# Finding solutions for Laplace Transform and its inverse using MathCAD professional software

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**Abstract:**

**Laplace Transform is an important mathematic device usually used in geometry, mathematics and statistics. For example, the transform gives an excellent method to solve many difference equations. First transform gives the first derivation of algebra equations system. For the importance of this transformation and its numerous applications, many methods are used to find Laplace Transform and its inverse depending on MathCAD professional software. This will be illustrated in detail with many illustrating examples.**

**المستخلص:**

**أن تحويل لابلاس يمثل أداة رياضة مهمة تستخدم عادةً في الهندسة والرياضات والإحصاء، فالتحويل على سبيل المثال يعطي طريقة ممتازة لحل مجموعة من معادلات الفروق Difference equations حيث أن الأول يعطي المشتقة الأولى لمنظومة المعادلات الجبرية. ونظراً لأهمية هذا التحويل وأستخداماته المتعددة، تم أستخدام عدة طرائق لإيجاد تحويل لابلاس ومعكوسهُ، وذلك بالإعتماد على البرنامج الجاهز MathCAD professional ،وهذا ما سيتم عرضهُ بشيء من التفصيل موضحاً بالأمثلة التطبيقية.**

## Introduction:

**The proofs of the properties have been omitted and all operations are assumed to be well defined.**

**Let $f(t)$ be a real-value function in $[0,\infty)$. We define the Laplace transform of $f(t)$ as:**

$$L\{f(t)\} = \phi(s) = \int_0^\infty e^{-st} f(t)dt \quad \text{Re}(s) > 0$$

**If $f(t)$ is piecewise continuous on every interval [0,n] and of exponential order α [i.e., there exist constants $M_1$,$M_2$,and α such that for all $t>M_2$ we have $|f(t)| \le M_1 e^{at}$, then it can be**

**shown that *L{f(t)}=Φ(s)* exists.**

**Definition: 1 let *F(t)* be a real-valued function; then we define the Laplace- stieltjes transform of *F(t)* as :** $\int_0^\infty e^{-st} dF(t)dt \quad \text{Re}(s) > 0$

***We note that if* F(t) *is absolutely continuous and its Laplace- stieltjes transform exists, then* F(t) *is a differentiable monotonically increasing function and* dF(t)=F′(t)dt *the Laplace- stieltjes transform of F(t) then equals the Laplace transform for this case the following properties apply only to Laplace transform, although analogous properties hold for Laplace- stieltjes transform, let* L{f(t)}=Φ(s) *and assume that all operations are well defined.***

## The Aim of this research:

**This research aims to deal with difficulties caused by counting Laplace Transform and its inverse for complex equations, and equation system of "n" equations and "m" limits, to have the accurate solutions in the least time.**

### 1- The Properties of Laplace Transform:

**1- If $L\{f(t)_i\}=\Phi(s)_i$ and $f(t) = \sum_{i=1}^{\infty} \xi_i \phi_i(s)$ where $\xi_i$ is constant (i=1,2,…), then the $\phi(s) = \sum_{i=1}^{\infty} \xi_i \phi_i(s)$**

**2-If $g(t) = e^{\xi t} f(t)$ then $L\{g(t)\} = \phi(s-\xi)$**

**3-If $g(t) = \begin{cases} f(t-\xi) & for\ t > \xi \\ 0 & for\ t \le \xi \end{cases}$ then $L\{g(t)\} = e^{-\xi s}\phi(s)$.**

**4-If $\xi \neq 0$ and $g(t) = f(\xi t)$, then $L\{f(\xi t)\} = \frac{1}{\xi}\phi(\frac{s}{\xi})$**

**5-If $g(t)=d^n[f(t)]/dt^n=f^{(n)}(t)$, then**

$L\{g(t)\} = s^n\phi(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots\ldots - f^{(n-1)}(0).$ **Here the continuity at 0 of $f^{(n)}(t)$ is assumed for each n .**

