The Operator $_r\Phi_s$ and the Polynomials K_n

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Abstract

Based on basic hypergeometric series, a new generalized q-operator $_r\Phi_s$ has been constructed and obtained some operator identities. Also, a new polynomial $K_n(a_1,\dots,a_r,b_1,\dots,b_s,c;a;q)$ is introduced. The generating function and its extension, Mehler's formula and its extension and the Rogers formula for the polynomials $K_n(a_1,\dots,a_r,b_1,\dots,b_s,c;a;q)$ have been achieved by using the operator $_r\Phi_s$. In fact, this work can be considered as a generalization of Liu work's by imposing some special values of the parameters in our results. Therefore the q^{-1}-Rogers-Szegö polynomials $h_n(a,b|q^{-1})$ can be deduced directly.

Keywords: q-operator, generating function, Mehler's formula, Rogers formula, the q^{-1}-Rogers-Szegö polynomials.

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1. Introduction

Throught this paper, the notations in [2] will be used here and assuming that $|q|<1$.

Definition 1.1. [2] . *Let a be a complex variable. The q-shifted factorial is defined by*

$$(a;q)_0=1,\qquad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\qquad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k).$$

The compact notation for the multiple q-shifted factorial will be adopted here

$$(a_1,\dots,a_r;q)_n=(a_1;q)_n\dots(a_r;q)_n,$$

where n is an integer or ∞.

Definition 1.2. [2] . *The basic hypergeometric series $_r\phi_s$ is defined by*

$$
{}_r\phi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,x) = {}_r\phi_s\left(\begin{matrix}a_1,\dots,a_r\\ \\ b_1,\dots,b_s\end{matrix};q,x\right)
$$

$$
= \sum_{k=0}^{\infty} \frac{(a_1;q)_k(a_2;q)_k\cdots(a_r;q)_k}{(q;q)_k(b_1;q)_k\cdots(b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} x^k,
$$

where $r,s \in \mathbb{N}$; $a_1,\dots,a_r,b_1,\dots,b_s \in \mathbb{C}$; and none of the denominator factors evaluate to zero. The above series is absolutely convergent for all $x \in \mathbb{C}$ if $r < s+1$, for $|x|<1$ if $r = s+1$ and for $x = 0$ if $r > s+1$.

Definition 1.3. [2] . *The q-binomial coefficient is defined by*

$$
\begin{bmatrix}n\\k\end{bmatrix} = \begin{cases}\dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & if\ \ 0<k<n;\\ 0, & otherwise,\end{cases} \tag{1.1}
$$

where $n,k \in \mathbb{N}$.

The following equations will be used in this paper [2]:

$$
(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{1.2}
$$

$$
(q/a;q)_k = (-a)^{-k} q^{\binom{k+1}{2}}(aq^{-k};q)_\infty/(a;q)_\infty. \tag{1.3}
$$

$$
\binom{n-k}{2} = \binom{n}{2} + \binom{k}{2} + k - kn, \tag{1.4}
$$

$$
\binom{n+k}{2} = \binom{n}{2} + \binom{k}{2} + kn, \tag{1.5}
$$

where n and k are integers. Cauchy identity is given by [2]

$$
\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x|<1. \tag{1.6}
$$

The special case of Cauchy identity was founded by Euler [2] which is

$$
\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} x^k = (x;q)_\infty. \tag{1.7}
$$

Definition 1.4. [3] . *The operator θ is defined by*

$$
\theta\{f(a)\} = \frac{f(aq^{-1}) - f(a)}{aq^{-1}}. \tag{1.8}
$$

Theorem 1.5. [3]. (Leibniz rule for θ). *Let* θ *be defined as in* (1.8), *then*

$$\theta^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \theta^k\{f(a)\}\theta^{n-k}\{g(aq^{-k})\}. \tag{1.9}$$

The following identities are easy to prove:

Theorem 1.6. [4, 5, 6] . *Let* θ *be defined as in* (1.8), *then*

$$\theta^k\{a^n\} = \frac{(q;q)_n}{(q;q)_{n-k}} a^{n-k} q^{\binom{k}{2}+k(1+n)}. \tag{1.10}$$

$$\theta^k\{(at;q)_\infty\} = (-t)^k (at;q)_\infty. \tag{1.11}$$

$$\theta^k\left\{\frac{(at;q)_\infty}{(av;q)_\infty}\right\} = v^k q^{-\binom{k}{2}} (t/v;q)_k \frac{(at;q)_\infty}{(av/q^k;q)_\infty}, \quad |av| < 1. \tag{1.12}$$

In 1998, Chen and Liu [4] defined the q-exponential operator $E(b\theta)$ as follows:

Definition 1.7. [4] . *The* q*-exponential operator* $E(b\theta)$ *is defined as follows:*

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q;q)_n}. \tag{1.13}$$

Chen and Liu proved the following result:

Theorem 1.8. [4] . *Let* $E(b\theta)$ *be defined as in* (1.13), *then*

$$E(b\theta)\{(at;q)_\infty\} = (at, btq)_\infty. \tag{1.14}$$

$$E(b\theta)\{(as, at;q)_\infty\} = \frac{(as, at, bs, btq)_\infty}{(abst/q;q)_\infty}, \quad |abst| < 1. \tag{1.15}$$

They used the q-exponential operator $E(b\theta)$ to present an extension for the Askey beta integral.

