# Error estimate of the discontinuous Galerkin finite element method for convection-diffusion problems

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**Abstract**

This paper presents the theory analysis of the discontinuous Galerkin finite element (DGFE) method with respect to space variables only is applied whereas, time remains continuous oflinear convection – diffusion problem .The properties of the bilinear form $A(u,v)$(v-elliptic and continuity) of (DGFE), stability are proved and the approximate solution is converges with error of order $(h)$.

**Keywords:** linear convection–diffusion equation, discontinuous Galerkin method, error estimate, convergent and stability.

## 1. Introduction

A number of complex problems from science and technology(aerospace engineering, turbo machinery, oil recovery, meteorology, environmental protection etc.) require to apply an efficient, robust, reliable and highly accurate numerical methods. It is necessary to develop the techniques that allow realizing numerical approximationof strongly nonlinear singularly perturbed systems in domains with complex geometry whose solution contains internal or boundary layers. An excellent candidate to overcome these difficulties is the DGFE method, which becomes more popular in the solutiona number of problems. The DGFE method uses piecewise polynomial approximations of the sought solution on a finite elements mesh without any requirement on continuity between neighboring elements and can be considered a generalization of the finite volume and finite element methods. It allows to construct a higherorder schemes in a natural way and is suitable for approximation of discontinuous solutions of conservation laws or solutions of singularly perturbed convection-diffusion problems having steep gradients. This method exploits' advantages of the finite element method and finite volume schemes with an approximate Riemann solver and can be applied on unstructured grids which are generated for most complex geometries. The original DGFE method was introduced in (1) for the solution of a neutron transport linear equation and analyzed theoretically in (2) and later in (3). Almost simultaneously the DGFE techniques were developed for the numerical solution of elliptic problems(4)and space semi discretization of parabolic problems(5,6), using the interior penalty Galerkin methods. In these works the symmetric approximation of the diffusion terms is used, called the SIPG(symmetric interior penalty). Theoretical analysis of this type of the DGFE method applied to elliptic problems can be found e.g., in, (7,8,9) The DGFE method found very soon a number of applications . Let us mention in particular the solution nonlinear conservation laws(10,11,12,13,14) , and compressible flow(15,16,17,18). A survey of DGFE methods.

Techniques and some applications can be found in (19,20). In the discretizationof non-stationary problems, one often uses the space semi discretization, also calledthe method of lines. In this approach, the DGFE discretization with respect to spacevariables only is applied , whereas time remains continuous . This leads to a large system of ordinary differential equations which can be solved numerically by a suitable ODE solver(9, 21, 22, 23, 24, 12, 13).

This paper is organized as follows. In section 2 we present the convection-diffusion equation. Some definitions and important lemmas are presents in section 3. The discretization is shown in section 4. In section 5 derive the weak form. In section 6 we proved the properties of the bilinear form and stability. The error estimate are presented in section 7. Finally the conclusion is shown in section 8.

## 2. The Convection-Diffusion Equation.

Let$\Omega \subset R^d, (d = 2\ or\ 3)$be a bounded polyhedral domain and T>0. The convection-diffusion problem is considered : Find $u \in Q_T = \Omega \times (0, \mathrm{T}) \to R$such that

$$u_t - a\Delta u + b\nabla u = f \qquad in\ Q_T \quad (2.1)$$

$$u = u_D \qquad on \quad \partial\Omega_D \times (0, T) \quad (2.2)$$

$$a\frac{\partial u}{\partial n} = u_N \qquad on \quad \partial\Omega_N \times (0, T) \quad (2.3)$$

$$u(x, 0) = u^0(x)\ , \quad x \in \Omega\ . \quad (2.4)$$

We assume that $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$

$$b \cdot n < 0 \quad on \quad \partial\Omega_D \qquad (2.5)$$

$$b \cdot n \geq 0 \quad on \quad \partial\Omega_N \text{for all } t \in [0\ .T]; \quad (2.6)$$

here $n$is the unit outer normal to the boundary$\partial\Omega$of $\Omega$,$\partial\Omega_D$ is the inflow boundary $\partial\Omega_N$is the outflow boundary, $b$ is convection coefficient and $a$ is diffusioncoefficient(25).

## 3-Definitions and some important lemmas:

It is beneficial to mention the definitions of the vector space that we used during this study. The vector space $L^2(\Omega)$ is the space of square-integrable functions on$\Omega \subset R^n$,$L^2(\Omega) = \{v: \Omega \to R s.t. \int v^2\, d\Omega \leq \infty\}$,

Indeed $L^2(\Omega)$ is Hilbert space with respect to the following inner product

$$(u, v) = \int_\Omega u(x)v(x)\, dx \text{andnorm} \|v\|_{L^2(\Omega)} = \left(\int_\Omega v^2 d\,\Omega\right)^{\frac{1}{2}}$$

for$p = \infty$, $L^\infty(\Omega)$denotes the space of all functions which are bounded for almost all

$$x \in \Omega:$$

$$L^\infty(\Omega) = \{u: |u(x)| < \infty\ for\ almost\ all\ x \in \Omega\},$$

this space is equipped with the norm

$$\|v\|_{L^\infty(\Omega)} = \{ess\sup\{|v(x)|: x \in R\}.$$

We introduce the sobolev space

$$H^1(\Omega) = \left\{v \in L^2(\Omega): \frac{\partial v}{\partial x_i} \in L^2(\Omega), i = 1,2 \dots \dots d\right\},$$

and the corresponding norm,

$$\|v\|_{H^1(\Omega)} = \left(\int_\Omega (\mathrm{v}^2 + (\nabla \mathrm{v})^2 \mathrm{d}\Omega)\right)^{\frac{1}{2}},$$

also,

$H_0^1(\Omega)=\{v\in H^1(\Omega): v=0\ on\partial\Omega\}$,

with the same scalar product and norm as $H^1(\Omega)$ .