**6-If $g(t)=t^n f(t)$, then $L\{g(t)\} = (-1)^n \phi^{(n)}(s)$**

**7-When the indicated limit exists, we have $\lim_{t\to\infty} \phi(s) = 0$**

$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\phi(s)$

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\phi(s)$

**8-Let *f(t)* be the probability density function of a continuous random variable T, then $E(T) = -\phi^{(1)}(0)$**

**9-let** $f(t) = f(t)_1 * f(t)_2 = \int_0^x f_1(\tau) f_2(\tau - \pi) d\tau$ **and** $L\{f(t)_i\} = \Phi(s)_i$ $(i=1,2);$

**Then** $\phi(s) = \phi_1(s) * \phi_2(s)$ **.**

**10-if** $f(t)$ **is such that** $\int_0^x f(t)dt = 0$ **for all** $x>0$ **, then** $f(t)$ **is called a null function and** $\phi(s) = 0$.

**Perhaps a word about the uniqueness of Laplace transform of** $f(t)$ **is in order . Suppose** $f_2(t)$ **is a null function and** $f(t)=f_1(t)*f_2(t);$ **then by properties (1) and (10),we have** $\phi(s) = \phi_1(s) + \phi_2(s) = \phi_i(s) = L\{f_i(t)\}$ **one have can see that several different functions many have the same Laplace transform ; but if we do not consider null function , the Laplace transform of a function is unique.**

***THEOREM 1.1.***

**Suppose that the function** $f(t)$ **is continuous and piecewise smooth (** $f'(t)$ **is piecewise continuous) for all t > 0 and there are constants (*M, a*) such that** $|f(t)| \le Me^{\alpha t}$ **For** $t \ge T$ **, Then** $\phi'(s)$ **is defined for all s > α and** $\phi'(s) = s\phi(s) - f(0)$**. Now if nth derivative** $f^n(t)$ **is piecewise smooth, then**

$$L\{g(t)\} = s^n \phi(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots\dots$$

$$\dots\dots - f^{(n-1)}(0).$$

**For example:**

- **When n = 2**

$\phi^{(2)}(s) = s^2 \phi(s) - sf(0) - f'(0).$

- **When n = 3**

$\phi^{(3)}(s) = s^3 \phi(s) - s^2 f(0) - sf'(0) - f''(0).$

- **When n = 4**

$\phi^{(4)}(s) = s^4 \phi(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - f^{(3)}(0).$

**The following table list most commonly used functions, and**

$u_a(t) = u(t-a) = \begin{cases} 0 & if \quad t>a \\ 1 & if \quad t \ge a \end{cases}$ **is the Heaviside function.**

| $f = L^{-1}(f^\wedge)$ | $\phi(s)$ | $f = L^{-1}(f^\wedge)$ | $\phi(s)$ |
|---|---|---|---|
| *1* | *1/s (s>0)* | *t* | $1/s^2$ *(s>0)* |
| $T^n$ *(n>=0)* | $n!/(s^{n+1})$ *(s>0)* | $t^a$*(a>-1)* | $\Gamma(a+1)/s^{a+1}$*(s>0)* |
| $e^{at}$ | *1/s-a (s>0)* | *Sin(kt)* | $k/(s^2+k^2)$ *(s>0)* |
| *Cos(kt)* | $s/s^2+k^2$ *(s>0)* | *Sinh(kt)* | $k/(s^2-k^2)$ *(s>\|k\|)* |
| *Cosh(kt)* | $s/(s^2-k^2)$ *(s>\|k\|)* | *u(t-a)* | $e^{as}/s$ *(s>0)* |

**EXAMPLE 1.1**. **Find** $\phi(s)$ **if** $x'' + 2x' - 3x = e^{t}, x(0) = 1, x'(0) = -1.$

**Solution: Apply Laplace transform on both sides of the equation,**

$$L(x'' + 2x' - 3x)(s) = L(e^{t})(s)$$

**Using the linear property** $\phi(s) = \phi_1(s) + \phi_2(s) = \phi_i(s) = L\{f_i(t)\}$ **we have**

$$L(x'' + 2x' - 3x)(s) = \phi''(s) + 2\phi'(s) - 3\phi(s).$$

**Together with,** $\phi''(s) = s^2\phi(s) - sx(0) - x'(0)$ **and**

$\phi'(s) = s\phi(s) - x(0)$ **We have,**

**due to .** $x(0) = 1, x'(0) = -1$,

$$L(x'' + 2x' - 3x)(s) = \phi''(s) + 2\phi'(s) - 3\phi(s)$$

$$= \left[s^2\phi(s) - sx(0) - x'(0)\right] + 2\left[s\phi(s) - x(0)\right] - 3\phi(s)$$