In 2006, Zhang and Liu [6] used $E(d\theta)$ to prove the following result:

Theorem 1.9. [6] . *Let* $E(d\theta)$ *be defined as in* (1.13), *then*

$$E(d\theta)\{a^n(as;q)_\infty\} = a^n(as;q)_\infty \; {}_2\phi_1\left(\begin{matrix} q^{-n}, q/as \\ 0 \end{matrix}; q, ds\right), \quad |ds| < 1. \tag{1.16}$$

In 2007, Fang [7] defined the Cauchy operator $_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right)$ as follows:

Definition 1.10. [7] . *The Cauchy operator* $_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right)$ *is defined by*

$$_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}(-c\theta)^n. \tag{1.17}$$

Fang proved the following result:

Theorem 1.11. [7] . *Let* $_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right)$ *be defined as in* (1.17), *then*

$$_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right)\{(as;q)_\infty\} = \frac{(bcs, as;q)_\infty}{(cs;q)_\infty}, \quad |cs| < 1. \tag{1.18}$$

Fang used Cauchy operator $_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right)$ to obtain an extension for the q-Chu-Vandermonde identity.

In 2010, Zhang and Yang [8] introduced the finite q-exponential operator with two parameters $_2\mathcal{E}_1\left[\begin{matrix} q^{-N}, v \\ w \end{matrix}; q, c\theta\right]$ as follows:

Definition 1.10. [8] . The finite q-exponential operator $_2\mathcal{E}_1\left[\begin{matrix} q^{-N}, v \\ w \end{matrix}; q, c\theta\right]$ *is defined by*

$$_2\mathcal{E}_1\left[\begin{matrix} q^{-N}, v \\ w \end{matrix}; q, c\theta\right] = \sum_{n=0}^{\infty} \frac{(q^{-N}, v;q)_n}{(q, w;q)_n}(c\theta)^n.$$

By using this operator, , Zhang and Yang found an extension for q-Chu-Vandermonde summation formula.

In 2010, Liu [1] defined the q^{-1}-Rogers-Szegö polynomial as follows:

Definition 1.12. [1] . *The* q^{-1}*-Rogers-Szegö polynomial* $h_n(a, b|q^{-1})$ *is defined by*

$$h_n(a, b|q^{-1}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2 - nk} a^k b^{n-k}. \tag{1.19}$$

Liu used the q-difference equation to prove the following:

Theorem 1.13. [1] . *Let* $h_n(a,b|q^{-1})$ *be defined as in* (1.19), *then*

• *The generating function for* $h_n(a,b|q^{-1})$

$$\sum_{n=0}^{\infty} h_n(a,b|q^{-1}) \frac{(-t)^n q^{\binom{n}{2}}}{(q;q)_n} = (at,bt;q)_\infty. \tag{1.20}$$

• *Mehler's formula for* $h_n(a,b|q^{-1})$

$$\sum_{n=0}^{\infty} h_n(a,b|q^{-1}) h_n(c,d|q^{-1}) \frac{(-t)^n q^{\binom{n}{2}}}{(q;q)_n} = \frac{(act,adt,bct,bdt;q)_\infty}{(abcdt^2/q;q)_\infty}, \tag{1.21}$$

provided that $|abcdt^2/q| < 1$.

This paper is organized as follows: In section 2, a generalized q-operator ${}_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, -c\theta\right)$ and some of its identities will be definded and studied. In section 3, we define a polynomial $K_n(a_1,\dots,a_r,b_1,\dots,b_s,c;a;q)$ and represent it by the operator ${}_r\Phi_s$. The generating function and its extension for $K_n(a_1,\dots,a_r,b_1,\dots,b_s,c;a;q)$is obtained. In section 4, the Mehler's formula and its extension for $K_n(a_1,\dots,a_r,b_1,\dots,b_s,c;a;q)$ is derived . while, in section 5, the Rogers formula for $K_n(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)$ is constructed. Finally, section 6 is focused on the summary of the results and the conclusions.

2. The Operator ${}_r\Phi_s$ and it's Identities

In this section, we define the generalized q-operator ${}_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, -c\theta\right)$ as follows:

Definition 2.1. *The generalized* q*-operator* ${}_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, -c\theta\right)$ *is defined by*

$${}_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, -c\theta\right) = \sum_{k=0}^{\infty} \frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k} \frac{(-c\theta)^k}{(q;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}. \tag{2.1}$$

When $r=s=0$, we get the q-exponential operator $E(c\theta)$ defined by Chen and Liu [4] in 1998. Also when $r=1, s=0$, $a_1=b$, we obtain the q-exponential operator ${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, -c\theta\right)$ defined by Fang [7] in 2007. And when $r=2, s=1$, $a_1=q^{-N}$, $a_2=v$, $b_1=w$ we obtain the finite q-exponential operator with two parameters $_2\mathcal{E}_1\begin{bmatrix} q^{-N}, v \\ ; q, c\theta \\ w \end{bmatrix}$ defined by Zhang and Yang [8] in 2010. Finally, when $r = 2, s = 1$, $a_1 = u$, $a_2 = v$, $b_1 = w$, we get the generalized q-exponential operator with three parameters $\mathbb{E}\begin{bmatrix} u, v \\ |q; c\theta \\ w \end{bmatrix}$ defined by Li and Tan [9] in 2016.

