann}_{R_S}(x)_S = (\text{ann}_R(x))_S \)
for some \( n \in \mathbb{Z}_+ \). Hence, there exists \( r_1 \in \text{ann}_R(x) \) and \( t \in S \) such that \( r^n/s^n=r_1/t \) and so there exists \( t_1 \in S \) such that \( t_1r^n=t_1r_1s^n \in \text{ann}_R(x) \), which implies that \( t_1r^n \in \text{ann}_R(x) \) and so \( t_1r \in \sqrt{\text{ann}_R(x)}_S \). Therefore,

\( m=r/s=t_1r_1t_2s \in \left( \sqrt{\text{ann}_R(x)}_S \right) \) and

\( \sqrt{\text{ann}_{R_S}(x)_S} \subseteq \left( \sqrt{\text{ann}_R(x)}_S \right) \)

...(2)

By (1) and (2) we get

\( \left( \sqrt{\text{ann}_R(x)}_S \right) = \sqrt{\text{ann}_{R_S}(x)_S} \).

Now, the following proposition has been stated and proved:

Proposition 1.13: Let \( M \) be a finitely generated semi-bounded \( R \)-module and \( S \) be a multiplicatively closed subset of \( R \), then \( M_S \) is a semi-bounded \( R_S \)-module.

Proof: Since \( M \) is a semi-bounded \( R \)-module, then there exists \( x \in M \) such that \( \sqrt{\text{ann}_R M} = \sqrt{\text{ann}_R(x)} \), and so

\( \left( \sqrt{\text{ann}_{R_S}(x)_S} \right) = \left( \sqrt{\text{ann}_R(x)}_S \right) \). And since \( M \) is a finitely generated, so \( \text{ann}_M = \text{ann}_{R_S} M \) by [6, proposition 3.14, p.43]. Hence

\( \left( \sqrt{\text{ann}_R M} \right)_S = \sqrt{\text{ann}_{R_S} M}_S \)

by [7, lemma 2.1.24, ch.2].

On the other hand,

\( \sqrt{\text{ann}_R(x)_S} = \sqrt{\text{ann}_{R_S}(x)_S} \)

by lemma (1.12). Therefore

\( \sqrt{\text{ann}_{R_S} M}_S = \sqrt{\text{ann}_{R_S}(x)_S} \) and \( M_S \) is a semi-bounded \( R_S \)-module.

The following corollary follows immediately from proposition (1.13).

If \( P \) is a prime ideal of \( R \) and \( M \) is a finitely generated semi-bounded \( R \)-module, then \( M_P \) is a semi-bounded \( R_P \)-module.

2. Some Relations Between Semi-Bounded Modules and Other Modules

In this section, we study the relationships between semi-bounded modules and other modules such as bounded modules, prime, quasi-Dedekined, cyclic and multiplication modules. As we have mentioned in (1.2(1)), that bounded module is a semi-bounded module and the converse need not be true in general. However the following result shows that the converse is true. But first the following definition is needed. Recall that a submodule \( N \) of an \( R \)-module \( M \) is said to be semi-prime if for every \( r \in R, x \in M, k \in \mathbb{Z}_+ \), such that \( r^kx \in N \), then \( rx \in N \), see [7].

Proposition 2.1: If \( M \) is a semi-bounded and \( (0) \) is a semi-prime submodule of \( M \), then \( M \) is a bounded \( R \)-module.

Proof: To prove \( M \) is a bounded module, we must prove \( \text{ann}_R(M) = \text{ann}_R(x) \) for some \( x \in M \). It is clear that \( \text{ann}_R(M) \subseteq \text{ann}_R(x) \). Let \( r \in \text{ann}_R(x) \), hence \( r \in \sqrt{\text{ann}_R(x)} \). But \( M \) is a semi-bounded module, so

\( \sqrt{\text{ann}_R M} = \sqrt{\text{ann}_R(x)} \).

Hence, \( r \in \sqrt{\text{ann}_R M} \), which implies that \( r^n \in \text{ann}_R(M) \) for some \( n \in \mathbb{Z}_+ \). Thus, \( r^n m = 0 \) for each \( m \in M \). But \( (0) \) is a semi-prime submodule of \( M \), then \( rm = 0 \) and hence \( r \in \text{ann}_R(M) \), so that \( \text{ann}_R(x) \subseteq \text{ann}_R(M) \). Therefore, \( \text{ann}_R(x) = \text{ann}_R(M) \), that is \( M \) is bounded \( R \)-module.

Next, we study the relationship between semi-bounded modules and prime modules. And we give a condition under which the two concepts are equivalent. Recall that
an R-module M is said to be prime module if \(\text{ann}_R M = \text{ann}_R N\) for every non-zero submodule N of M, [8]. It is clear that every prime R-module is bounded and hence it is semi-bounded, but the converse need not be true in general, for example: Let \(M = \mathbb{Z}_8\) as a Z-module is bounded and so semi-bounded, but not prime module since \(\text{ann}_\mathbb{Z} M = \text{ann}_\mathbb{Z} (\mathbb{Z}_8) = 8\mathbb{Z}\) but \(\text{ann}_\mathbb{Z} (2) = 4\mathbb{Z}\). In order we can give the following result. But first we need the following definition. Recall that a submodule N of an R-module M is called a bounded if there exists \(x \in N\) such that \(\text{ann}_R N = \text{ann}_R (x)\), see [2].

