The Approximate Solutions for Volterra Integro-Differential Equations within Local Fractional Integral Operators

Hassan Kamil Jassim[1] , Hussein Khashan Kadhim[2]
[1 and 2] Department of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar, Nasiriyah, Iraq
[1] hassan.kamil@yahoo.com , [2] khashanhussein@gmail.com

الملخص
في هذا البحث نطبق طريقة تحويل يانك - لابلاس على المعادلات التكاملية- التفاضلية نوع فولترة من الصنف الثاني ضمن المؤثرات التفاضلية الكسرية المحلية للحصول على الحلول التقريبية. الإجراءات التكرارية تستند على المؤثرات التفاضلية الكسرية المحلية. هذا الاسلوب يوفر لنا طريق ملائم لإيجاد حل مع أقل حساب بالمقارنة مع طريقة التكرار المتغاير الكسرية المحلية. بعض الأمثلة التوضيحية نوقشت .أظهرت النتائج أن الطريقة هي أداة فعالة جدا وبسيط لحل المعادلات التكاملية.

Abstract

In this paper, we use the Yang-Laplace transform on Volterra integro-differential equations of the second kind within the local fractional integral operators to obtain the nondifferentiable approximate solutions. The iteration procedure is based on local fractional derivative operators. This approach provides us with a convenient way to find solution with less computation as compared with local fractional variational iteration method. Some illustrative examples are discussed. The results show that the methodology is very efficient and simple tool for solving integral equations.

Keywords: Volterra integro-differential equations, Local fractional Laplace transform method, Local fractional derivative operator, Local fractional integral operator.

1. Introduction

The Yang-Laplace transform in fractal space is a generalization of Laplace transforms derived from the local fractional calculus. Several analytical and numerical techniques were successfully applied to deal with integral equations within local fractional derivative operators such as local fractional variational iteration method (Neamah, (2014)), Adomian decomposition method (Yang and Zhang, (2012)), Picard's successive approximation method (Yang(5th reference) ,(2012)) and another method.

The standard $k\alpha$ order local fractional Volterra integro-differential equation of the second kind is given by (Neamah, (2014)):

$$\psi^{(k\alpha)}(x) = f(x) + \frac{1}{\Gamma(1+\alpha)} \int_0^x K(x,t)\psi(t)(dt)^{\alpha} \quad , \tag{1.1}$$

with the initial conditions

$$\psi^{(m\alpha)}(0) = a_m , m = 0,1,....., k-1, \tag{1.2}$$

where $k(x,t)$ is the kernel of the local fractional integral equation, and $f(x)$ is a local fractional continuous function.

The Laplace transform is an essential mathematical tool for the design, analysis and monitory of systems and shows insight into the transient behavior the steady state behavior, and the stability of continuous time systems. However, the classical Laplace-transform does not deal with fractal functions, which are local fractional continuous non-differential functions (Yang(6th reference), (2012)).In this paper, we investigate the application of local fractional Laplace transform method for solving the local fractional Volterra integro-differential equation of the second kind. This paper is organized as follows: In section 2, the concept of local fractional calculus and integrals are given. In section 3, Laplace transform method is proposed based on local fractional integrals. Illustrative examples are shown in section 4. Conclusions are given in section 5.

2. Mathematical Fundamentals

In this section we present some basic definitions and notations of the local fractional calculus [see (Yan , et al, (2014), Yang(7th reference),(2012), Yang(8th reference),(2014) and Jassim, et al,(2015))].

Definition 1. The local fractional derivative of $\psi(x)$ of order α at $x = x_0$ is given by

$$\left.\frac{d^{\alpha}}{dx^{\alpha}}\psi(x)\right|_{x=x_0} = \psi^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{\Delta^{\alpha}(\psi(x) - \psi(x_0))}{(x-x_0)^{\alpha}} , \tag{2.1}$$

where $\Delta^{\alpha}(\psi(x) - \psi(x_0)) \cong \Gamma(\alpha+1)(\psi(x) - \psi(x_0))$.