In this paper, we will denote to $\frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k}$ by W_k. Then the generalized q-operator $_r\Phi_s$ can be written as follows:

$$ {}_r\Phi_s\begin{pmatrix} a_1, \dots, a_r \\ ; q, -c\theta \\ b_1, \dots, b_s \end{pmatrix} = \sum_{k=0}^{\infty} W_k \frac{(-c\theta)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}. \tag{2.2} $$

Theorem 2.2. *Let* $_r\Phi_s\begin{pmatrix} a_1, \dots, a_r \\ ; q, -c\theta \\ b_1, \dots, b_s \end{pmatrix}$ *be defined as in* (2.2), *then*

$$ \begin{aligned} {}_r\Phi_s\begin{pmatrix} a_1, \dots, a_r \\ ; q, -c\theta \\ b_1, \dots, b_s \end{pmatrix}\{(au, at; q)_\infty\} &= (au, at; q)_\infty \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} W_{k+j}\frac{(ct)^k}{(q;q)_k} \\ &\times \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\frac{(q/at;q)_j}{(q;q)_j}(actu/q)^j\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} q^{kj(s-r)}. \end{aligned} \tag{2.3} $$

Proof. From the definition of the operator $_r\Phi_s\begin{pmatrix} a_1, \dots, a_r \\ ; q, -c\theta \\ b_1, \dots, b_s \end{pmatrix}$ and by using Leibniz rule (1.9), we have

$$ {}_r\Phi_s\begin{pmatrix} a_1, \dots, a_r \\ ; q, -c\theta \\ b_1, \dots, b_s \end{pmatrix}\{(au, at; q)_\infty\} $$

$$ = \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q,q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\theta^k\{(au, at; q)_\infty\} $$

$$ = \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q,q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k} kj\theta^j\{(au;q)_\infty\}\theta^{k-j}\{(atq^{-j};q)_\infty\} $$

$$ = \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q,q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k}\frac{(q,q)_k}{(q,q)_j(q,q)_{k-j}}(-u)^j(au;q)_\infty $$

$$ \times (-tq^{-j})^{k-j}(atq^{-j};q)_\infty $$

$$= \sum_{k=0}^{\infty} W_k \frac{(-c)^k}{(q,q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} \sum_{j=0}^{k} \frac{(q,q)_k}{(q,q)_j (q,q)_{k-j}} (-u)^j (au;q)_\infty$$

$$\times (-t)^{k-j} q^{-kj+j^2} (-at)^j q^{-\binom{j}{2}-j} (q/at;q)_j (at;q)_\infty \qquad (by\ using\ (1.3))$$

$$= (at,au;q)_\infty \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} W_{k+j} \frac{(-c)^{k+j}}{(q,q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r} q^{kj(1+s-r)}$$

$$\times (-u)^j (-t)^k q^{-kj-j^2+j^2} (-at)^j q^{-\binom{j}{2}-j} \frac{(q/at;q)_j}{(q,q)_j} \qquad (by\ using\ (1.5))$$

$$= (au,at;q)_\infty \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} W_{k+j} \frac{(ct)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} \frac{(q/at;q)_j}{(q;q)_j} (actu/q)^j \left[(-1)^j q^{\binom{j}{2}}\right]^{s-r}$$
$$\times q^{kj(s-r)}.$$

■

By setting $r = s = 0$ in (2.3), we get Theorem 2.11. obtained in Chen and Liu [4] (equation (1.15)).

Putting $u = 0$ in (2.3), we get the following corollary:

Corollary 1. *Let* ${}_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, -c\theta\right)$ *be defined as in* (2.2), *then*

$$ {}_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, -c\theta\right)\{(at;q)_\infty\} = (at;q)_\infty \sum_{k=0}^{\infty} W_k \frac{(ct)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}. \qquad (2.4)$$

Setting $r = s = 0$ in (2.4), we get Theorem 2.9. obtained by Chen and Liu [4] (equation (1.14)). Setting $r = 1$ and $s = 0$ in (2.4), we get Theorem 1.3. obtained by Fang [7] (equation (1.18)).

Theorem 2.3. *Let* ${}_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, -c\theta\right)$ *be defined as in* (2.2) *and* $n \in \mathbb{Z}^+$, *then*

$$ {}_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, -c\theta\right)\{a^n(at,q)_\infty\} = a^n (at,q)_\infty \sum_{j=0}^{n}\sum_{k=0}^{\infty} W_{k+j} \frac{(ct)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}$$

$$\times \frac{(q^{-n}, q/at;q)_j}{(q;q)_j}(ct)^j\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} q^{kj(s-r)}. \tag{2.5}$$

Proof. From (2.2), we have

$$_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s\end{matrix};q,-c\theta\right)\{a^n(at,q)_\infty\} = \sum_{k=0}^{\infty} W_k\,\frac{(-c)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\theta^k\{a^n(at,q)_\infty\}.$$

By using Leibniz rule (1.9), we have

$$_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ \\ b_1,\dots,b_s\end{matrix};q,-c\theta\right)\{a^n(at,q)_\infty\}$$

$$= \sum_{k=0}^{\infty} W_k\,\frac{(-c)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k} kj\theta^j\{a^n\}\theta^{k-j}\left\{(atq^{-j};q)_\infty\right\}$$

$$= \sum_{k=0}^{\infty} W_k\,\frac{(-c)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k}\frac{(q;q)_k}{(q;q)_j(q;q)_{k-j}}(-1)^j a^{n-j}q^j(q^{-n};q)_j$$

$$\times\,\theta^{k-j}\left\{(atq^{-j};q)_\infty\right\} \qquad (by\ \ using\ \ (1.1)\ \ and\ \ (1.10))$$

$$= \sum_{k=0}^{\infty} W_k(-c)^k\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k}\frac{1}{(q;q)_j(q;q)_{k-j}}(-1)^j a^{n-j}q^j(q^{-n};q)_j(-tq^{-j})^{k-j}$$

