Another Type of Separation Axioms
Depend on an θg-open set

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تقديم البحث:- 2011/7/4

قبول النشر:- 2011/11/30

Abstract: This paper is devoted to introduce another type of separation axioms which we call θg -separation axioms, where we have get that θg - separation axioms give g- separation axioms. Besides, we give examples to show that the converse may not be true. Also,We discuss the relation among θg-separation axioms and give enough examples about it. On the other hand,We have proved some theorems and have given examples to discuss the property of θg -separation axioms. Finally,we get that there is no relation between separation Axioms and θg-separation Axioms.

نوع اخر من بديهيات الفصل

الخلاصة: كرس هذا البحث لتقديم نوع جديد من بديهيات الفصل اسميناه بديهيات الفصل – θgحيث حصلنا على ان بديهيات الفصل-θg تعطي بديهيات الفصل-g .اعطينا امثلة توضح بأن العكس غير صحيح. وايضا ناقشنا العلاقة بين بديهيات الفصل-θg ببعضها واعطينا امثلة كافية عنها. من ناحية اخرى برهنا بعض النظريات واعطينا امثلة لتناقش خواص بديهيات الفصل-θg . واخيرا توصلنا الى انه لا توجد علاقة بين بديهيات الفصل وبديهيات الفصل-θg .

(1) §1 Introduction: Separation Axioms was studied by Munkers in [4]. The first step of generalizing closed sets was done by Levin in 1970 [3]. He defined a set A to be generalized closed if its closure belongs to every open subset of A. generalized separation axioms was studied by Al-Meklafi in 2002[1],[6], where she defines g-T_0 , g-T_1 , g-T_2 ,and g-T_4 spaces. Dontchev and Maki [2] introduced θ-generalized closed set. θ-generalized Hausdorff is introduced in [5]. The aim of this paper is to introduce another type of separation axioms which we call θ-generalized separation axioms where we study the relation between θ-generalized separation axioms and generalized separation axioms . In this paper a topological space X will denote to the topological space (X,τ), and use a symbole θg to denote θ-generalized. We have get that θg separation axioms give g- separation axioms. Besides, we give examples to show that the converse may not be true. Beside,We discuss the relation among θg -separation axioms and give enough examples about it. Also,We have proved some theorems and have given examples to discuss the property of θg -separation axioms. Finally, we get that

There is no relation between separation Axioms and θg-separation Axioms.

§2 Preliminiaries [2]: The θ-interior of a subset A of a topological space X is the union of all open sets of X whose closures are contained in A, and is denoted by $int_\theta(A)$.The subset A is called θ-open if A= $int_\theta(A)$. The complement of a θ-open set is called θ-closed. Alternatively, a subset A of a topological space X is called θ-closed if A=$cl_\theta(A)$,where $cl_\theta(A)=\{x\epsilon X|\bar{U} \cap A\neq \phi, U\epsilon\ \tau$ and $x\epsilon U\}$.The family of all θ-open sets forms a topology on X is denoted by τ_θ . GC(X) (TGC(X)) will denote the family of all g-closed (θg-closed) subsets of X. Besides, GO(X) (TGO(X)) will denote the family of all g-open (θg-open) subsets of X.

Definition 2.1[2].(i) A subset A of a topological space X is called θ-generalized closed (written θg-closed) if $cl_\theta(A) \subseteq U$, whenever $A\subseteq U$ and $U\epsilon\ \tau$.

(ii)A subset A of a topological space X is called θ-generalized open (written θg-open) if X-A is θ-generalized closed.

Observation 2.2.[2] Every θ-generalized closed set is generalized closed.But the converse may not be true as in the follwing example. Let X= {a,b,c} and let $\tau = \{\phi,\{a\},\{a,b\},\{a,c\},X\}$. Set A={c}. Clearly, A is closed and hence g-closed. Next, set U={a,c}. Note that X= $cl_\theta(A) \not\subset U\in\tau$. Thuse A is not θg-closed.

Theorem 2.3.[2] (i)A finite union of θg-closed sets always a θg-closed set.

(ii) A countable union of θg-closed sets need not be θg-closed set.

(iii) A finite intersection of θg-closed sets may fail to be θg-closed set.

Definition 2.4.[5] A topological space X is said to be a θg -T_2 space if given any two distinct points x and y ,there are θg -

open sets U and V such that $x \in U, y \in V$ and $U \cap V = \phi$.

Theorem 2.5.[2] Let B⊆H⊆(X,τ).
(i) If B is θg-closed relative to H ,H∈TGC(X),and H is open in X ,then B∈TGC(X).
(ii) If B is θg-closed in X, then B is θg-closed relative to H.

