Lie Symmetries to Ordinary Differential Equations (ODEs)

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Abstract

In this paper, we have provided some examples motivating the use of Lie symmetries. Also, we have discussed the properties of one- parameter groups of transformations (Lie groups) and introduced the ideas of invariants of the group of transformations. In the case of ordinary differential equations use of Lie symmetries resulted in a separable ODE when the governing DE was first order and a reduction of order when the governing DE was of order greater than one.

Keywords: *Lie symmetries, the group of transformations*, the *infinitesimals,ordinarydifferential equations, canonical coordinates.*

الملخص:

في هذا البحث،جهزنا بعضا الأمثلة لاستخدام تناظرات لي . كذلك، ناقشنا خصائص مجموعات التحويلات ذات البارامبتر الواحد وتسمى (مجموعات لي). كما ناقشنا حل المعادلات التفاضلية الاعتيادية باستخدام تناظرات لي نتيجة لطريقة فصل المتغيرات وذلك عندما تكون المعادلة التفاضلية الاعتيادية من الرتبة الاولى او اختزال رتبتها اذا كانت اكبر من واحد.

1.1 Introduction

In this section, we discuss the construction of a one-parameter group of transformations which leaves a given ordinary differential equation (ODE) unchanged. We shall find that if an ODE is invariant under a one-parameter group of transformations then use of an invariant of the group results in a simplification of the ODE. If the differential equation is of first order then the equation will become a separable differential equation. For a higher order differential equation, the use of an invariant leads to the reduction in the order of the equation by one. We firstly illustrate finding infinitesimals for an ODE from first principles and then quote a general result for a general first –order ODE.

1.2 (one- parameter transformation groups)

Definition: In the (x, y) plane, the transformation

$$x_1 = f(x, y, \epsilon) y_1 = g(x, y, \epsilon) \qquad (1.1)$$

is a **one- parameter group of transformations** if the following properties hold:

(i) (identity) the value $\epsilon = 0$ characterises the identity transformation,

$$x = f(x, y, 0), \qquad y = g(x, y, 0)$$

(ii) (inverse) the parameter $-\epsilon$ characterises the inverse transformation,

$$x = f(x_1, y_1, -\epsilon), \qquad y = g(x_1, y_1, -\epsilon)$$

(iii) (closure) if $x_2 = f(x_1, y_1, \delta)$, $\quad y_2 = g(x_1, y_1, \delta)$ then the product of the two transformation is also a member of the set of the transformation (1.1) and moreover is characterised by the parameter $\epsilon+\delta$, that is $x_2=f(x,y,\epsilon+\delta)$, $y_2=g(x,y,\epsilon+\delta)$, see [1],[5].

Example 1.1

Show that the transformation

$x_1=\frac{x}{1+\epsilon x}$, $y_1=(1+\epsilon x)^2y$ (1.2)

does indeed form a one-parameter group of transformations as defined above.

1) Firstly when $\epsilon=0$, $x_1=x$, $y_1=y$, so (i) is satisfied.

2) On rearranging (1.2) we obtain

$$(1+\epsilon x)x_1=x$$

$\Rightarrow x(\epsilon x_1-1)+x_1=0$

$\Rightarrow x=\frac{x_1}{1-\epsilon x_1}$

And $y=\frac{y_1}{(1+\epsilon x)^2}$

$\Rightarrow y=\frac{y_1}{(1+\epsilon\frac{x_1}{1-\epsilon x_1})^2}$

$\Rightarrow y=y_1(1-\epsilon x_1)^2$

So, $x=\frac{x_1}{1-\epsilon x_1}$, $y=y_1(1-\epsilon x_1)^2$

so that $-\epsilon$ characterizes the inverse and (ii) is satisfied .

3) we see that if

$x_2 = \frac{x_1}{1+\delta x_1}$, $\quad y_2 = (1+\delta x_1)^2 y_1$ then we have

$$x_2 = \frac{\frac{x}{1+\epsilon x}}{1+\delta(\frac{x}{1+\epsilon x})}$$

$\Rightarrow x_2 = \frac{x}{1+(\epsilon+\delta)x}$

and

$$y_2 = (1+\delta.\frac{x}{1+\epsilon x})^2.(1+\epsilon x)^2 y$$

$\Rightarrow y_2 = \left[(1+\delta.\frac{x}{1+\epsilon x}).(1+\epsilon x)\right]^2 y$

$\Rightarrow y_2 = [1+\epsilon x+\delta x]^2 y$

$\Rightarrow y_2 = (1+(\epsilon+\delta)x)^2 y,$

so that x_2 and y_2 are members of the group of transformations, characterised by $\epsilon+\delta$. Therefore (iii) is satisfied.

Remark:

The transformations (1.1) are called point transformations because the transformed values only depend on the dependent and independent variables x *and* y (and not derivatives of variables) and the parameter ϵ.The functions $f(x,y,\epsilon)$ *and* $g(x,y,\epsilon)$ are referred to as *the global form* of the group,[7]. For small values of ϵ, we can expand f *and* g and since

$$x_1 = x, \qquad y_1 = y \qquad at \ \ \epsilon = 0$$

we have

$$x_1 = x + \epsilon(\frac{df}{d\epsilon})_{\epsilon=0} + O(\epsilon^2),\ y_1 = y + \epsilon(\frac{dg}{d\epsilon})_{\epsilon=0} + O(\epsilon^2) \quad (1.3)$$

where $O(\epsilon^2)$ indicates terms of order ϵ^2 and higher.Defining $X(x,y)$ *and* $Y(x,y)$ by

$$X(x,y) = (\frac{df}{d\epsilon})_{\epsilon=0} \ \ and \quad Y(x,y) = (\frac{dg}{d\epsilon})_{\epsilon=0} \qquad (1.4)$$

then we obtain

$$x_1 = x + \epsilon X(x,y) + O(\epsilon^2),\ y_1 = y + \epsilon Y(x,y) + O(\epsilon^2)(1.5)$$

and (1.5) is referred to as ***the infinitesimal form*** of the group. $X(x,y), Y(x,y)$ are often referred to as "***the infinitesimal***", see [1],[5].

1.3 Invariants of a group

Definition:

In [2],[4], A differential function $F(x,y)$ is called an ***invariant function*** of a group G where$x^* = f(x,y,\epsilon),\quad y^* = g(x,y,\epsilon)$,

if $\qquad F(x^*,y^*) = F(x,y)$,(1.6)

Definition:

The ***infinitesimal generator*** of a one-parameter Lie group of transformations where

$$\bar{x} = x + \epsilon X(x,y) + O(\epsilon^2),\quad \bar{y} = y + \epsilon Y(x,y) + O(\epsilon^2)(1.7)$$

is given by $\qquad \Gamma = X(x,y)\frac{\partial}{\partial x} + Y(x,y)\frac{\partial}{\partial y}. \qquad (1.8)$

see [4],[6].