We introduce the norm for both continuous time $t\in[0,T]$ and space $\Omega$by:

$$\|v\|_{L^{\infty}(H^r(\Omega))} = \max_{0\le t\le T}\|v\|_r \ and\ \ \|v_t\|_{L^2(L^2(\Omega))} = \left(\int_0^t \|v_t\|^2\right)^{\frac{1}{2}}.$$

## 4. Discretization of the problem:

Let $\tau_h$ be a partition of $\overline{\Omega}$ (the closure of the domain $\Omega$) into a finite number of closed triangles with mutually disjoint interiors. let$\tau_h$ a triangulation of $\Omega$.We denote conforming properties of $\tau_h$ used in the finite element method. This means that we admit the so-called hanging nodes. If two elements $E^i$, $E^j\in\tau_h$contain a nonempty open part oftheir sides, we call them neighbors. Let $\partial E$ denote the boundary of an element $E\in\tau_h$, if we set $\partial E^1\cap\partial E^2$ has a positive $(d-1)$- dimensional measure, we say that $e\in E$ is the edge of $E$, if it is a maximal connected open subset either of$E^1\cap E^2$, where $E^1$ is a neighbor of $E^2$ or subset of $\partial E\cap\partial\Omega$. By $\partial\tau_h$ denote the system of all edges of all elements $\in\tau_h$. Further, define the set of all inner and boundary edges by

$$\partial\tau_h^I=\{e\in\partial\tau_h, e\subset\Omega\},$$
$$\partial\tau_h^B=\{e\in\partial\tau_h, e\subset\partial\Omega\},$$
$$\Gamma_D=\{e\in\partial\tau_h^B, e\subset\partial\Omega_D\},$$
$$\Gamma_N=\{e\in\partial\tau_h^B, e\subset\partial\Omega_N\}.$$

For $\varphi\in H^1(\Omega,\tau_h)$ we introduce the following notation. Obviously $\partial\tau_h=\partial\tau_h^I\cup\partial\tau_h^B$, $\partial\tau_h^B=\Gamma_D\cup\Gamma_N$ for each $e\in\partial\tau_h$,

$$\{\varphi\}_e=\frac{1}{2}(\varphi^+ + \varphi^-)\quad(4.1)$$
$$[\varphi]_e=(\varphi^+ - \varphi^-)\quad(4.2)$$

Moreover,

$$\partial E_-^i=\{x\in\partial E^i: b\cdot n<0\},\quad(4.3)$$
$$\partial E_+^i=\{x\in\partial E^i: b\cdot n\ge 0\},\quad(4.4)$$

where $n$denotes the unit outer normal to $\partial E^i$.

### 4.1 Assumptions:

a)$f\in C([0,T];L^2(\Omega)), u^0, u, u_t\in L^2(\Omega)$ ,

b)$u_D$is the trace of some $u\in C([0,T];H^1(\Omega))\cap L^{\infty}(QT)$on $\partial\Omega_D\times(0,T)$

c)$u_N\in C([0,T];L^2(\partial\Omega_N))$,

d) $|E|$is the area of$E\in\tau_h$, and $\sigma=\frac{\sigma^0}{|e|^{\beta_0}}$, $\beta_0\ge(d-1)^{-1}$

e) Define $h_E$is the length of the longest side of the triangle$E\in\tau_h$ and put $h_E$ = diameter of $E$,$h=max_{E\in\tau_h} h_E$.

## 5. The weak form of problem.

Multiply equation (2.1) by the test function $v\in V=H^1(\Omega)$ such that.

$$(u_t,v)+\sum_E a(\nabla u,\nabla v) - \sum_{e\in\partial\tau_h}\int \nabla u\cdot n\, v\, ds + \sum_E (b\cdot\nabla u, v) - \sum_{e\in\partial\tau_h}\int |b\cdot n| uv\, ds = (f,v),$$

where $n$denotes the outward normal to each element edge. The third and fifth termson the left-hand side contain the integrals over the element edges,Then have,

$$(u_t,v)+\sum_E a(\nabla u,\nabla v)-\sum_{e\in\partial\tau_h}\int[\nabla u\cdot n\, v]\,ds+\sum_E(b\cdot\nabla u,v)-\sum_{e\in\partial\tau_h}\int|b\cdot n|[uv]\,ds$$
$$=(f,v).$$

Since, $[uv]=\{u\}[v]+\{v\}[u]$, we have

$$(u_t,v)+\sum_E a(\nabla u,\nabla v)-\sum_{e\in\partial\tau_h}\int([\nabla u\cdot n\,]\{v\}+[v\,]\{\nabla u\cdot n\})\,ds+\sum_E(b\cdot\nabla u,v)$$
$$-\sum_{e\in\partial\tau_h}\int|b\cdot n|([u]\{v\}+\{u\}[v])\,ds=(f,v).$$

Since $u$ is continuous then $[u]$ and $[\nabla u\cdot n]=0$, we get

$$(u_t,v)+\sum_E a(\nabla u,\nabla v)-\sum_{e\in\partial\tau_h}\int[v\,]\{\nabla u\cdot n\}\,ds+\sum_E(b\cdot\nabla u,v)-\sum_{e\in\partial\tau_h}\int|b\cdot n|\{u\}[v]\,ds$$
$$=(f,v),$$

Note that, the left hand side of the above equation is still non-symmetric and non-positivity with respect to argument $u$ and $v$ [26], to rectify these properties, we add the terms

$$\varepsilon\sum_{e\in\partial\tau_h}\int[u\,]\{\nabla v\cdot n\}\,ds$$ and $$\sigma\sum_{e\in\partial\tau_h}\int[u][v]ds\,.$$

Then,

$$(u_t,v)+\sum_E a(\nabla u,\nabla v)-\sum_{e\in\partial\tau_h}\int(\{a\nabla u\cdot n\}[v]-\varepsilon\{a\nabla v\cdot n\}[u])\,ds+\sum_E(b\cdot\nabla u,v)$$
$$-\sum_{e\in\partial\tau_h}\int|b\cdot n|\{u\}[v]\,ds+\sigma\sum_{e\in\partial\tau_h}\int[u][\,v]ds=(f,v).$$

$$(u_t,v)+\sum_E a(\nabla u,\nabla v)-\sum_{e\in\partial\tau_h}\int(\{a\nabla u\cdot n\}[v]-\varepsilon\{a\nabla v\cdot n\}[u])\,ds+\sum_E(b\cdot\nabla u,v)$$
$$-\sum_{e\in\partial\tau_h}\int|b\cdot n|\{u\}[v]\,ds+\sigma\sum_{e\in\partial\tau_h}\int[u][\,v]ds=(f,v)+\sum_{e\in\Gamma_N}\int u_N\,v ds-\sum_{e\in\Gamma_D}\int\varepsilon a\nabla v\cdot n u_D\,ds+\sum_{e\in\Gamma_D}\int|b\cdot n|u_D v\,ds$$
$$-\sigma\sum_{e\in\Gamma_D}\int u_D v ds. \qquad (5.1)$$