$$= (s^2 + 2s - 3)\phi(s) - s - 1$$

**So we have an algebraic equation for** $\phi(s)$,

$$(s^2 + 2s - 3)\phi(s) - s - 1 = \hat{e}^{t}(s) = \frac{1}{s-1}$$

**Solve this equation for** $\phi(s)$,

$$(s^2 + 2s - 3)\phi(s) - s - 1 = \hat{e}^{t}(s) = \frac{1}{s-1}$$

$$(s^2 + 2s - 3)\phi(s) - s - 1 = \frac{1}{s-1} + s + 1$$

$$(s^2 + 2s - 3)\phi(s) - s - 1 = \frac{1}{s-1} + \frac{(s+1)(s-1)}{s-1}$$

$$(s^2 + 2s - 3)\phi(s) - s - 1 = \frac{1 + (s+1)(s-1)}{s-1}$$

$$(s^2 + 2s - 3)\phi(s) - s - 1 = \frac{1 + (s^2 - 1)}{s-1}$$

$$(s^2 + 2s - 3)\phi(s) - s - 1 = \frac{s^2}{s-1}$$

**We get** $\phi(s) = \dfrac{s^2}{(s-1)(s^2 + 2s - 3)}$

## 2- Using MathCAD:

**MathCAD can help us in finding both Laplace transform and inverse Laplace transform.**

**To use MathCAD to find Laplace transform, we first enter the expression of the function, then press [Shift] [Ctrl] [.], in the place holder type the key word Laplace followed by comma (,) and the variable name. For example, to find the Laplace of** $f(t) = t^2 \sin(at)$, **you first enter the expression** $t^2 \sin(at)$, **by typing.**

**t^2*sin(a*t),**

**Then press [Shift] [Ctrl] [.], and entering keyword Laplace followed by comma (,) and t, you will get**

$$t^2 \sin(at)\, laplace, t \longrightarrow \frac{8as^2}{(s^2 + a^2)^3} - \frac{2a}{(s^2 + a^2)^2}$$

**If you want a simplified result, you type comma (,) after entering the variable name and keyword simplify, so the following input**

***t^2*sin(a*t)[Shift][Ctrl][.]Laplace , t , simplify***

**will produce the result**

$$t^2\sin(at)laplace,t,simplify \longrightarrow \frac{2a(a^2-3s^2)}{(s^2+a^2)^3}$$

**Similarly, to use MathCAD to find inverse Laplace transform, we first, enter the expression, then press [Shift] [Ctrl] [.], in the place holder type the key word invlaplace followed by comma (,) and the variable name. For example, to find the inverse Laplace of $\frac{8as^2}{(s^2+a^2)^3}$, you first enter the expression $\frac{8as^2}{(s^2+a^2)^3}$, as**

***8a*s^2/(s^2+a^2)^3,***

**Then press [Shift] [Ctrl] [.], and entering keyword invlaplace followed by comma (,) and s, and for simplifying the result, by another comma (,) and keyword simplify, we get**

$$\frac{8as^2}{(s^2+a^2)^3} invlaplace,s,simplify \longrightarrow$$

$$\frac{1}{a^2}(\sin(at)-ta\cos(at)+t^2a^2\sin(at))$$

**When using MathCAD together with Laplace transform to solve an ODE (Ordinary Differential Equation)**

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \ldots\ldots + a_1 x' + a_0 x = f(t)$$

**We follow these steps.**

**• Step One: Apply Laplace to both sides of equation. Using MathCAD to find Laplace transform of f(t).**

**• Step Two: Using the linear property and**

$$L\{g(t)\} = s^n\phi(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots\ldots\ldots - f^{(n-1)}(0).$$