Definition 2.6.[2] (i)A function f from a topological space X into a topological space Y is θg - irresolute if $f^{-1}(V)$ is an θg - closed in X for each θg -closed V of Y.
(ii) A function f from a topological space X into a topological space Y is θg -continuous if $f^{-1}(V)$ is an θg -closed in X for each closed V of Y.

Remark 2.7. The results above is also true on a θg-open sets by taking the complement.

§3 θ-generalized Separation Axioms

Definition 3.1.(i) A topological space X is called a θg-T_0 space if ∀x,y∈X , x≠y ∃ U a θg-open s.t. x∈U,y∉U or x∉U,y∈U.

(ii) A topological space X is said to be θg -T_1 space if for any two distinct points x and y of X, there are a θg -open set U of x which does not contain y and θg -open set V of y which does not contain x ,that is, $x \in U, y \notin U$ and $x \notin V, y \in V$.

(iv) A topological space X is said to be an θg -regular space if given any closed subset F of X and any point x of X which is not in F ,there are θg -open sets U and V such that $x \in U, F \subset V$,and $U \cap V = \phi$.

(v) A topological space X is said to be a θg-T_3 Space if and only if X is an θg-regular and also a θg-T_1 space.

(vi) A topological space X is said to be a θg-normal space if and only if for every two disjoint closed sets F_1 and F_2 in X there exist two θg-open sets U and V such that $F_1 \subset U, F_2 \subset V, U \cap V = \phi$.

(vii) A topological space X is called a θg- T_4 space if X is θg-normal and θg -T_1 at the same time.

Theorem 3.2. Every θg-T_i space is g-T_i space, i=0,1,2,3,4.

Proof: Let i= 0. Then a topological space X satisfies definition 3.1(i) and by 2.2 we get that the proof is complete. And by the same way other cases.

Observation 3.3. The following examples show that the converse of 3.2 may not be true .

(i) Let X={a,b,c}, τ={ϕ,X,{a},{b},{a,b},{a,c}}. Then:
GC(X)={ϕ,X,{c},{a,c},{b,c}}

TGC(X)={ϕ,X,{b,c}}

So X is a g-T_0 space but not θg-T_0 space. Notice that X is a T_0 space.

(ii) Let X={a,b,c,d}, τ={ϕ,X,{a},{a,c},{a,b,d}}

GO(X)
$=\{\phi,X,\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{a,b,d\},\{a,c,d\},\{a,b,c\}\}$
TGO(X) $=\{\phi,X,\{d\},\{b\},\{a\}\}$
So X is a g- T_2 space but it is neither θg-T_2 nor θg-T_1 spaces.

Theorem 3.4. Every θg -T_i space is θg-T_{i-1} space, i=1,2,3,4.

Proof: Let i=1. Then a topological space X satisfies definition 3.1(ii), which leads to definition 3.1(i). And by the same way other cases.

The following examples show that the converse of 3.4 may not be true.

Examples 3.5. (i) Let X={a,b,c}, $\tau = \{\phi,X,\{a\},\{a,b\}\}$.
TGO(X) $=\{\phi,X,\{a\},\{b\},\{a,b\}\}$
Then X is a θg -T_0 space but not θg-T_1 space.

(ii) Let X={a,b,c}, $\tau = \{\phi,X,\{a\}\}$.
TGO(X) $=\{\phi,X,\{a\},\{b\},\{c\},\{a,b\},\{a,c\}\}$
Then X is a θg -T_2 space but not θg-T_3 space (since it is not θg -regular). Notice that X is neither T_0 ,T_1 nor T_2 spaces

Remarks 3.6. (A) The following examples show that θg-T_i , i=0,1,2,3,4 space is not hereditary property .

(i) Let X={a,b,c}, $\tau = \{\phi,X,\{a\},\{b\}\}$. Let Y={a,c}, $\tau_y = \{\phi,Y\}$. Then:

TGO(X) $=\{\phi,X,\{a\},\{b\},\{c\},\{a,b\},\{b,c\}\}$
TGO(Y) $=\{\phi,Y\}$
So X is a θg -T_0 space but Y is not θg-T_0 space.

(ii) Let X={a,b,c}, $\tau = \{\phi,X\}$. Let Y={a,b}, $\tau_y = \{\phi,Y,\{b\}\}$. Then:

TGO(X) ={ϕ,X,{a},{b},{c},{a,b},{a,c},{b,c}}
TGO(Y) ={ϕ,Y,{b}}
So X is a θg –T_1 space but Y is not θg-T_1 space.

(B) There is no relation between separation Axioms and θg-separation Axioms as we have seen in examples 3.3(i) and 3.5(ii)

Theorem 3.7. A topological space X is an θg-T_1 space if every singleton set is θg-closed .