$$\times\,(atq^{-j};q)_\infty \qquad (by\ \ using\ \ (1.11))$$

$$= \sum_{j=0}^{n}\sum_{k=0}^{\infty} W_{k+j}(-c)^{k+j}\left[(-1)^{k+j} q^{\binom{k+j}{2}}\right]^{1+s-r}\frac{1}{(q;q)_j(q;q)_k}(-1)^j a^{n-j}q^j(q^{-n};q)_j(-tq^{-j})^k$$

$$\times\,(-at)^{\mathrm{j}}q^{-j_2\,-j}(q/at;q)_j(at;q)_\infty \qquad (by\ \ using\ (1.3))$$

$$= a^n(at,q)_\infty\sum_{j=0}^{n}\sum_{k=0}^{\infty} W_{k+j}\frac{(ct)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\frac{(q^{-n},q/at;q)_j}{(q;q)_j}(ct)^j\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r}$$

$$\times\,q^{kj(s-r)}. \qquad (by\ \ using\ (1.5))$$

■

Setting $r = s = 0$ in (2.5), we get Corollary 2.4. obtained in Zhang and Liu [6] (equation(1.16)).

3. The Generating Function for $K_n(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)$

In this section we define a polynomial $K_n(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)$. By using the operator

${}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ ;q,-c\theta \\ b_1,\ldots,b_s \end{matrix}\right)$, we get the generating function and its extension for the polynomials K_n. We give some special values to the parameters in the generating function and its extension for $K_n(a_1,\ldots,a_r;b_1,\ldots,b_s,c;a;q)$ to obtain the generating function and its extension for the q^{-1}-Rogers-Szegö polynomials $h_n(a,b|q^{-1})$.

Definition 3.1. *The polynomial* $K_n(a_1,\ldots,a_r;b_1,\ldots,b_s,c;a;q)$ *is defined by*

$$K_n(a_1,\ldots,a_r;b_1,\ldots,b_s,c;a;q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix} W_k\, c^k\left[(-1)^k q^{\binom{k}{2}}\right]^{2+s-r} q^{k(1-n)}a^{n-k}, \tag{3.1}$$

where $W_k=\frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k}$.

Setting $r=s=0$, $a=b$, $c=a$ in (3.1), we get the q^{-1}-Rogers-Szegö polynomials $h_n(a,b|q^{-1})$ (2.12) defined by Liu [1] (equation (1.19)).

Theorem 3.2. *Let the polynomials* $K_n(a_1,\ldots,a_r;b_1,\ldots,b_s,c;a;q)$ *be defined as in* (3.1), *then*

$${}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ ;q,-c\theta \\ b_1,\ldots,b_s \end{matrix}\right)\{a^n\}=K_n(a_1,\ldots,a_r;b_1,\ldots,b_s,c;a;q). \tag{3.2}$$

Proof.

$$\begin{aligned}
&{}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ ;q,-c\theta \\ b_1,\ldots,b_s \end{matrix}\right)\{a^n\}\\
&=\sum_{k=0}^{\infty} W_k\frac{(-c\theta)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\{a^n\}\\
&=\sum_{k=0}^{\infty} W_k\frac{(-c)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\theta^k\{a^n\}\\
&=\sum_{k=0}^{\infty} W_k\frac{(-c)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\frac{(q;q)_n}{(q;q)_{n-k}}a^{n-k}q^{\binom{k}{2}-nk+k} \qquad (by\ \ using\ \ (1.10))\\
&=\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix} W_k\, c^k\left[(-1)^k q^{\binom{k}{2}}\right]^{2+s-r} q^{k(1-n)}a^{n-k}\\
&=K_n(a_1,\ldots,a_r;b_1,\ldots,b_s,c;a;\mathrm{q}).
\end{aligned}$$

■

Theorem 3.3. (The generating function for K_n). *Let* $K_n(a_1,\ldots,a_r;b_1,\ldots,b_s,c;a;q)$ *be defined as in* (3.2), *then*

$$\sum_{n=0}^{\infty} K_n(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)\frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n} = (au;q)\ {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ \\ b_1,\dots,b_s\end{matrix};q,cu\right), \tag{3.3}$$

provided that the series is absolutely convergent $\forall\ cu\in\mathbb{C}\ if\ s>r-1, cu=0\ if\ s<r-1\ and\ |cu|<1\ if\ s=r-1.$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} K_n(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)\frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n}\\
&= \sum_{n=0}^{\infty} {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ \\ b_1,\dots,b_s\end{matrix};q,-c\theta\right)\{a^n\}\frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n} && (by\ \ using\ \ (3.2))\\
&= {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ \\ b_1,\dots,b_s\end{matrix};q,-c\theta\right)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}(au)^n\right\}\\
&= {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ \\ b_1,\dots,b_s\end{matrix};q,-c\theta\right)\{(au;q)_\infty\} && (\ by\ \ using\ \ (1.7)\ )\\
&= (au;q)_\infty\sum_{k=0}^{\infty} W_k\frac{(cu)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} && (\ by\ \ using\ \ (2.4)\ )\\
&= (au;q)\ {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ \\ b_1,\dots,b_s\end{matrix};q,cu\right).
\end{aligned}$$

■

Setting $r=s=0,\ a=b,\ c=a$ in (3.3) we obtain the generating function for the polynomials $h_n(a,b|q^{-1})$ (2.13) obtained by Liu [1] (equation (1.20)).