**To find an algebraic equation for $\phi(s)$,**

**• Step Three: Solve the equation obtained in Step Two for**

**$\phi(s)$, And using MathCAD to find inverse transform which will be the solution $x(t)$.**

**<u>EXAMPLE 2.1.</u> Find general solution to $x'' + 2x' + 3x = t^2e^t$.**

**Solution: Since we want to find general solution, we set, $x(0) = a$. and $x'(0) = b$**

**• Step One Apply Laplace transform to both sides of the equation and find Laplace transform for $t^2e^t$.**

**$L(x''+2x'+3x)(s)=L(t^2e^2)(s)$ Type.**

***T^2*e^t[Shift][Ctrl][.]Laplace ,t ,simplify*** **We find**

$$L(t^2e^t)(s) = \frac{2}{(s-1)^3}$$

**• Step Two: Using linear property to find an equation for $\phi(s)$.**

$$L(x''+2x'+3x)(s) = (\phi''+2\phi'+3\phi)(s)$$

$$= s^2\phi(s) - as - b + 2(s\phi(s) - a) + 3\phi(s)$$

$$= (s^2+2s+3)\phi(s) - sa - b - 2a$$

**The equation for $\phi(s)$, is**

$$(s^2+2s+3)\phi(s)-sa-b-2a=\frac{2}{(s-1)^3}$$

**Hence,**

$$\phi(s)=\frac{sa+b+2a}{(s^2+2s+3)}+\frac{2}{(s-1)^3(s^2+2s+3)}$$

**• Step Three: Using MathCAD to find inverse Laplace transform and** $x(t)$, **we enter.**

***s*a+b+a/(s^2+2s+3)+2/(s-l)^3(s^2+2s+3)[Shift] [Ctrl] [,]laplace,s***

**And get**

$$x(t)=ae^{-t}\left(\frac{\sqrt{2}}{2}\sin(\sqrt{2}t)+\cos(\sqrt{2}t)\right)+b\frac{\sqrt{2}}{2}e^{-t}\sin(\sqrt{2}t)$$

$$+e^{t}\left(\frac{1}{3}t+\frac{2}{9}+\frac{2}{9}\cos(\sqrt{2}t)\right)+\frac{\sqrt{2}}{18}e^{-t}\sin(\sqrt{2}t)$$

**<u>2.1. Solving System of Differential Equations</u>: We can use Laplace transform method to solve system of differential equations. The procedure is the same as solving a higher order ODE. But, after applying Laplace transform to each equation, we get a system of linear equations whose unknowns are the Laplace transform of the unknown functions. The following example shows how we can use Laplace method with MathCAD to solve system of differential equations.**

<u>EXAMPLE 2.2.</u> **Find solution to**

$$x' = 3x - 4y + t^2$$
$$y' = -x + 5y + e\hat{\ }t$$
$$x(0) = 1,\ y(0) = -1$$

**Solution**

**• Step One: Apply Laplace transform to both sides of the equations and use MathCAD to find Laplace transform of $t^2$ and $e^t$,**

$L(x') = L(3x - 4y + t^2)$

$L(y') = L(-x+5y+e^t)$

**and** $L(t^2)(s) = \frac{2!}{s^3}$ **and** $L(e^t)(s) = \frac{1}{s-1}$

**• Step Two: Apply linear property to get, a system of equations for $\hat{x}(s)$, and $\hat{y}(s)$, due to $L(x')=s\hat{x}(s)-x(0)$ and $L(\hat{y})=s\hat{y}(s)-y(0)$,**

$$s\hat{x}(s)-1=3\hat{x}(s)-4\hat{y}(s)+\frac{6}{s^3}$$

$$s\hat{y}+1=-\hat{x}(s)+5\hat{y}(s)+\frac{1}{s-1}$$

**From which we get**

$$(s-3)\hat{x}(s)+4\hat{y}(s)=1+\frac{6}{s^2}$$

$$\hat{x}(s)+(s-5)\hat{y}=-1+\frac{1}{s-1}$$

**Fine** $\hat{x}(s)=\begin{bmatrix}\hat{x}(s)\\ \hat{y}(s)\end{bmatrix}, A=\begin{bmatrix}s-3 & 4\\ 1 & s-5\end{bmatrix},$