The formulas of local fractional derivatives of special functions used in the paper are as follows:

$$D_x^{(\alpha)} a\psi(x) = aD_x^{(\alpha)}\psi(x), \tag{2.2}$$

$$\frac{d^\alpha}{dx^\alpha}\left(\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\right) = \frac{x^{(n-1)^\alpha}}{\Gamma(1+(n-1)\alpha)}, \quad n \in N \tag{2.3}$$

Definition 2. The local fractional integral of $\psi(x)$ of order α in the interval $[a,b]$ is given by

$${}_aI_b^{(\alpha)}\psi(x) = \frac{1}{\Gamma(1+\alpha)}\int_a^b \psi(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}\psi(t_j)(\Delta t_j)^\alpha, \tag{2.4}$$

where the partitions of the interval $[a,b]$ are denoted as (t_j, t_{j+1}), with $\Delta t_j = t_{j+1} - t_j$, $t_0 = a$, $t_N = b$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, ...\}$, $j = 0,...,N-1$.

The formulas of local fractional integrals of special functions used in the paper are as follows:

$${}_0I_x^{(\alpha)} a\psi(x) = a\, {}_0I_x^{(\alpha)}\psi(x), \tag{2.5}$$

$${}_0I_x^{(\alpha)}\left(\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\right) = \frac{x^{(n+1)^\alpha}}{\Gamma(1+(n+1)\alpha)}, \quad n \in N \tag{2.6}$$

Definition 3. Let $\frac{1}{\Gamma(1+\alpha)}\int_0^\infty |\psi(x)|(dx)^\alpha < k < \infty$. The Yang-Laplace transforms of $\psi(x)$ is given by

$$L_\alpha\{\psi(x)\} = \Psi_s^{L,\alpha}(s) = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha(-s^\alpha x^\alpha)\psi(x)(dx)^\alpha, \; 0 < \alpha \le 1 \tag{2.7}$$

where the latter integral converges and $s^\alpha \in R^\alpha$.

Definition 4. The inverse formula of the Yang-Laplace transforms of $\psi(x)$ is defined as

$$L_\alpha^{-1}\{\Psi_s^{L,\alpha}(s)\} = \psi(x) = \frac{1}{(2\pi)^\alpha}\int_{\beta-i\omega}^{\beta+i\omega} E_\alpha(s^\alpha x^\alpha)\Psi_s^{L,\alpha}(s)(ds)^\alpha, \; 0 < \alpha \le 1 \tag{2.8}$$

where $s^\alpha = \beta^\alpha + i^\alpha\omega^\alpha$, i^α is the fractal imaginary unit and $\mathrm{Re}(s) = \beta > 0$.

Definition 5. The convolution of two functions is defined symbolically by

$$\psi_1(x) * \psi_2(x) = \frac{1}{\Gamma(1+\alpha)}\int_0^x \psi_1(t)\psi_2(x-t)(dt)^\alpha, \tag{2.9}$$

or

$$\psi_2(x)*\psi_1(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^x \psi_2(t)\psi_1(x-t)(dt)^\alpha \,. \tag{2.10}$$

Theorem 1. (The convolution theorem)

Let $L_\alpha\{\psi_1(x)\}=\Psi_{s,1}^{L,\alpha}(s)$ and $L_\alpha\{\psi_2(x)\}=\Psi_{s,2}^{L,\alpha}(s)$, then

$$L_\alpha\{\psi_1(x)*\psi_2(x)\}=\Psi_{s,1}^{L,\alpha}(s)\,\Psi_{s,2}^{L,\alpha}\,. \tag{2.11}$$

Theorem 2.

$$L_\alpha\left\{\psi^{(k\alpha)}(x)\right\}=s^{k\alpha}\Psi_s^{L,\alpha}(s)-s^{(k-1)\alpha}\psi(0)-s^{(k-2)\alpha}\psi^{(\alpha)}(0)-\cdots-\psi^{((k-1)\alpha)}(0)\,, \tag{2.12}$$

$$L_\alpha\left\{E_\alpha(a^\alpha x^\alpha)\right\}=\frac{1}{s^\alpha-a^\alpha}\,. \tag{2.13}$$

3. Local Fractional Laplace Transform Method

In view of the convolution theorem for the Yang Laplace transform, if the kernel $K(x,t)$ in equation (1.1) be a difference kernel. Then the local fractional Volterra integro-differential equation can thus be written as

$$\psi^{(k\alpha)}(x)=f(x)+\frac{1}{\Gamma(1+\alpha)}\int_0^x K(x-t)\psi(t)(dt)^\alpha\,. \tag{3.1}$$