**Proposition 2.2:**

Let M be an R-module and let \(0 \neq x \in M\) such that:

1. \(\sqrt{\text{ann}_R M} = \sqrt{\text{ann}_R (x)}\)
2. \((0)\) is a semi-prime submodule of M.
3. Every non-zero submodule N of M is bounded. Then M is a prime R-module.

**Proof:** Let N be a non-zero R-submodule of M, to prove \(\text{ann}_R M = \text{ann}_R N\). Since every non-zero submodule N of M is bounded, then \(\text{ann}_R N = \text{ann}_R (x)\) for some \(x \in N\). Therefore (by condition 1)

\[
\sqrt{\text{ann}_R M} = \sqrt{\text{ann}_R N}
\]

and M is a primary R-module by [7, theorem (2.1.3), ch.2]. But \((0)\) is a semi-prime submodule of M (by condition 2), then M is a prime R-module by [7, corollary (2.3.3), ch.2].

The following corollary, we give a condition under which a semi-bounded module is prime.

**Corollary 2.3:**

If M is a semi-bounded R-module such that every non-zero submodule N of M is bounded and \((0)\) is a semi-prime submodule of M, then M is a prime R-module. Next, we study the relationship between semi-bounded and quasi-Dedekind module. Now, the following definitions are needed. Let M be an R-module. A submodule N of M is called quasi-invertible if \(\text{Hom}_R (M, N, M) = 0\) [9, definition 1.1.1, ch.1]. And M is called quasi-Dedekind R-module if every submodule N of M is quasi-invertible, [9, definition 2.1.1, ch.2].

**Remark 2.4:**

Every quasi-Dedekind R-module is a semi-bounded R-module.

**Proof:** By [9, theorem 1.7, ch.2] every quasi-Dedekind is prime and hence it is semi-bounded. However, the converse is not true in general, for example: \(Z_8\) as Z-module is semi-bounded. But it is not prime (since \(\text{ann}_\mathbb{Z} Z_8 = 8\mathbb{Z}\) and \(\text{ann}_\mathbb{Z} (2) = 4\mathbb{Z}\)). So it is not quasi-Dedekind. In the following proposition, we give a condition under which the converse of remark (2.4) is true.

**Proposition 2.5:**

If M is a uniform semi-bounded R-module such that \((0)\) is a semi-prime submodule of M and every non-zero submodule of M is bounded, then M is a quasi-Dedekind.

**Proof:** By proposition (2.3), M is a prime R-module. But M is uniform, so by [9, theorem 11, ch.2] we obtain the result. As we mentioned in (1.2,(6)), that cyclic module is a semi-bounded and the converse need not be true in general. However the following result shows that the converse is true. But first the following definition is needed. Recall that an R-module M is said to be fully stable if \(\text{ann}_M (\text{ann}_R (x)) = (x)\) for each \(x \in M\). [10, corollary 3.5]. In the following proposition, we give a condition under which the converse of (1.2,(6)) is true.

**Proposition 2.6:**

If M is a fully stable semi-bounded R-module and \((0)\) is a semi-prime submodule, then M is cyclic R-module.

**Proof:** Since M is a semi-bounded R-module and \((0)\) is a semi-prime submodule, then M is a bounded by proposition (2.1). But M is a fully stable, so by [2, proposition 1.1.4, ch.1]
we obtain the result. Now, the relationship between semi-bounded modules and multiplication modules has been studied. And we give a condition under which the two concepts are equivalent. Recall that an R-module M is said to be multiplication module if for every submodule N of M, there exists an ideal I in R such that N=IM, [11]. Note that it is not necessary that every semi-bounded is multiplication for example: Q as a Z-module is semi-bounded, but not multiplication, since Z is a submodule of Q, but $\mathbb{Z}/I$ an ideal of Z such that IQ=Z. In the following corollary, we give a sufficient condition for semi-bounded module is multiplication.

Corollary 2.7:
If M is a fully stable semi-bounded R-module and (0) is a semi-prime submodule, then M is a multiplication R-module.

Proof: By proposition (2.6), we obtain that M is a cyclic R-module. Then it is clear that M is a multiplication R-module. In the following proposition, we give some condition under which the converse of corollary (2.7) is true. But first we need the following definition. Recall that an R-module M is called a quasi-prime R-module if and only if annihilator N is a prime ideal for each non-zero submodule N of M, [6].

Proposition 2.8:
If M is a multiplication quasi-prime R-module, then M is a semi-bounded R-module.

Proof: Since M is a multiplication quasi-prime R-module, so M is a prime module by [6, theorem 1.4.1, ch.1], hence it is a bounded. Therefore M is a semi-bounded R-module by (1.2,(1)).

References:
الخلاصة:

لتكن حلقة ابدالية ذات محايد ولتكن $R$. اطلق على $R$ مقاساً احادياً. إذا وجد $x$ عنصراً، $M$ مقاساً مقيداً، فإن $R$ مقاساً احادياً. في المصدر [1]. كما يلي: يطلق على $M$ مقاساً شبه مقيداً. إذا وجد $x$ عنصراً، ينتمي إلى $M$ مقاساً شبه مقيداً، حيث $ann_R(M) = ann_R(x)$ في المصدر [1].

تم تقديم مفهوم مقاس شبه مقيداً. إذا وجد $x$ عنصراً، ينتمي إلى $M$ مقاساً شبه مقيداً، حيث $ann_R(M) = ann_R(x)$ في المصدر [1].

في هذا البحث، تم تعريف بعض الخواص والتميزات حول المقاسات شبه المقيدة. بالإضافة إلى هذا، تم دراسة بعض العلاقات بين المقاسات شبه المقيدة مع أنواع أخرى من المقاسات.