Theorem 3.4. (Extension of the generating function for K_n).
Let $K_n(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)$ *be defined as in* (3.2), *then*

$$\sum_{n=0}^{\infty} K_{n+l}(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)\frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n} = a^l(au;q)_\infty$$

$$\times\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(q^{-l},q/au;q)_j}{(q;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r}(cu)^j W_{i+j}\frac{(cu)^i}{(q;q)_i}\left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(s-r)}. \tag{3.4}$$

Proof.

$$\sum_{n=0}^{\infty} K_{n+l}(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)\frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} {}_r\Phi_s\left(\begin{matrix} a_1, \dots, a_r \\ \\ b_1, \dots, b_s \end{matrix}; q, -c\theta\right)\{a^{l+n}\}\frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n} \qquad (by\ \ using\ \ (3.2))$$

$$= {}_r\Phi_s\left(\begin{matrix} a_1, \dots, a_r \\ \\ b_1, \dots, b_s \end{matrix}; q, -c\theta\right)\left\{a^l \sum_{n=0}^{\infty} \frac{(-1)^n (au)^n q^{\binom{n}{2}}}{(q;q)_n}\right\}$$

$$= {}_r\Phi_s\left(\begin{matrix} a_1, \dots, a_r \\ \\ b_1, \dots, b_s \end{matrix}; q, -c\theta\right)\{a^l (au;q)_\infty\} \qquad (by\ \ using\ \ (1.7))$$

$$= a^l (au;q)_\infty \sum_{j=0}^{l}\sum_{i=0}^{\infty} \frac{(q^{-l}, q/au;q)_j}{(q;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} (cu)^j W_{i+j}\frac{(cu)^i}{(q;q)_i}$$

$$\times\left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(s-r)}. \qquad (by\ \ using\ \ (2.5))$$

■

Setting $r = s = 0,\ a = b,\ c = a$ in (3.4) we obtain an extension of the generating function for the polynomials $h_n(a,b|q^{-1})$ as follows:

$$\sum_{n=0}^{\infty} h_{n+l}(a,b|q^{-1})\frac{(-t)^n q^{\binom{n}{2}}}{(q;q)_n} = b^l (bu;q)_\infty \sum_{j=0}^{l}\sum_{i=0}^{\infty} \frac{(q^{-l}, q/bu;q)_j}{(q;q)_j}(au)^j \frac{(au)^i}{(q;q)_i}(-1)^i q^{\binom{i}{2}}$$

$$= b^l (bu;q)_\infty \sum_{j=0}^{l} \frac{(q^{-l}, q/bu;q)_j}{(q;q)_j}(au)^j \sum_{i=0}^{\infty}\frac{(au)^i}{(q;q)_i}(-1)^i q^{\binom{i}{2}}$$

$$= b^l (au, bu;q)_\infty \sum_{j=0}^{l} \frac{(q^{-l}, q/bu;q)_j}{(q;q)_j}(au)^j.$$

4. Mehler's Formula for $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$

In the section, we will derive Mehler's formula and its extension for the polynomials K_n by using the operator ${}_r\Phi_s$. We give some special values to the parameters in the Mehler's formula and its extension for K_n to obtain Mehler's formula and its extension for the q^{-1}-Rogers-Szegö polynomials $h_n(a,b|q^{-1})$.

Theorem 4.1. (Mehler's formula for K_n). *Let* $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$ *be defined as in* (3.2), *then*

$$\sum_{n=0}^{\infty} K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q) K_n(a_{1\prime}, \dots, a_{r\prime}; b_{1\prime}, \dots, b_{s\prime}, c'; a'; q)\frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= (aua';q)_\infty \sum_{k=0}^{\infty} W_k \frac{(cua')^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{(q^{-k},q/a'au;q)_j}{(q;q)_j}$$
$$\times \left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} (cau)^j W_{i+j} \frac{(cau)^i}{(q;q)_i}\left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(s-r)}, \tag{4.1}$$

provided that $|cua'|<1$.

Proof.

$$\sum_{n=0}^{\infty} K_n(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q) K_n(a_{1'},\dots,a_{r'};b_{1'},\dots,b_{s'},c';a';q) \frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n}$$
$$= \sum_{n=0}^{\infty} K_n(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)\, {}_r\Phi_s\left(\begin{matrix} a_{1'},\dots,a_{r'} \\ b_{1'},\dots,b_{s'}\end{matrix};q,-c'\theta\right)\{(a')^n\} \frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n}$$
$$= {}_r\Phi_s\left(\begin{matrix} a_{1'},\dots,a_{r'} \\ b_{1'},\dots,b_{s'}\end{matrix};q,-c'\theta\right)\left\{\sum_{n=0}^{\infty} K_n(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q) \times \frac{(-a'u)^n q^{\binom{n}{2}}}{(q;q)_n}\right\}$$
$$= {}_r\Phi_s\left(\begin{matrix} a_{1'},\dots,a_{r'} \\ b_{1'},\dots,b_{s'}\end{matrix};q,-c'\theta\right)\left\{(aua';q)_\infty\, {}_r\Phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s\end{matrix};q,cua'\right)\right\} \qquad (by\ using\ (3.3))$$
$$= {}_r\Phi_s\left(\begin{matrix} a_{1'},\dots,a_{r'} \\ b_{1'},\dots,b_{s'}\end{matrix};q,-c'\theta\right)\left\{(aua';q)_\infty \sum_{k=0}^{\infty} W_k \frac{(cua')^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\right\}$$
$$= \sum_{k=0}^{\infty} W_k \frac{(cu)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} {}_r\Phi_s\left(\begin{matrix} a_{1'},\dots,a_{r'} \\ b_{1'},\dots,b_{s'}\end{matrix};q,-c'\theta\right)\{(a')^k(aua';q)_\infty\}$$
$$= (aua';q)_\infty \sum_{k=0}^{\infty} W_k \frac{(cua')^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} \sum_{j=0}^{k}\sum_{i=0}^{\infty} \frac{(q^{-k},q/a'au;q)_j}{(q;q)_j}$$
$$\times \left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} (c'au)^j W_{i+j} \frac{(c'au)^i}{(q;q)_i}\left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(s-r)}. \qquad (by\ using\ (2.5))$$