**And** $b(s)=\begin{bmatrix} 1+\dfrac{6}{s^3} \\ -1+\dfrac{1}{s-1} \end{bmatrix}$

**We have, in matrix form,** $A\hat{x}(s)=b(s)$

**• Step Three: Solve the algebraic equation and apply the inverse Laplace transform, using MathCAD we have**

$$\hat{x}(s)=A^{-1}b(s)=\frac{1}{s^2-8s+11}\begin{bmatrix} \dfrac{(s-5)(s^3+6)}{s^3}-4\dfrac{2-s}{s-1} \\ \dfrac{(s-3)(2-s)}{(s-1)}-\dfrac{s^3+6}{s^3} \end{bmatrix} \qquad x(t)=\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$=\begin{bmatrix} -\frac{15}{11}t^2-\frac{147}{121}t-\frac{1062}{1331}+e^{4t}(\frac{3742}{1331}\cosh(\sqrt{5}t)-\frac{2334}{6655}\sinh(\sqrt{5}t))-e^t \\ -\frac{3}{11}t^2-\frac{48}{121}t-\frac{318}{1331}-e^{4t}(\frac{695}{2662}\cosh(\sqrt{5}t)+\dfrac{8143}{1331}\sinh(\sqrt{5}t))-\dfrac{1}{2}e^t \end{bmatrix}$$

**<u>EXAMPLE 2.3</u>. Express**

$$f(t)=\begin{cases} t^2-t+3 \; if \; 0<t<2 \\ e^t \quad if \; 2<t<5 \\ 2t\sin(t) \; if \quad t\geq 5 \end{cases}$$

**As linear combination of step functions.**

**Solution Notice *f(t)* has three different expresses over interval [0, 2), [2, 5) and [5, ∞), so we have two difference, $u(t)-u_2(t)$ and $u_2(t)-u_5(t)$ and three products.**

**• $(t^2 - t + 3)(u(t) - u_2(t))$ for $(t^2 - t + 3)$ defined on [0,2);**

**• $e^t u_2(t) - u_5(t))$ for $e^t$ defined on [2, 5);**

**• $2t\sin(t)u_{5,}(t)$ for $2t$ defined on [5. ∞).**

***f(t)* is sum of these three products.**

$$f(t) = (t^2 - t + 3)(u(t) - u_2(t)) + e^t u_2(t)) - u_5(t))) + 2t\sin(t)u_5(t)$$
$$= (t^2 - t + 3)u(t) + (e^t - t^2 + t - 3)u_2(t) + (2t\sin(t) - e^t)u_5(t).$$

## <u>3- The Piecewise defined Functions:</u>

**When solving an ODE $a_n x^{(n)} + a_{n-1}x^{(n-1)} + ......... + a_1 x' + a_0 x = f(t)$ with multiple defined function *f{t),* we need to change *f(t)* into linear combinations of step functions $U_a(t)$.**

**Here** $u_a(t)=u(t-a)=\begin{cases} 0 \quad if \quad t>a \\ 1 \quad if \quad t\geq a \end{cases}$

**And**

$$u(t)=\begin{cases} 0 \quad if \quad t<0 \\ 1 \quad if \quad t\geq 0 \end{cases}$$

**is called Heaviside function or unit step function.**

**We have graphed some step functions in fig.1**

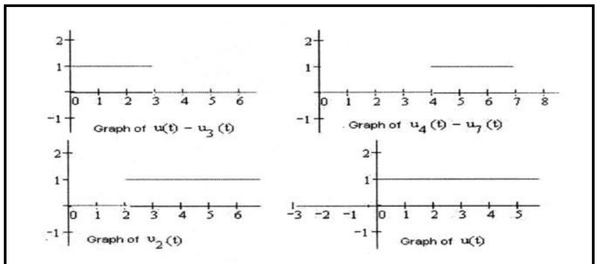


FIGURE 1. Step Functions

**FIGURE 1. Step Functions**

**In general, if** $b > a$ **,** $u_a(t) - u_b(t) = \begin{cases} 0\, if\ 0 < t < a \\ 1\, if\ a < t < b \\ 0\, if\ \ t > b \end{cases}$