Thus the weak form is: find $u \in V$ such that

$$(u_t, v) + A(u, v) = B(v), \qquad (5.2)$$

where,

$$A(u,v) = \sum_E a(\nabla u, \nabla v) - \sum_{e\in\partial\tau_h} \int (\{a\nabla u \cdot n\}[v] - \varepsilon\{a\nabla v \cdot n\}[u])\, ds + \sum_E (b\cdot\nabla u, v) - \sum_{e\in\partial\tau_h} \int |b\cdot n|\{u\}[v]\, ds + \sigma \sum_{e\in\partial\tau_h} \int [u][\, v] ds. \qquad (5.3)$$

And

$$B(v) = (f,v) + \sum_{e\in\Gamma_N} \int u_N\, v ds - \sum_{e\in\Gamma_D} \int \varepsilon a\nabla v\cdot n u_D\, ds + \sum_{e\in\Gamma_D} \int |b\cdot n| u_D v\, ds - \sigma \sum_{e\in\Gamma_D} \int u_D v ds.$$

The DGFE method: find $u_h \in V_h$ such that

$$(u_{h,t}, v) + A(u_h, v) = B(v), \qquad (5.5)$$

where,

$$A(u_h,v) = \sum_E a(\nabla u_h, \nabla v) - \sum_{e\in\partial\tau_h} \int (\{a\,\nabla u_h \cdot n\}[v] - \varepsilon\{a\nabla v \cdot n\}[u_h])\, ds + \sum_E (b\cdot\nabla u_h, v) - \sum_{e\in\partial\tau_h} \int |b \cdot n|\{u_h\}[v]\, ds + \sigma \sum_{e\in\partial\tau_h} \int [u_h][\, v] ds.$$

and

$$B(v) = (f,v) + \sum_{e\in\Gamma_N} \int u_N\, v ds - \sum_{e\in\Gamma_D} \int \varepsilon a\nabla v\cdot n u_D\, ds + \sum_{e\in\Gamma_D} \int |b\cdot n| u_D\, v\, ds - \sigma \sum_{e\in\Gamma_D} \int u_D v ds.$$

Where,

$$V_h = \{v \in L^2(\Omega);\, v|_E \in P^k(E), \forall E \in \tau_h\} \qquad (5.4)$$

and $P^k(E)$ = set of polynomials of degree at most $k$ on $E$ and $k \geq 1$ is an integer.

## 6. Properties of the bilinear form $A(u, v)$.

Let $V$ be Hilbert space with scalar product $(\cdot,\cdot)_V$, $(V = H^1(\Omega))$, and corresponding norm $\|u\|_{H^1(\Omega)}$. Suppose that $A(u, v)$ is bilinear form on $V \times V$. We prove the properties of the bilinear form (V- elliptic and continuous).

**Lemma 1**.(V$-$*elliptic*). Assume that the penalty value $\sigma$ is sufficiently large and that, $\beta_0 \geq (d-1)^{-1}$, there exist a positive constant $k$ independent of $h$ such that,

$$A(u,u) \geq k\|u\|^2_{H^1(E)},\ \forall u \in V.$$

**Proof:** Put $v = u$ in equation (5.3), we get

$$A(u,u) = \sum_E a(\nabla u, \nabla u) + (\varepsilon - 1)\sum_{e\in\partial\tau_h}\int\{a\nabla u \cdot n\}[u]\,ds + \sum_E (b\cdot\nabla u, u) - \sum_{e\in\partial\tau_h}\int |b\cdot n|\{u\}[u]\,ds + \sigma\sum_{e\in\partial\tau_h}\int [u]^2\,ds = \sum_{i=1}^{4} A^{(i)}. \qquad (6.1)$$

Define the energy norm,

$$A(u,u) = \left(\sum_E \left\|a^{\frac{1}{2}}\nabla u\right\|^2_{L^2(E)} + \sigma\sum_{e\in\partial\tau_h}\|[u]\|^2_{L^2(e)}\right)^{\frac{1}{2}} = \|u\|_{H^1(\tau_h)}.$$

To estimate $A^{(1)}$,

$$A^{(1)} = \sum_E a(\nabla u, \nabla u) = \sum_E \left\|a^{\frac{1}{2}}\nabla u\right\|^2_{L^2(E)}. \qquad (6.2)$$

for $A^{(2)}$, by Schwartz inequality we have,

$$A^{(2)} = (\varepsilon - 1)\sum_{e\in\partial\tau_h}\int\{a\nabla u\cdot n\}[u]ds \leq \sum_{e\in\partial\tau_h}\|\{a\nabla u\cdot n\}\|_{L^2(e)}\|[u]\|_{L^2(e)}$$

$$= (\varepsilon - 1)\sigma^{\frac{1}{2}}\sigma^{-\frac{1}{2}}\sum_{e\in\partial\tau_h}\|\{a\nabla u\cdot n\}\|_{L^2(e)}\|[u]\|_{L^2(e)}$$

$$= (\varepsilon - 1)\sigma^{-\frac{1}{2}}\sum_{e\in\partial\tau_h}\|\{a\nabla u\cdot n\}\|_{L^2(e)}\sigma^{\frac{1}{2}}\|[u]\|_{L^2(e)}.$$

Since, $\sigma^{-\frac{1}{2}}\|\{a\nabla u\cdot n\}\|_{L^2(e)} = \left|\frac{a}{2}\right|\sigma^{-\frac{1}{2}}\left\|(\nabla u\cdot n)_{E_1} + (\nabla u\cdot n)_{E_2}\right\|_{L^2(e)}$

$$\leq \left|\frac{a}{2}\right|\left(\frac{1}{h}\right)^{-\frac{1}{2}}\left(\left\|(\nabla u\cdot n)_{E_1}\right\|_{L^2(e)} + \left\|(\nabla u\cdot n)_{E_2}\right\|_{L^2(e)}\right).$$