■

Setting $r=s=0$, $c'=c$, $a=b$, $a'=d$, $c=a$ and $u=t$ in equation (4.1) we get Mehler's formula for the polynomials $h_n(a,b|q^{-1})$ (2.14) obtained by Liu [1] (equation (1.21)) as we see in the following corollary:

Corollary 2. (Mehler's formula for $h_n(a,b|q^{-1})$). *Let* $h_n(a,b|q^{-1})$ *be defined as in* (1.19), *then*

$$\sum_{n=0}^{\infty} h_n(a,b|q^{-1}) h_n(c,d|q^{-1}) \frac{(-t)^n q^{\binom{n}{2}}}{(q;q)_n} = \frac{(act,adt,bct,bdt;q)_\infty}{(abcdt^2/q;q)_\infty},$$

provided that $|abcdt^2/q| < 1$.

Proof. Setting $r = s = 0,\ c' = c,\ a = b,\ a' = d,\ c = a$ and $u = t$ in equation (4.1) we get

$$\sum_{n=0}^{\infty} h_n(a,b|q^{-1})h_n(c,d|q^{-1})\frac{(-t)^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= (btd;q)_\infty \sum_{k=0}^{\infty}\frac{(atd)^k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}\sum_{j=0}^{k}\sum_{i=0}^{\infty}\frac{(q^{-k},q/dbu;q)_j}{(q;q)_j}(cbt)^j\frac{(cbt)^i}{(q;q)_i}(-1)^{\mathrm{i}}q^{\binom{i}{2}}$$

$$= (btd;q)_\infty \sum_{k=0}^{\infty}\frac{(atd)^k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}\sum_{j=0}^{k}\frac{(q/dtb;q)_j}{(q;q)_j}(-1)^j q^{\binom{j}{2}-kj}\frac{(q;q)_k}{(q;q)_{k-j}}(cbt)^j(cbt;q)_\infty$$

$$= (btd,cbt;q)_\infty \sum_{j=0}^{\infty}\frac{(q/dbt;q)_j}{(q;q)_j}(cbt)^{\mathrm{j}}(-1)^j q^{\binom{j}{2}-j^2}\sum_{k=0}^{\infty}\frac{(dta)^{k+j}}{(q;q)_k}(-1)^{k+j}q^{\binom{k+j}{2}-kj}$$

$$= (btd,cbt;q)_\infty \sum_{j=0}^{\infty}\frac{(q/dbt;q)_j}{(q;q)_j}(cbt)^j(atd)^j q^{-j}\sum_{k=0}^{\infty}\frac{(dcu)^k}{(q;q)_k}(-1)^k q^{\binom{k}{2}} \quad (by\ \ using\ \ (1.5))$$

$$= (btd,cbt;q)_\infty \frac{(cat;q)_\infty}{(acbdt^2/q;q)_\infty}(adt;q)_\infty \qquad (by\ \ using\ (1.6)\ \ and\ (1.7))$$

$$= \frac{(btd,btc,atc,atd;q)_\infty}{(acbdt^2/q;q)_\infty}.$$

■

Theorem 4.2. (Extension of Mehler's formula for K_n). *Let* $K_n(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)$ *be defined as in* (3.1), *then*

$$\sum_{n=0}^{\infty} K_{n+m}(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)K_n(a_{1'},\dots,a_{r'};b_{1'},\dots,b_{s'},c';a';q)\frac{(-u)^n q^{\binom{k}{2}}}{(q;q)_n}$$

$$= a^m \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(q^{-m},q/aa'u;q)_j}{(q;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r}(cu)^j W_{i+j}\frac{(cu)^i}{(q;q)_i}\left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r}$$
$$\times\, q^{ij(s-r)}(a')^{i+j}(a'au,q)_\infty \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^{-(i+j)},q/a'au;q)_l}{(q;q)_l}\left[(-1)^l q^{\binom{l}{2}}\right]^{s-r}(c'au)^l W_{k+l}$$

$$\times \frac{(c'au)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} q^{kl(s-r)}. \tag{4.2}$$

Proof.