***f(t) = cos(2t) (u(t)) - u*** $_{2\pi}$ ***(t))*** **Apply Laplace transform to both sides of the equation and using MathCAD we get**

$$(s^2+4)\hat{x}(s) = -1 + (1 - e^{-\pi s})\frac{s}{s^2+4}$$

$$\hat{x}(s) = \frac{-1}{s^2+4} + (1 - e^{-\pi s})\frac{s}{(s^2+4)^2}$$

**3.1. Heaviside Function in MathCAD: You can get Heaviside function from Insert => functions, click on all and the box of the function list, type Heaviside. Click Ok, to insert the function.**

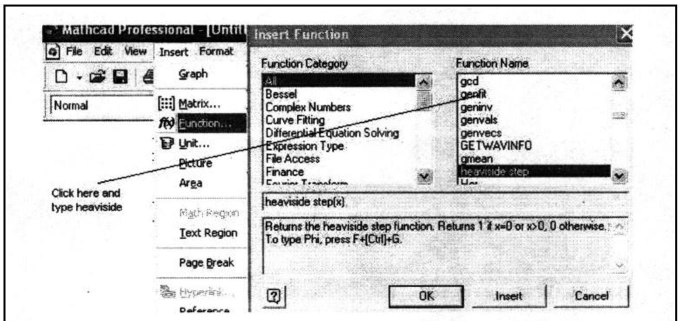


FIGURE 2. Insert Heaviside Function

**The step function is given by** $u_a(t) = \Phi(t-a).$ **Hence the function**

$$f(t) = \begin{cases} t^2 e^{-\frac{1}{3}}\ if\ 0 < t < 3 \\ \cos(t)\, if\ 3 \le t < 6 \\ 2t\ \ if\ \ t \ge 6 \end{cases}$$

**Can be defined in MathCAD as**

$$f(t) = t^2 e^{-\frac{1}{3}}\Phi(t) + (\cos(t) - t^2 e^{-\frac{1}{3}})\Phi(t-3) + (2t - \cos(t))\Phi(t-6).$$

**The following graph shows this function in MathCAD and its graph,**

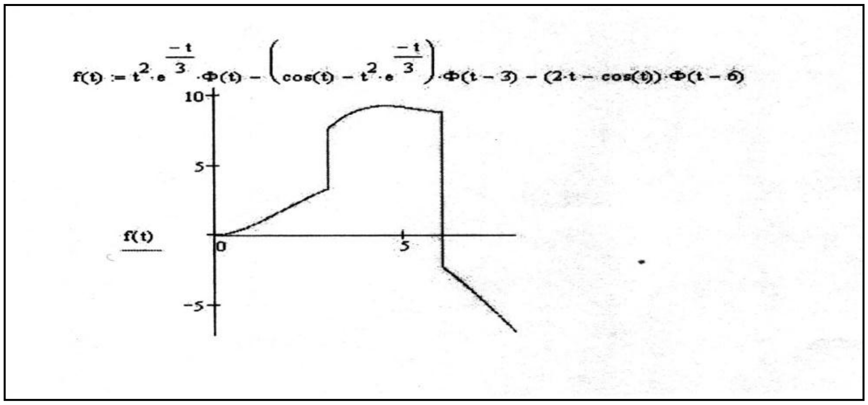


**FIGURE 3.The graph of function in MathCAD**

**3.2. Solve ODE With Piecewise Input Function:**

**EXAMPLE 3.1. Find solution to** $x'' + 4x = f(t), x(0) = 0, x'(0) = -1$

**Where** $f(t) = \begin{cases} \cos(2t) \text{ if } 0 \le t < \pi \\ 0 \quad \text{if} \quad t \ge \pi \end{cases}$

**Solution First we express *f(t)* as linear combination of step function. Using inverse Laplace transform, we get,**

$$x(t) = \begin{cases} -\frac{1}{2}\sin(2t) + \frac{1}{4}t\sin(2t) \text{ if } & 0 \le t \le \pi \\ 0 & \text{if } \frac{\pi-2}{4}\sin(2t) \ge \pi \end{cases}$$