Where, $E_1 = E^+$ and $E_2 = E^-$, from the trace inequality [25], we have

$$\leq \frac{|a|c_t}{2}h^{\frac{1}{2}}\left(h_{E_1}^{\frac{-1}{2}}\|\nabla u\|_{L^2(E_1)} + h_{E_2}^{\frac{-1}{2}}\|\nabla u\|_{L^2(E_2)}\right)$$

$$\leq \frac{|a|c_t}{2}\left(\|\nabla u\|_{L^2(E_1)} + \|\nabla u\|_{L^2(E_2)}\right)$$

$$\leq |a|c_t\|\nabla u\|_{L^2(E)} = c_t\|a\nabla u\|_{L^2(E)}.$$

Where $\forall e \in \partial E, |e| \leq h_E^{d-1} \leq h^{d-1}$ [27], and $c_t$ is a constant function [26]. then

$$A^{(2)} \leq (\varepsilon - 1) \sum_{e\in\partial\tau_h} c_t \|a\nabla u\|_{L^2(E)} \sigma^{\frac{1}{2}} \|[u]\|_{L^2(e)}.$$

Using young inequality we have,

$$A^{(2)} \leq \frac{\mu}{2} \sum_E \left\| a^{\frac{1}{2}} \nabla u \right\|^2_{L^2(E)} + \frac{{c_t}^2(\varepsilon-1)^2}{2\mu} \sum_{e\in\partial\tau_h} \sigma \, \|[u]\|^2{}_{L^2(e)}$$

$$\leq \gamma \left( \sum_E \left\| a^{\frac{1}{2}} \nabla u \right\|^2_{L^2(E)} + \sigma \sum_{e\in\partial\tau_h} \|[u]\|^2{}_{L^2(e)} \right) = \gamma \|u\|^2_{H^1(\mathrm{E})},$$

$$A^{(2)} \leq \gamma \|u\|^2_{H^1(\mathrm{E})}. \qquad (6.3)$$

To estimate $A^{(3)}$,

$$A^{(3)} = \sum_E (b\cdot\nabla u, u) - \sum_{e\in\partial\tau_h} \int |b\cdot n|\{u\}[u]\, ds = A^{(31)} + A^{(32)}. \qquad (6.4)$$

For $A^{(31)}$, by Schwartz and young inequality, we have

$$A^{(31)} = \sum_E \int b\cdot\nabla u u dx \leq \sum_E |b| \, \|\nabla u\|_{L^2(E)} \|u\|_{L^2(E)}$$

$$\leq \frac{\mu}{2} \|u\|^2_{L^2(E)} + \frac{b^2}{2\mu} \|\nabla u\|^2_{L^2(E)} \leq \gamma \left( \|u\|^2_{L^2(E)} + \|\nabla u\|^2_{L^2(E)} \right) = \gamma \|u\|^2_{H^1(E)},$$

$A^{(31)} \gamma \|u\|^2_{H^1(E)}$.

Where, $\gamma = \max\left\{\frac{\mu}{2}, \frac{{c_t}^2(\varepsilon-1)^2}{2\mu}, \frac{b^2}{2\mu}, \right\}$.

To estimate $A^{(32)}$,

$$A^{(32)} = -\sum_{e\in\partial\tau_h} \int |b\cdot n|[u]\{u\}\, ds \leq \sum_{e\in\partial\tau_h} |b\cdot n| \|[u]\|_{L^2(e)} \|\{u\}\|_{L^2(e)}$$

$$= \sigma^{\frac{1}{2}-\frac{1}{2}} \sum_{e\in\partial\tau_h} |b\cdot n| \|[u]\|_{L^2(e)} \|\{u\}\|_{L^2(e)}$$

$$= \sum_{e\in\partial\tau_h} |b\cdot n| \sigma^{\frac{1}{2}} \|[u]\|_{L^2(e)} \sigma^{-\frac{1}{2}} \|\{u\}\|_{L^2(e)}.$$

Since, $\sigma^{-\frac{1}{2}} \|\{u\}\|_{L^2(e)} \leq \frac{1}{2} h^{\frac{1}{2}} \left( \left\|(u)_{E_1}\right\|_{L^2(e)} + \left\|(u)_{E_2}\right\|_{L^2(e)} \right)$

from the trace and Poincare inequality we have,

$$A^{(32)} \leq \frac{1}{2} h^{\frac{1}{2}} C_t \left( h_{E_1}^{-\frac{1}{2}} \left\|(u)_{E_1}\right\|_{L^2(E_1)} + h_{E_2}^{-\frac{1}{2}} \left\|(u)_{E_2}\right\|_{L^2(E_2)} \right)$$

$$= \frac{1}{2} C_t \left( \left\|(u)_{E_1}\right\|_{L^2(\mathrm{E}_1)} + \left\|(u)_{E_2}\right\|_{L^2(\mathrm{E}_2)} \right)$$

$$\leq \frac{1}{2} C_t\left(\|u\|_{L^2(E)} + \|u\|_{L^2(E)}\right) = C_t\|u\|_{L^2(E)} \leq C_t\|u\|_{H^1(\mathrm{E})}.$$

Then,

$$A^{(32)} \leq \sum_{e\in\partial\tau_h} C_t|b\cdot n|\sigma^{\frac{1}{2}}\|[u]\|_{L^2(e)}\|u\|_{H^1(E)} \leq \delta\sum_{e\in\partial\tau_h}\sigma^{\frac{1}{2}}\|[u]\|_{L^2(e)}\|u\|_{H^1(\mathrm{E})}$$

$$\leq \frac{\mu}{2}\sum_{e\in\partial\tau_h}\sigma\|[u]\|^2{}_{L^2(e)} + \frac{\delta^2}{2\mu}\|u\|^2_{H^1(E)}. \qquad (6.6)$$

Where, $\delta \geq C_t|b\cdot n|$.