$$\sum_{n=0}^{\infty} K_{n+m}(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)K_n(a_{1'},\dots,a_{r'};b_{1'},\dots,b_{s'},c';a';q)\frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} K_{n+m}(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)\ {}_r\Phi_s\begin{pmatrix} a_{1'},\dots,a_{r'} \\ & ;q,-c'\theta \\ b_{1'},\dots,b_{s'} \end{pmatrix}\{(a')^n\}$$

$$\times \frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n} \qquad (by\ \ using\ \ (3.2))$$

$$= {}_r\Phi_s\begin{pmatrix} a_{1'},\dots,a_{r'} \\ & ;q,-c'\theta \\ b_{1'},\dots,b_{s'} \end{pmatrix}\left\{\sum_{n=0}^{\infty} K_{n+m}(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)\frac{(-a'u)^n q^{\binom{n}{2}}}{(q;q)_n}\right\}$$

$$= {}_r\Phi_s\begin{pmatrix} a_{1'},\dots,a_{r'} \\ & ;q,-c'\theta \\ b_{1'},\dots,b_{s'} \end{pmatrix}\left\{a^m (aa'u;q)_\infty \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(q^{-m},q/aa'u;q)_j}{(q;q)_j}\right.$$

$$\left.\times\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r}(cua')^j W_{i+j}\frac{(cua')^i}{(q;q)_i}\left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(s-r)}\right\} \qquad \big(by\ \ using\ \ (3.4)\big)$$

$$= a^m \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(q^{-m},q/aa'u;q)_j}{(q;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r}(cu)^j W_{i+j}\frac{(cu)^i}{(q;q)_i}\left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(s-r)}$$

$$\times\ {}_r\Phi_s\begin{pmatrix} a_{1'},\dots,a_{r'} \\ & ;q,-c'\theta \\ b_{1'},\dots,b_{s'} \end{pmatrix}\left\{(a')^{i+j}(aa'u;q)_\infty\right\}$$

$$= a^m \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(q^{-m},q/aa'u;q)_j}{(q;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r}(cu)^j W_{i+j}\frac{(cu)^i}{(q;q)_i}\left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(s-r)}$$

$$\times (a')^{i+j}(a'au,q)_\infty \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^{-(i+j)},q/a'au;q)_l}{(q;q)_l}\left[(-1)^l q^{\binom{l}{2}}\right]^{s-r}(c'au)^l W_{k+l}$$

$$\times\frac{(c'au)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} q^{kl(s-r)}. \qquad (by\ \ using\ \ (2.5))$$

■

Setting $r = s = 0$, $a = b$, $c = a$, $c' = c$ and $a' = d$ in equation (4.2) we get an extension of Mehler's formula for the polynomials $h_n(a, b|q^{-1})$ as we see in the following corollary:

Corollary 3. (Extension of Mehler's formula for $h_n(a, b|q^{-1})$). *Let* $h_n(a, b|q^{-1})$ *be defined as in* (1.19), *then*

$$\sum_{n=0}^{\infty} h_{n+m}(a, b|q^{-1}) h_n(c, d|q^{-1}) \frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n}$$
$$= b^m (aud, cbu, dbu, q)_\infty \sum_{j=0}^{\infty} \frac{(q^{-m}, q/bdu; q)_j}{(q;q)_j} (aud/q)^j \sum_{l=0}^{\infty} \frac{(q^{-(i+j)}, q/dbu; q)_l}{(q;q)_l} (cbu)^l.$$

Proof.

Setting $r = s = 0$, $a = b$, $c = a$, $c' = c$ and $a' = d$ in equation (4.2) we get

$$\sum_{n=0}^{\infty} h_{n+m}(a, b|q^{-1}) h_n(c, d|q^{-1}) \frac{(-u)^n q^{\binom{n}{2}}}{(q;q)_n}$$

$$= b^m (dbu, q)_\infty \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(q^{-m}, q/bdu; q)_j}{(q;q)_j} (aud/q)^j \frac{(aud)^i}{(q;q)_i} (-1)^i q^{\binom{i}{2}}$$

$$\times \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{-(i+j)}, q/dbu; q)_l}{(q;q)_l} (cbu)^l \frac{(cbu)^k}{(q;q)_k} (-1)^k q^{\binom{k}{2}}$$

$$= b^m (dbu, q)_\infty \sum_{j=0}^{\infty} \frac{(q^{-m}, q/bdu; q)_j}{(q;q)_j} (aud/q)^j \sum_{i=0}^{\infty} \frac{(aud)^i}{(q;q)_i} (-1)^i q^{\binom{i}{2}}$$

$$\times \sum_{l=0}^{\infty} \frac{(q^{-(i+j)}, q/dbu; q)_l}{(q;q)_l} (cbu)^l \sum_{k=0}^{\infty} \frac{(cbu)^k}{(q;q)_k} (-1)^k q^{\binom{k}{2}}$$

$$= b^m (aud, cbu, dbu, q)_\infty \sum_{j=0}^{\infty} \frac{(q^{-m}, q/bdu; q)_j}{(q;q)_j} (aud/q)^j \sum_{l=0}^{\infty} \frac{(q^{-(i+j)}, q/dbu; q)_l}{(q;q)_l} (cbu)^l.$$
$$\big(by\ using\ (1.7)\big)$$

■

5. Rogers Formula for $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$

We will derive, in this section, Roger's formula for the polynomials K_n by using the operator ${}_r\Phi_s$. We give some special values to the parameters in Rogers formula for $K_n(a_1, \dots, a_r; b_1, \dots, b_s, c; a; q)$ to obtain Rogers formula for the q^{-1}-Rogers-Szegö polynomials

$h_n(a,b|q^{-1})$.

Theorem 5.1. (Rogers formula for K_n). *Let* $K_n(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)$ *be defined as in* (3.2), *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} K_{n+m}(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)\frac{(-t)^n q^{\binom{n}{2}}}{(q;q)_n}\frac{(-u)^m q^{\binom{m}{2}}}{(q;q)_m}$$

$$= (at,au;q)_\infty \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q/au;q)_j}{(q;q)_j}\,(actu/q)^j\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} W_{k+j}\frac{(cu)^k}{(q;q)_k}$$
$$\times\left[(-1)^k q^{(k)}\right]^{1+s-r} q^{kj(s-r)}, \tag{5.1}$$

provided that $|actu/q|<1$.