**The equation** $x'' + 4x = f(t)$ **models a spring system that with one spring, one end of the spring is fixed and a object with 1 unit mass attached at one end. The Hook's constant** $k = 4$. ***f(t)*** **is external force applied to the system. In the Example 3.1, the force *sin(2t)* only applied at first Π unit of time (second).**

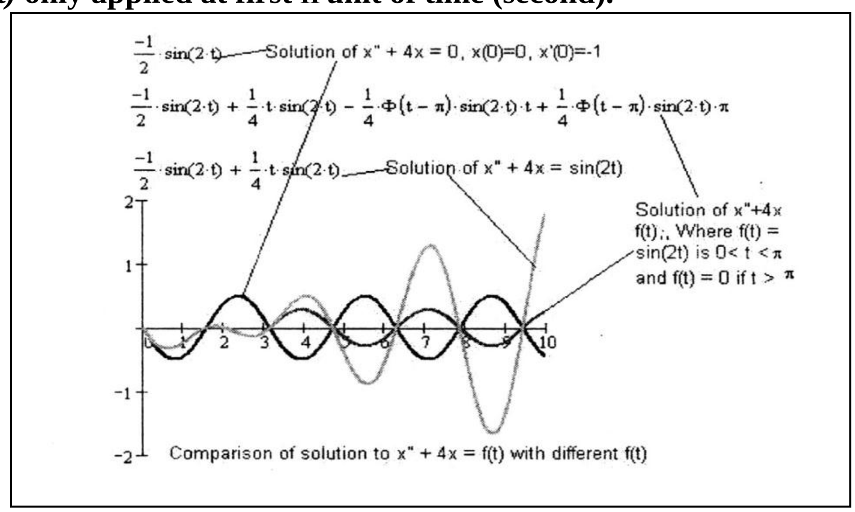


**Figure 4. The diagram compares the solution of this equation with different input f(t).**

**The graph contains solutions for the following three cases,**

**(1) No external force is applied (blue curve).**

**(2) Force of *sin(2t)* is applied at, first TT seconds and turnoff'(red curve).**

**(3) Force of *sin(2t)* is constantly applied (green curve). The initial configuration is that the spring is at equilibrium position *x(0) = 0* and is compressed at 1 unit, speed (feet/s) x'(0)= -1. From the graph we can clearly see that when the force is turnoff at *t* =- *π* its effect is immediately gone (as shown below Fig.5) .**

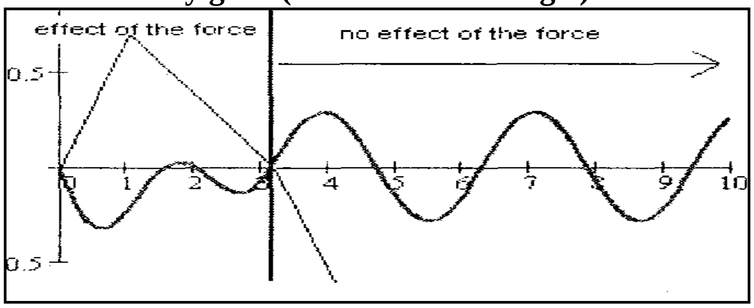


**Figure 5. The graph contains solutions for the three cases**

**When force *sin(2t)* turn off at *t*= *π* , its effect is gone immediately.**

**4. Delta Function:**

**Delta function is one of so-called generalized functions, which are not functions in ordinary sense but, as operators that sometimes can be represented by ordinary functions.**

**DEFINITION 4.1.** **The Dirac delta function at a, $\delta_a(t)$, is an operator that satisfies**

$$\int_0^\infty g(t)\delta_a(t)dt = g(a)$$

**For any continuous function, *g(t)*.**

**In Physics, if *f(t)*, $a \le t \le b$ is a force that acts only during a short period of time interval [a,b], the impulse *p* of force *f(t)* is computed as**

$$\int_a^b f(t)dt$$

**$\delta_a(t)$ can be viewed as an instantaneous unit impulse that, occurs precisely at, the instant t = a. And $p\delta_a(t)$ is an instantaneous *p* units impulse that occurs precisely at, the time t = a.**