To estimate $A^{(4)}$,

$$A^{(4)} = \sigma\sum_{e\in\partial\tau_h}\int [u]^2\,ds \leq \sigma\sum_{e\in\partial\tau_h}\|[u]\|^2{}_{L^2(e)}. \qquad (6.7)$$

Substituting (6.2), (6.3), (6.4) and (6.7) in (6.1) we have,

$$A(u\,,u) = \sum_E\left\|a^{\frac{1}{2}}\nabla u\right\|^2_{L^2(E)} + \gamma\|u\|^2_{H^1(E)} + \sigma\sum_e\|[u]\|^2{}_{L^2(e)} + \gamma\|u\|^2_{H^1(E)} + \frac{\mu}{2}\sigma\sum_e\|[u]\|^2{}_{L^2(e)} + \frac{\delta^2}{2\mu}\|u\|^2_{H^1(E)}$$

$$= \sum_E\left\|a^{\frac{1}{2}}\nabla u\right\|^2_{L^2(E)} + \left(1+\frac{\mu}{2}\right)\sigma\sum_e\|[u]\|^2{}_{L^2(e)} + \left(2\gamma+\frac{\delta^2}{2\mu}\right)\|u\|^2_{H^1(E)}$$

$$\geq \alpha\left(\sum_E\left\|a^{\frac{1}{2}}\nabla u\right\|^2_{L^2(E)} + \sigma\sum_e\|[u]\|^2{}_{L^2(e)}\right) + \lambda\|u\|^2_{H^1(E)}$$

$$= (\alpha+\lambda)\|u\|^2_{H^1(E)} \geq \mathrm{k}\|u\|^2_{H^1(E)},$$

then

$$A(u\,,u) \geq \mathrm{k}\|u\|^2_{H^1(E)}. \qquad (6.8)$$

Where, $\alpha = min\left\{1,\left(1+\frac{\mu}{2}\right)\right\}$, $\lambda \leq \left(2\gamma+\frac{\delta^2}{2\mu}\right)$ and $\mathrm{k} \leq (\alpha+\lambda)$

**Lemma 2**.(*countinuity*), a bilinear form defined on $V$ space equipped with norm $\|\cdot\|_V$ is continuous if there is a positive constant $\zeta$ such that:

$$\forall\, u,v \in V = H^1(E),\quad A(u\,,v) \leq \zeta\|u\|_{H^1(E)}\|v\|_{H^1(E)}$$

$$A(u,v) = \sum_E a(\nabla u,\nabla v) + \sum_E(b\cdot\nabla u, v) - \sum_{e\in\partial\tau_h}\int\{a\nabla u\cdot n\}[v]ds$$

$$+\varepsilon \sum_{e\in\partial\tau_h} \int \{a\nabla v\cdot n\}[u]ds - \sum_{e\in\partial\tau_h} \int |b \cdot n|\{u\}[v]ds + \sigma \sum_{e\in\partial\tau_h} \int [u][\, v]\, ds$$

$$= \sum_{i=1}^{5} A^{(i)}. \qquad (6.9)$$

To estimate $A^{(1)}$,

$$A^{(1)} = \sum_{\mathrm{E}} \big(a\,(\nabla u\ ,\nabla v) + \ (b\cdot\nabla u\ ,v)\big)$$

$$\le \sum_{E} \big(|a|_{\mathrm{L}\infty} \|\nabla u\|_{L^2(E)} \|\nabla v\|_{L^2(E)} + |\mathrm{b}|_{\mathrm{L}\infty} \|\nabla u\|_{L^2(E)} \|v\|_{L^2(E)}\big)$$

$$\le c \sum_{E} \big(\|\nabla v\|_{L^2(E)} + \|v\|_{L^2(E)}\big) \|\nabla u\|_{L^2(E)}$$

$$\le c \sum_{E} \big(\|\nabla v\|_{L^2(E)} + \|v\|_{L^2(E)}\big) \big(\|\nabla u\|_{L^2(E)} + \|u\|_{L^2(E)}\big)$$

$$A^{(1)} \ = c \sum_{E} \|u\|_{H^1(E)} \|v\|_{H^1(E).} \ (6.10)$$

Where, $c = \max\{|a|_{L^\infty}, |b|_{L^\infty}\}$.

To estimate $A^{(2)}$,

$$A^{(2)} = \sum_{e\in\partial\tau_h} \int \{a\nabla u\cdot n\}[v]\, ds \le \sum_{e\in\partial\tau_h} \|\{a\nabla u \cdot n\}\|_{L^2(e)} \|[v]\|_{L^2(e)}$$

$$= \sigma^{\frac{1}{2}}\sigma^{-\frac{1}{2}} \sum_{e\in\partial\tau_h} \|\{a\nabla u \cdot n\}\|_{L^2(e)} \|[v]\|_{L^2(e)}$$

$$= \sigma^{-\frac{1}{2}} \sum_{e\in\partial\tau_h} \|\{a\nabla u \cdot n\}\|_{L^2(e)} \sigma^{\frac{1}{2}} \|[v]\|_{L^2(e)},$$

since, $\sigma^{-\frac{1}{2}}\|\{a\nabla u\cdot n\}\|_{L^2(e)}$

$$= \left|\frac{a}{2}\right| \sigma^{-\frac{1}{2}} \big\|(\nabla u\cdot n)_{E_1} + (\nabla u\cdot n)_{E_2}\big\|_{L^2(e)}$$

$$\le \left|\frac{a}{2}\right| \left(\frac{1}{h}\right)^{-\frac{1}{2}} \big\|(\nabla u \cdot n)_{E_1}\big\|_{L^2(e)} + \big\|(\nabla u\ \cdot n)_{E_2}\big\|_{L^2(e)}.$$

Fromthe trace inequality we have

$$\left|\frac{a}{2}\right| h^{\frac{1}{2}} \Big(\big\|(\nabla u\ \cdot n)_{E_1}\big\|_{L^2(e)} + \big\|(\nabla u\ \cdot n)_{E_2}\big\|_{L^2(e)}\Big) \le \frac{|a|C_t}{2} \big(\|\nabla u\|_{L^2(E_1)} + \|\nabla u\|_{L^2(E_2)}\big)$$

$$\le |a|C_t \|\nabla u\|_{L^2(E).} (6.11)$$

Similarly for

$$\sigma^{\frac{1}{2}}\|[v]\|_{L^2(e)} = \sigma^{1-\frac{1}{2}}\|[v]\|_{L^2(e)} = \sigma^{-\frac{1}{2}}\sigma\|[v]\|_{L^2(e)} \le \sigma C_t \|v\|_{L^2(E).} \qquad (6.12)$$

From (6.11) and (6.12), we get

$$A^{(2)} \le |a|\sigma {C_t}^2 \sum_{e\in\partial\tau_h} \|\nabla u\|_{L^2(E)} \, \|v\|_{L^2(E)}$$

$$\leq |a|\sigma {C_t}^2 \sum_{e\in\partial\tau_h} \left(\|\nabla u\|_{L^2(E)} + \|u\|_{L^2(E)}\right)\left(\|v\|_{L^2(E)} + \|\nabla v\|_{L^2(E)}\right)$$

$$= |a|\sigma {C_t}^2 \|u\|_{H^1(E)}\|v\|_{H^1(E).} \quad (6.13)$$

To estimate $A^{(3)}$,

$$A^{(3)} = \sum_{e\in\partial\tau_h} \int \{a\nabla u\cdot n\}[u]\, ds$$
$$\leq |a|\sigma {C_t}^2 \|u\|_{H^1(E)}\|v\|_{H^1(E).} \quad (6.14)$$