Proof:

Let every singleton set in X is θg-closed .To prove X is θg-T_1 space.

Suppose $x, y \in Y$ such that $x \neq y$.

Since $\{x\}$ is θg-closed ,thus X-$\{x\}$ θg-open set such that $x \notin X - \{x\}, y \in X - \{x\}$.

Similarly $X - \{y\}$ is a θg-open set. Thus $y \notin X - \{y\}, x \in X - \{y\}$.

Hence X is a θg-T_1 space.

Theorem 3.8. (i)Let X be a topological space and Y is an θg -T_2 space. If $f : X \to Y$ is injective and θg -irresolute ,then X is θg -T_2 space.

(ii) Let X be a topological space and Y is a θg -T_2 space. If $f : X \to Y$ injective and θg -continuous ,then X is a θg -T_2 space.

Proof:

(i) Suppose that $x, y \in X$ such that $x \neq y$.

Since f is injective , then $f(x) \neq f(y)$.

Since Y is a θg -T_2 space, then there are two θg -open sets U and V in Y such that $f(x) \in U, f(y) \in V$ and $U \cap V = \phi$.

Since f is θg-irresolute then $f^{-1}(U), f^{-1}(V)$ are two θg-open sets in X, $x\in f^{-1}(U), y\in f^{-1}(V)$, $f^{-1}(U)\cap f^{-1}(V)=\phi$.

Hence X is θg -T_2.

(ii) By the same way in (i),and using f as θg-continuous function instead of θg-irresolute function.

Theorem 3.9. X is an θg -T_2 space if for each x in X ,there is a set U such that $x\in U$ where U is closed, θg -open , θg -closed ,and also θg -T_2 space in X.

Proof:

Let $x, y\in X$ such that $x\neq y$.Now, we have three cases:

1.$x\notin U$ and $y\in U$ .2.$x\in U$ and $y\notin U$ .3.$x, y\in U$.

1. If $x\notin U$ and $y\in U$,then $x\in U^c$ and U^c is θg -open sets and U is θg -open set in X ,and $U\cap U^c=\phi$.Hence X is θg -T_2 space.

2.By the same way if $x\in U$ and $y\notin U$,then X is θg -T_2 space.

3.If $x, y\in U$.
Since U is θg -T_2 space, then there are two θg -open sets W,V in U such that $x\in V, y\in W, V\cap W=\phi$,but V and W are θg -open sets in X by 2.5(i).Hence X is a θg -T_2 space.

Theorem3.10. Let X and Y be topological spaces and Y is a regular space.If $f: X\rightarrow Y$ is closed , θg -irresolute, and one to one ,then X is an θg -regular space.

Proof

Let F be closed set in X,$x\notin F$.Since f is closed mapping ,then $f(F)$ is closed set in Y. $f(x)=y\notin f(F)$.But Y is θg -

regular space, then there are two θg -open sets U and V in Y such that $f(F)\subseteq V, y\in U, U\cap V=\phi$.

Since f is θg -irresolute mapping and one to one ,so $f^{-1}(U), f^{-1}(V)$ are two θg -open sets in X and $x\in f^{-1}(U), F\subset f^{-1}(V)$, $f^{-1}(U)\cap f^{-1}(V)=\phi$

Hence X is a θg -regular space.

Theorem 3.11. Let X be a θg -normal space ,and $Y\subseteq X$ .If Y is open and closed ,then Y is θg -normal.

Proof

Let F_1 ,F_2 be closed sets in Y.

Since Y is a closed set , then F_1 and F_2 are closed in X.

Since X is θg -normal space, then there exist two θg -open sets U and V such that $F_1\subset U, F_2\subset V, U\cap V=\phi$.

By theorem 2.5 (ii), we have U ,V are two θg -open sets in Y .

Hence Y is θg -normal.

Theorem3.12. Let f be a closed and θg - irresolute mapping from a topological space X into a topological space Y .If Y is θg -normal, so is X.

Proof

Let F_1, F_2 be closed sets in X such that $F_1\cap F_2=\phi$.

Since f is closed map ,we have $f(F_1), f(F_2)$ are two closed sets in Y and $f(F_1)\cap f(F_2)=\phi$.

Since Y is θg -normal and f is θg - irresolute ,then there exist two θg -open sets U,V in Y such that $f(F_1)\subset U, f(F_2)\subset V, U\cap V=\phi$,also $f^{-1}(U), f^{-1}(V)$ are θg -open sets in X and $F_1\subset f^{-1}(U), F_2\subset f^{-1}(V)$, $f^{-1}(U)\cap f^{-1}(V)=\phi$.

Hence is X θg -normal.

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