By taking Yang Laplace transform of both sides of (3.7)

$$s^{k\alpha}\Psi_s^{L,\alpha}(s)-s^{(k-1)\alpha}\psi(0)-s^{(k-2)\alpha}\psi^{(\alpha)}(0)-\cdots-\psi^{((k-1)\alpha)}(0)=F_s^{L,\alpha}(s)+K_s^{L,\alpha}(s)\Psi_s^{L,\alpha}(s)$$
(3.2)

where

$$\Psi_s^{L,\alpha}(s)=L_\alpha\{\psi(x)\},\quad F_s^{L,\alpha}(s)=L_\alpha\{f(x)\},\quad K_s^{L,\alpha}(s)=L_\alpha\{K(x)\} \tag{3.3}$$

Substituting the initial conditions into (3.2), we obtain

$$s^{k\alpha}\Psi_s^{L,\alpha}(s)-a_0\,s^{(k-1)\alpha}-a_1\,s^{(k-2)\alpha}-\cdots-a_{k-1}=F_s^{L,\alpha}(s)+K_s^{L,\alpha}(s)\Psi_s^{L,\alpha}(s) \tag{3.4}$$

Solving (3.4) for $\Psi_s^{L,\alpha}(s)$ gives

$$\Psi_s^{L,\alpha}(s)=\frac{F_s^{L,\alpha}(s)+a_0\,s^{(k-1)\alpha}+a_1 s^{(k-2)\alpha}+\cdots+a_{k-1}}{s^{k\alpha}-K_s^{L,\alpha}(s)}\,. \tag{3.5}$$

The solution $\psi(x)$ is obtained by applying the inverse Yang Laplace transform of both sides of (3.5). Therefore, we obtain

$$\psi(x)=L_\alpha^{-1}\left\{\frac{F_s^{L,\alpha}(s)+a_0\,s^{(k-1)\alpha}+\cdots+a_{k-1}}{s^{k\alpha}-K_s^{L,\alpha}(s)}\right\}. \tag{3.6}$$

4. Illustrative Examples

Example 1. Let us consider the following Volterra integro-differential equation of the second kind involving local fractional derivative:

$$u^{(\alpha)}(x)=1+\frac{1}{\Gamma(1+\alpha)}\int_0^x u(t)(dt)^{\alpha}, \qquad u(0)=1. \tag{4.1}$$

Taking the Yang Laplace transform of (4.1) gives

$$L_{\alpha}\left\{u^{(\alpha)}(x)\right\}=L_{\alpha}\{1\}+L_{\alpha}\{1*u(x)\}, \tag{4.2}$$

so that

$$s^{\alpha}U_s^{L,\alpha}(s)-u(0)=\frac{1}{s^{\alpha}}+\frac{1}{s^{\alpha}}U_s^{L,\alpha}(s), \tag{4.3}$$

Using the given initial condition and solving for $U_s^{L,\alpha}(s)$ we find

$$U_s^{L,\alpha}(s)=\frac{1}{s^{\alpha}-1} \tag{4.4}$$

By taking the Yang-Laplace inverse of the equation (4.4), the nondifferentiable solution is given by

$$u(x)=E_{\alpha}(x^{\alpha}), \tag{4.5}$$

which is equal to the result based on the local fractional variational iteration method (Neamah, (2014)).

Example 2: Consider the following Volterra integro-differential equation of the second kind involving local fractional derivative:

$$u^{(2\alpha)}(x)=1+\frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}u(t)(dt)^{\alpha} \quad ,u(0)=1,\ u^{(\alpha)}(0)=0. \tag{4.6}$$

The Yang-Laplace transform of equation (4.6) yields

$$L_{\alpha}\left\{u^{(2\alpha)}(x)\right\}=L_{\alpha}\{1\}+L_{\alpha}\left\{\frac{x^{\alpha}}{\Gamma(1+\alpha)}\right\}L_{\alpha}\{u(x)\}, \tag{4.7}$$

so that

$$s^{2\alpha}U_s^{L,\alpha}(s)-s^{\alpha}u(0)-u^{(\alpha)}(0)=\frac{1}{s^{\alpha}}+\frac{1}{s^{2\alpha}}U_s^{L,\alpha}(s). \tag{4.8}$$

Using the given initial condition and solving for $U_s^{L,\alpha}(s)$ we find

$$U_s^{L,\alpha}(s)=\frac{1}{2}\frac{1}{s^{\alpha}-1}+\frac{1}{2}\frac{1}{s^{\alpha}+1}. \tag{4.9}$$

By taking the inverse Yang-Laplace transform of both sides (4.9), the nondifferentiable solution is given by

$$u(x)=\frac{1}{2}E_\alpha(x^\alpha)+\frac{1}{2}E_\alpha(-x^\alpha)=\cosh_\alpha(x^\alpha), \tag{4.10}$$

which is equal to the result based on the local fractional variational iteration method (Neamah, (2014)).