Similarly for $A^{(4)}$,

$$A^{(4)} = \sum_{e\in\partial\tau_h} \int |b\cdot n|\{u\}[v]ds$$
$$\leq \sigma {C_t}^2 \|u\|_{H^1(E)}\|v\|_{H^1(E).} \quad (6.15)$$

To estimate $A^{(5)}$,

$$A^{(5)} = \sigma \sum_{e\in\partial\tau_h} \int [u][v]ds$$
$$\leq \sigma \sum_{e\in\partial\tau_h} \|[u]\|_{L^2(e)}\|[v]\|_{L^2(e)}$$

$$= \sigma^2 \sum_{e\in\partial\tau_h} \sigma^{-\frac{1}{2}}\|[u]\|_{L^2(e)}\sigma^{-\frac{1}{2}}\|[v]\|_{L^2(e)},$$

since, $\sigma^{-\frac{1}{2}}\|[u]\|_{L^2(e)} \leq C_t\|u\|_{L^2(E)}$, and $\sigma^{-\frac{1}{2}}\|[v]\|_{L^2(e)} \leq C_t\|v\|_{L^2(E)}$,

then,

$$A^{(5)} \leq \sigma^2 {C_t}^2 \|u\|_{L^2(E)}\|v\|_{L^2(E)}$$
$$\leq \sigma^2 {C_t}^2 \|u\|_{H^1(E)}\|v\|_{H^1(E)}. \quad (6.16)$$

Substituting (6.10), (6.13), (6.14), (6.15) and (6.16) in (6.9), we have

$$A(u,v) = \sum_E a(\nabla u,\nabla v) - \sum_{e\in\partial\tau_h}\int\{a\nabla u\cdot n\}[v]\,ds + \varepsilon\sum_{e\in\partial\tau_h}\int\{a\nabla v\cdot n\}[u]\,ds$$
$$+ \sum_E (b\cdot\nabla u, v) - \sum_{e\in\partial\tau_h}\int|b\cdot n|\{u\}[v]\,ds + \sigma\sum_{e\in\partial\tau_h}\int[u]\,[v]ds$$

$$\leq c\|u\|_{H^1(E)}\|v\|_{H^1(E)} - |a|\sigma {C_t}^2\|u\|_{H^1(E)}\|v\|_{H^1(E)} + \varepsilon|a|\sigma {C_t}^2\|u\|_{H^1(E)}\|v\|_{H^1(E)}$$
$$-\sigma {C_t}^2\|u\|_{H^1(E)}\|v\|_{H^1(E)} + \sigma^2 {C_t}^2\|u\|_{H^1(E)}\|v\|_{H^1(E)}$$

$$= \left(c + |a|\sigma {C_t}^2(\varepsilon - 1) + \sigma {C_t}^2(\sigma - 1)\right)\|u\|_{H^1(E)}\|v\|_{H^1(E)}$$

$$\leq \zeta\|u\|_{H^1(E)}\|v\|_{H^1(E)},$$

then,

$$A(u,v) \leq \zeta\|u\|_{H^1(E)}\|v\|_{H^1(E).} \quad (6.17)$$

Where, $\zeta \geq \left(c + |a|\sigma {C_t}^2(\varepsilon - 1) + \sigma {C_t}^2(\sigma - 1)\right)$.

**Lemma3**.(*stability*): there exist a constant $\alpha > 0$ independent of $h$ such that

$$\|u_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \delta\int_0^T \|u_h\|^2_{H^1(E)} \le \alpha\left(\|u_0\|^2_{L^2(\Omega)} + \sum_E \|f\|^2_{L^2(0,T;L^2(E))}\right) + \alpha\left(\sum_{e\in\partial\tau_h} \left\||b\cdot n|^{\frac{1}{2}}u_D\right\|^2_{L^2(0,T;L^2(\Gamma_D))} + \sigma\sum_{e\in\partial\tau_h}\|u_D\|^2_{L^2(0,T;L^2(\Gamma_D))} + \sum_{e\in\partial\tau_h}\|u_N\|^2_{L^2(0,T;L^2(\Gamma_N))}\right)$$

Proof: choose $v = u_h$ in equation (5.2), we get

$$(u_{h,t}, u_h) + A(u_h, u_h) = \sum_E (f, u_h) - \varepsilon\sum_{e\in\Gamma_D}\int a\nabla u_h\cdot n\, u_D ds + \sum_{e\in\Gamma_N}\int u_N\, u_h ds + \sum_{e\in\Gamma_D}\int |b\cdot n| u_D u_h ds - \sum_{e\in\Gamma_D}\sigma\int u_D u_h\ ds = \sum_{i=1}^{5} I^{(i)}. \quad (6.18)$$

From lemma(1), we have

$$(u_{h,t}, u_h) + A(u_h, u_h) \ge \frac{1}{2}\frac{d}{dt}\|u_h\|^2_{L^2(\Omega)} + k\|u_h\|^2_{H^1(E)}. \quad (6.19)$$

Using young inequality, we get

$$I^{(1)} = (f, u_h) \le \frac{\mu}{2}\|f\|^2_{L^2(E)} + \frac{1}{2\mu}\|u_h\|^2_{L^2(E)} \le C\left(\|f\|^2_{L^2(E)} + \|u_h\|^2_{L^2(E)}\right). \quad (6.20)$$

To estimate $I^{(2)}$,

$$I^{(2)} = -\varepsilon\sum_{e\in\Gamma_D}\int a\nabla u_h\cdot n u_D ds \le \varepsilon\sum_{e\in\partial\tau_h}\|a\nabla u_h\cdot n\|_{L^2(e)}\|u_D\|_{L^2(e)} \le \sum_{e\in\partial\tau_h}\left(\frac{\mu}{2}\|\nabla u_h\|^2_{L^2(e)} + \frac{\varepsilon^2}{2\mu}\|u_D\|^2_{L^2(\Gamma_D)}\right)$$

$$= C\sum_{e\in\partial\tau_h}\left(\|u_h\|^2_{H^1(E)} + \|u_D\|^2_{L^2(\Gamma_D)}\right). \quad (6.21)$$