**$\delta_a(t)$ is an important function used in modeling real phenomena, for example.**

**• In modeling the movement of a baseball after being hit by bat.**

**• In spring-mass system, where a object with given mass it, attached to one end of a spring whose other end is attached to a steady object (such as wall), modeling the movement when hits the object with a hammer.**

**• Modeling the current in a closed circuit when the switch is turn on and off instantly.**

**It, turns out, that we can stretch our self a little and compute the Laplace transform of $\delta_a(t)$. Let $\delta(t)= \delta_0(t)$then $\delta_a(t)= \delta(t - a)$.**

**THEOREM 4.1** $\hat{\delta}(s) = 1 and \hat{\delta}_a(s) = e^{-as}$

**EXAMPLE 4.1.** **A mass m = 1 is attached to a spring with constant k = 4; there is not dashpot. The mass is released from the rest with *x(0) = 3*. At the instant I = *π* the mass is struck with a hammer, providing an impulse p = 8. Determine the motion of the mass.**

**Solution: In general *mx" + kx = f(t)* models the movement, of a mass attached to the end of spring with no dashpot. So we need to solve**
***x" + 4x = 8δ*** $_\pi$ ***(t), x(0) =3, x'(0) =0*. Applying Laplace transform to both sides of the equation,**

$$\hat{x}''(s) + 4\hat{x}(s) = 8\delta_\pi(s)$$

$$s^2\hat{x} - sx(0) - x'(0) + 4\hat{x}(s) = 8e^{-\pi s}$$

$$(s^2+4)\hat{x}(s) - 3s = 8e^{-\pi s}$$

**Solve for** $\hat{x}(s)$, **we get**

$$\hat{x}(s) = \frac{3s}{s^2+4} + \frac{8}{s^2+4}e^{-\pi s}$$

**Using MathCAD , we can find inverse Laplace transform,**

$$x(t) = 2\cos(2t) + 4u_\pi(t)\sin(2t) = \begin{cases} 2\cos(2t) & if\ 0 \le t \le \pi \\ 2\cos(2t) + 4\sin(2t) & if\ \ t \ge \pi \end{cases}$$

**So clearly the impact of the impulse is felt by the spring and changes the it's movement, immediately. The solution curve is shown in fig.6.**
***Z(t)=3cos(2t)* Solution with no impulse, *x" + 4x =0,x(0)=3, x' (0)=0***

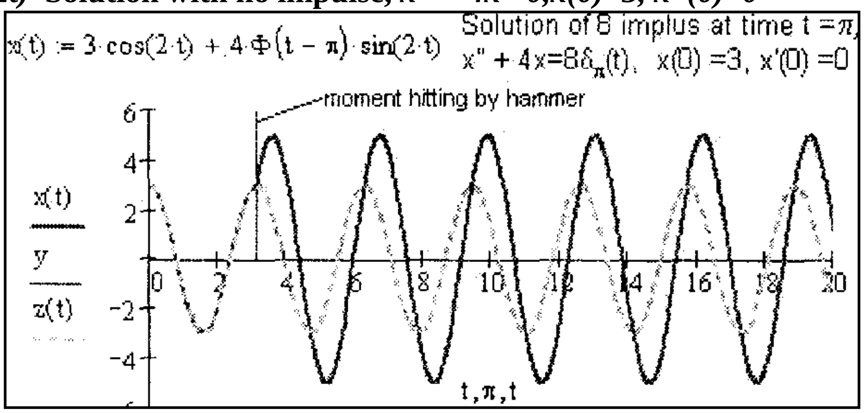


**Figure 6. The solution curve of models movement**

## Conclusions

1- **We can use Laplace transform method to solve system of complex differential equations, (higher order ODE).**
2- **MathCAD can help us in finding both Laplace transform and inverse Laplace transform.**
3- **We can get Heaviside function and graph of function can be gotten for any functions MathCAD professional.**
4- **By using MathCAD we can find inverse matrix for any order (m*n).**
5- **Simply we can get the results in the least time, for any functions or systems complex.**
6- **In this program (MathCAD) any errors in data entering cause either error in the results or not answering.**

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