Example 3. Consider the following local fractional Volterra integro-differential equation of the second kind with given initial condition

$$u^{(4\alpha)}(x)=\sin_\alpha(x^\alpha)+\cos_\alpha(x^\alpha)+\frac{2}{\Gamma(1+\alpha)}\int_0^x \sin_\alpha(x^\alpha-t^\alpha)u(t)(dt)^\alpha , \tag{4.11}$$

with given initial condition

$$u(0)=u^{(\alpha)}(0)=u^{(2\alpha)}(0)=u^{(3\alpha)}(0)=1 . \tag{4.12}$$

Taking the Yang Laplace transform of (4.11) gives

$$L_\alpha\left\{u^{(4\alpha)}(x)\right\}=L_\alpha\left\{\sin_\alpha(x^\alpha)\right\}+L_\alpha\left\{\cos_\alpha(x^\alpha)\right\}+2L_\alpha\left\{\sin_\alpha(x^\alpha)\right\}L_\alpha\left\{u(x)\right\}, \tag{4.13}$$

so that

$$s^{4\alpha}U_s^{L,\alpha}(s)-s^{3\alpha}u(0)-s^{2\alpha}u^{(\alpha)}(0)-s^{\alpha}u^{(2\alpha)}(0)-u^{(3\alpha)}(0)=\frac{1}{s^{2\alpha}+1}-\frac{s^\alpha}{s^{2\alpha}+1}+\frac{2}{s^{2\alpha}+1}U_s^{L,\alpha}(s),$$

(4.14)

Using the given initial conditions (4.12) and solving for $U_s^{L,\alpha}(s)$ we obtain

$$U_s^{L,\alpha}(s)=\frac{1}{s^\alpha-1}. \tag{4.15}$$

By taking the inverse Yang-Laplace transform of both sides (4.11), the nondifferentiable solution is given by

$$u(x)=E_\alpha(x^\alpha). \tag{4.16}$$

5. Conclusions

In this work, we considered the local fractional Laplace transform method to solve the Volterra integro-differential equations of the second kind within the local fractional operators and their nondifferentiable approximate solutions were obtained. The proposed method is a powerful tool for solving many integral equations within the local fractional derivatives.

6. References

Jassim, H. K., Ünlü, C., Moshokoa ,S. P. and Khalique ,C. M., (2015), Local Fractional Laplace Variational Iteration Method for Solving Diffusion and Wave Equations on Cantor Sets within Local Fractional Operators, Mathematical Problems in Engineering, vol. 2015, Article ID 309870 1-9.

Neamah, A. A., (2014) , Local Fractional Variational Iteration Method for Solving Volterra Integro-Differential equations within the local fractional operators, Journal of Mathematics and Statistics, vol. 10, no. 3 , 401-407.

Yan ,S. P., Jafari, H., and Jassim, H. K., (2014) , Local Fractional Adomian Decomposition and Function Decomposition Methods for Solving Laplace Equation within Local Fractional Operators, Advances in Mathematical Physics, vol. 2014, Article ID 161580 ,1-7.

Yang, X. J. and Zhang, Y. , (2012) , A new Adomian decomposition procedure scheme for solving local fractional Volterra integral equation, Advances in Information Technology and Management, vol.1, no. 4 ,158-161.

Yang ,X. J., (2012) , Picard's approximation method for solving a class of local fractional Volterra integral equations, Advances in Intelligent Transportation Systems, vol. 1, no. 3 ,67-70.

Yang, X. J., (2012), A New Approach to Solving Class of Local Fractional Volterra Integral Equation, Advances in Information Technology and Management, vol. 1, no. 4 ,183-186.

Yang, X. J.,(2012),Advanced Local Fractional Calculus and Its Applications, World Science, New York, NY, USA

Yang, Y. J., Wang, S. Q. and Jassim, H. K., (2014), Local Fractional Function Decomposition Method for Solving Inhomogeneous Wave Equations with Local Fractional Derivative, Abstract and Applied Analysis, Volume 2014, Article ID 176395 ,1-7.