Similarly for $I^{(3)}$,

$$I^{(3)} = \sum_{e\in\Gamma_N}\int u_N u_h ds \le C\sum_{e\in\partial\tau_h}\left(\|u_h\|^2_{H^1(E)} + \|u_N\|^2_{L^2(\Gamma_N)}\right). \quad (6.22)$$

$$I^{(4)} = \sum_{e\in\Gamma_D} \int |b\cdot n| u_D u_h \, ds$$
$$\leq C \sum_{e\in\partial\tau_h} \Big( \|u_h\|^2_{H^1(E)}$$
$$+ \|u_D\|^2_{L^2(\Gamma_D)} \Big). \qquad (6.23)$$
$$I^{(5)} = -\sum_{e\in\Gamma_D} \sigma \int u_D u_h \, ds$$
$$\leq C \sum_{e\in\partial\tau_h} \Big( \|u_h\|^2_{H^1(E)}$$
$$+ \|u_D\|^2_{L^2(\Gamma_D)} \Big). \qquad (6.24)$$

Substituting (6.19), (6.20), (6.21), (6.22), (6.23) and (6.24) in (6.18), we have

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2_{L^2(\Omega)} + k\|u_h\|^2_{H^1(E)}$$
$$\leq C \sum_E \big( \|f\|^2_{L^2(E)}$$
$$+ \|u_h\|^2_{L^2(E)} \big)$$
$$+C \sum_{e\in\partial\tau_h} \Big( \|u_h\|^2_{H^1(E)}$$
$$+ \|u_D\|^2_{L^2(\Gamma_D)} \Big)$$
$$+C \sum_{e\in\partial\tau_h} \Big( \|u_h\|^2_{H^1(E)}$$
$$+ \|u_N\|^2_{L^2(\Gamma_N)} \Big)$$
$$+C \sum_{e\in\partial\tau_h} \Big( \|u_h\|^2_{H^1(E)}$$
$$+ \|u_D\|^2_{L^2(\Gamma_D)} \Big)$$
$$+C \sum_{e\in\partial\tau_h} \Big( \|u_h\|^2_{H^1(E)}$$
$$+ \|u_D\|^2_{L^2(\Gamma_D)} \Big).$$

Re-arrange the terms in the right hand side we have

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2_{L^2(\Omega)} + k\|u_h\|^2_{H^1(E)}$$
$$\leq C \sum_E \big( \|f\|^2_{L^2(E)}$$
$$+ \|u_h\|^2_{L^2(E)} \big)$$
$$+ 4C \sum_{e\in\partial\tau_h} \|u_h\|^2_{H^1(E)}$$
$$+C \sum_{e\in\partial\tau_h} \Big( \|u_N\|^2_{L^2(\Gamma_N)}$$
$$+ 3\|u_D\|^2_{L^2(\Gamma_D)} \Big).$$

By integrating both sides from 0to $t$ we have

$$\|u_h\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)}$$
$$+ k \int_0^t \|u_h\|^2_{H^1(E)}$$
$$\leq C \sum_{E\in\tau_h} \left( \int_0^t \|f\|^2_{L^2(E)} \right.$$
$$\left. + \int_0^t \|u_h\|^2_{L^2(E)} \right)$$
$$+4C \sum_{e\in\partial\tau_h} \int_0^t \|u_h\|^2_{H^1(E)}$$
$$+C \sum_{e\in\partial\tau_h} \left( \int_0^t \|u_N\|^2_{L^2(\Gamma_N)} \right.$$
$$\left. + 3\int_0^t \|u_D\|^2_{L^2(\Gamma_D)} \right)$$

Since $\int_0^t \|u_h\|^2_{L^2(E)} \leq \gamma$,

and $\gamma$ is a constant depends on time, hence

$$\|u_h\|^2_{L^2(\Omega)} + (k-4C)\int_0^t \|u_h\|^2_{H^1(E)} \le C\left(\|u_0\|^2_{L^2(\Omega)} + \sum_E \|f\|^2_{L^2\left(0,T;L^2(E)\right)} + \gamma\right)$$
$$+C\sum_{e\in\partial\tau_h}\left(\|u_N\|^2_{L^2\left(0,T;L^2(\Gamma_N)\right)} + 3\|u_D\|^2_{L^2\left(0,T;L^2(\Gamma_D)\right)}\right)$$

Re-arrange the last inequality, we get

$$\|u_h\|^2_{L^2(\Omega)} + \delta\int_0^t \|u_h\|^2_{H^1(E)} \le \alpha\left(\|u_0\|^2_{L^2(\Omega)} + \sum_E \|f\|^2_{\left(0,T;L^2(E)\right)}\right)$$
$$+\alpha\sum_{e\in\partial\tau_h}\left(\|u_N\|^2_{L^2\left(0,T;L^2(\Gamma_N)\right)} + \|u_D\|^2_{L^2\left(0,T;L^2(\Gamma_D)\right)}\right), \qquad (6.25)$$

where,

$C = max\left\{\frac{\mu}{2}, \frac{\varepsilon^2}{2\mu}\right\}$, $\delta \le (k-4C)$ and $\alpha = max\{C, 3C, \gamma C\}$.

## 7. The error estimate.

**Theorem (1)**: suppose that $u\epsilon L^2\left(0,T;H^1(E)\right)$ and that $u_0$ belongs to $H^1(\Omega)$ and let $\sigma$ sufficiently large then there exist a constant $C$ such that :

$$\|u-u^h\|_{L^2(\Omega)} \le Ch$$

**Proof:** let $\tilde{u}$be the$L^2$ projection, and$e = u-u^h = u-\tilde{u}+\tilde{u}-u^h = \rho-\theta$, then

$\|u-u^h\|_{L^2(\Omega)} \le \|u-\tilde{u}\|_{L^2(E)} + \|u^h-\tilde{u}\|_{L^2(E)} = \|\rho\|_{L^2(E)} + \|\theta\|_{L^2(E)}$,(7.1)