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} K_{n+m}(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)\frac{(-t)^n q^{\binom{n}{2}}}{(q;q)_n}\frac{(-u)^m q^{\binom{m}{2}}}{(q;q)_m}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,-c\theta\right)\{a^{n+m}\}\frac{(-t)^n q^{\binom{n}{2}}}{(q;q)_n}\frac{(-u)^m q^{\binom{m}{2}}}{(q;q)_m}$$

$$= {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,-c\theta\right)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n(at)^n q^{\binom{n}{2}}}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(-1)^m(au)^m q^{\binom{m}{2}}}{(q;q)_\mathrm{m}}\right\}$$

$$= {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,-c\theta\right)\{(at;q)_\infty(au;q)_\infty\}. \qquad (by\ \ using\ \ (1.7))$$

$$= (at,au;q)_\infty \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q/ua;q)_j}{(q;q)_j}\,(actu/q)^j\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} W_{k+j}\frac{(cu)^k}{(q;q)_k}$$

$$\times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} q^{kj(s-r)}. \qquad by\ \ using\ \ (2.3))$$

■

Setting $r=s=0$, $a=b$ and $c=a$ in equation (5.1) we obtain Rogers formula for the polynomials $h_n(a,b|q^{-1})$ as we see in the following corollary:

Corollary 4. (Rogers formula for $h_n(a,b|q^{-1};q)$). *Let* $h_n(a,b|q^{-1})$ *be defined as in* (1.19), *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(a,b|q^{-1};q)\frac{(-t)^n q^{\binom{n}{2}}}{(q;q)_n}\frac{(-u)^m q^{\binom{m}{2}}}{(q;q)_m}=\frac{(at,au,bt,bu;q)_\infty}{(abtu/q;q)_\infty},$$

provided that $|abtu/q| < 1$.

Proof.

Setting $r = s = 0,\ a = b$ and $c = a$ in equation (5.1) we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(a,b|q^{-1};q)\frac{(-t)^n q^{\binom{n}{2}}}{(q;q)_n}\frac{(-u)^m q^{\binom{m}{2}}}{(q;q)_m}$$

$$= (bt,bu;q)_\infty \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q/bu;q)_j}{(q;q)_j}\,(batu/q)^{\mathrm{j}}\frac{(au)^k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}$$

$$= (bt,bu;q)_\infty \sum_{j=0}^{\infty}\frac{(q/au;q)_j}{(q;q)_j}\,(batu/q)^j\sum_{k=0}^{\infty}\frac{(au)^k}{(q;q)_k}(-1)^k q^{\binom{k}{2}}$$

$$= \frac{(bt,at,bu,au;q)_\infty}{(batu/q;q)_\infty}. \qquad (by\ using\ (1.6)\ and\ (1.7))$$

■

6. Conclusions

This paper devoted to study a new generalized q-operator ${}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,-c\theta\right)$. Also, a new polynomial $K_n(a_1,\dots,a_r,b_1,\dots,b_s,c;a;q)$ is constructed. The generating function and its extension for $K_n(a_1,\dots,a_r,b_1,\dots,b_s,c;a;q)$ is studied. Also, the Mehler's formula and its extension for $K_n(a_1,\dots,a_r,b_1,\dots,b_s,c;a;q)$ is investigated. While, the Rogers formula for $K_n(a_1,\dots,a_r;b_1,\dots,b_s,c;a;q)$ is constructed. In order to explore the results, one can imposing some special values of the parameters. So, by setting $r = s = 0,\ a = b,\ c = a$ in the generating function and its extension for $K_n(a_1,\dots,a_r,b_1,\dots,b_s,c;a;q)$, the generating function and its extension for the q^{-1}-Rogers-Szegö polynomials $h_n(a,b|q^{-1})$ is obtained directly. Also, by setting $r = s = 0$, $a = b$, $c = a$ in Mehler's formula and its extension for $K_n(a_1,\dots,\mathrm{a}_r,b_1,\dots,b_s,c;a;q)$, the Mehler's formula and its extension for the polynomials $h_n(a,b|q^{-1})$ is achieved directly. Finaly, by setting $r = s = 0,\ a = b,\ c = a$ in Rogers formula for $K_n(a_1,\dots,a_r,\mathrm{b}_1,\dots,b_s,c;a;q)$, the Rogers formula for the polynomials $h_n(a,b|q^{-1})$ is created.

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المؤثر ${}_r\Phi_s$ ومتعددة الحدود K_n

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المستلخص

باستخدام تعريف الدالة الهندسية الفوقية الاساسية، قمنا بتعريف الموثر-q العام ${}_r\Phi_s$ وحصلنا على بعض المتطابقات للمؤثر ${}_r\Phi_s$. أيضاً، عرفنا متعددة حدود جديدة $K_n(a_1,\dots,a_r,b_1,\dots,b_s,c;a;q)$. وجدنا الدالة المولدة وتوسيعها، صيغة Mehler وتوسيعها وصيغة Rogers لمتعددة الحدود $K_n(a_1,\dots,a_r,b_1,\dots,b_s,c;a;q)$ باستخدام المؤثر ${}_r\Phi_s$. في الحقيقة، يمكن اعتبار هذا العمل بمثابة تعميم لعمل Liu [1] عن طريق فرض بعض القيم الخاصة للمعلمات في نتائجنا. لذلك يمكن الحصول على متعددات حدود روجرز- زيجو-q^{-1} $h_n(a,b|q^{-1})$ مباشرة.