Since,[28]$\|\rho\|_{L^2(E)} \le ch^2\|u\|_{L^2(H^1).}$ (7.2)By subtracting (5.5) from (5.2) we have,

$$((u-u^h)_t, v) + A(u-u^h, v) = ((\rho-\theta)_t, v) + A(\rho-\theta, v) = 0$$

then$(\theta_t, v) + A(\theta, v) = (\rho_t, v) + a(\rho, v)$

for bound $\theta$

$(\theta_t, v) + A(\theta, v) = (\rho_t, v) + A(\rho, v)$ , let $v = \theta$, we have

$$(\theta_t, \theta) + A(\theta,\theta) = (\rho_t, \theta) + A(\rho, \theta). \qquad (7.3)$$

Since

$$(\theta_t, \theta) = \frac{1}{2}\frac{d}{dt}\|\theta\|^2{}_{L^2(\Omega)}. \qquad (7.4)$$

From lemma(1) we have,

$$(\theta_t, \theta) + A(\theta,\theta) \ge \frac{1}{2}\frac{d}{dt}\|\theta\|^2{}_{L^2(\Omega)} + k\|\theta\|^2{}_{L^2(E)}. \qquad (7.5)$$

By Schwartz and young inequality we have,

$$(\rho_t,\theta) \le \|\rho_t\|_{L^2(\Omega)}\|\theta\|_{L^2(E)} \le \frac{\epsilon}{2}\|\rho_t\|^2{}_{L^2(\Omega)} + \frac{1}{2\epsilon}\|\theta\|^2{}_{L^2(E)}. \quad (7.6)$$

Since

$$\|\rho_t\|^2{}_{L^2(\Omega)} = \int_0^T \|\rho_t\|^2\,dt \le ch^2\|u_t\|^2{}_{L^2(H^1)}.$$

Then,

$$(\rho_t,\theta) \le \frac{\epsilon}{2}\, ch^2\|u_t\|^2{}_{L^2(H^1)} + \frac{1}{2\epsilon}\|\theta\|^2{}_{L^2(E)}. \quad (7.7)$$

$$A(\rho,\theta) \le \beta\|\rho\|_{L^2(E)}\|\theta\|_{L^2(E)} \le \frac{\epsilon\beta_1}{2}\|\rho\|^2{}_{L^2(E)} + \frac{\beta_1}{2\epsilon}\|\theta\|^2{}_{L^2(E)}$$

$$\le \frac{\epsilon\beta_1}{2}ch^2\|u\|^2{}_{L^2(H^1)} + \frac{\beta_1}{2\epsilon}\|\theta\|^2{}_{L^2(E)}$$

Then,

$$A(\rho,\theta) \le \frac{\epsilon\beta_1}{2}ch^2\|u\|^2{}_{L^2(H^1)} + \frac{\beta_1}{2\epsilon}\|\theta\|^2{}_{L^2(E)}. \quad (7.8)$$

Substituting (7.5), (7.7) and (7.8) in (7.3) we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2{}_{L^2(\Omega)} + k\|\theta\|^2{}_{L^2(E)} \le \frac{\epsilon}{2}\, ch^2\|u_t\|^2{}_{L^2(H^1)} + \frac{1}{2\epsilon}\|\theta\|^2{}_{L^2(E)} + \frac{\epsilon\beta_1}{2}ch^2\|u\|^2{}_{L^2(H^1)} + \frac{\beta_1}{2\epsilon}\|\theta\|^2{}_{L^2(E)},$$

then,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2{}_{L^2(\Omega)} + (k - \frac{1}{2\epsilon} - \frac{\beta_1}{2\epsilon})\|\theta\|^2{}_{L^2(E)} \le \frac{\epsilon}{2}\, ch^2\|u_t\|^2{}_{L^2(H^1)} + \frac{\epsilon\beta_1}{2}ch^2\|u\|^2{}_{L^2(H^1)} \le \gamma ch^2\left(\|u_t\|^2{}_{L^2(H^1)} + \|u\|^2{}_{L^2(H^1)}\right).$$

Where$\gamma = max\left\{\frac{\epsilon}{2}, \frac{\epsilon\beta_1}{2}\right\}$,then

$$\frac{d}{dt}\|\theta\|^2{}_{L^2(\Omega)} + K\|\theta\|^2{}_{L^2(E)} \le \gamma ch^2\left(\|u_t\|^2{}_{L^2(H^1)} + \|u\|^2{}_{L^2(H^1)}\right).$$

Where$K \le \left(k - \frac{1}{2\epsilon} - \frac{\beta_1}{2\epsilon}\right)$.

By integrating both side in the last inequality from 0 to $t$, we have

$$\|\theta(t)\|^2{}_{L^2(\Omega)} - \|\theta^0\|^2{}_{L^2(\Omega)} \le \gamma ch^2 \int_0^t \left(\|u_t\|^2{}_{L^2(H^1)} + \|u\|^2{}_{L^2(H^1)}\right)$$

We know that,$\theta^0 = 0$, and from (7.2), we get

$$\leq ch^2\|u\|_{L^2(H^1)} + \gamma ch^2\left(\|u_t\|^2_{L^2\left(0,T;H^1(\Omega)\right)} + \|u\|^2_{L^2\left(0,T;H^1(E)\right)}\right)$$

$$\|u-u^h\|_{L^2(\Omega)} \leq Ch, \tag{7.9}$$

where $C \geq \sqrt{1+\gamma c}$.

## 8. Conclusion.

In this paper the two dimensional convection –diffusion problem is considered and solve it by DGFE method. There are three versions depend on the choices of the parameters $\varepsilon$and $\sigma$.

• If$\varepsilon = -1$ and $\sigma$is bounded below by a large enough constant, the resulting method is called the symmetric interior penalty Galerkin (SIPG) method.

•If$\varepsilon =$ +1 and $\sigma$ = 1, the resulting method is called the nonsymmetric interior penalty Galerkin (NIPG) method.

• If$\varepsilon$=0 and$\sigma > 0$ we obtain the incomplete interior penalty Galerkin (IIPG) method. We analyzed these versions and compare the numerical results with theexact solution in the future works.

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## تحليل الخطأ لطريقة كالركن غيرالمستمرة للعناصرالمحددة لمسألة الحمل والانتشار

### الخلاصة

في هذا البحث تم دراسة طريقة كالركن غير المستمرة للعناصر المحددة لمسألة الحمل و الانتشار ذات البعدين بتقطيع المجال مع بقاء الزمن في الحالة المستمرة حيث برهننا خصائص الثنائية الخطية $A(u,v)$ ( الاهليجية و الاستمراية) واثبتنا الحل التقريبي متقارب بنسبة خطأ من الرتبة $o(h)$.