SIMULTANEOUS APPROXIMATION BY A NEW SEQUENCE OF SZÃSZ –BETA TYPE OPERATORS

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ABSTRACT. In this paper, we study some direct results in simultaneous approximation for a new sequence of linear positive operators $M_n(f(t);x)$ of Szãsz - Beta type operators. First, we establish the basic pointwise convergence theorem and then proceed to discuss the Voronovaskaja-type asymptotic formula. Finally, we obtain an error estimate in terms of modulus of continuity of the function being approximated.

التقريب المتزامن بواسطة متتابعة مؤثرات جديدة من النمط
SZÃSZ–BETA
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المستخلص: في هذا البحث درسنا بعض النتائج المباشرة للتقريب المتزامن لمتتابعة جديدة من المؤثرات الخطية الموجبة $M_n(f(t);x)$ من لنمط Szãsz-Beta. أولا، أثبتنا مبرهنة التقارب النقطي ومن ثم ناقشنا الصيغة المقاربة لنمط Voronovaskaja. أخيرا، وجدنا الخطأ المخمن بصيغة مقياس التقارب للدالة المقربة.

1. INTRODUCTION

In [3] Gupta and others studied some direct results in simultaneous approximation for the sequence:

$$B_n(f(t);x)=\sum_{k=0}^{\infty} q_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\,f(t)\,dt\,,$$

where $x,t\in[0,\infty)$, $q_{n,k}(x)=\dfrac{e^{-nx}(nx)^k}{k!}$ and $\mathbf{b_{n,k}(t)}=\dfrac{\Gamma(\mathbf{n}+\mathbf{k}+1)}{\Gamma(\mathbf{n})\Gamma(\mathbf{k}+1)}\mathbf{t^k}(1+\mathbf{t})^{-(\mathbf{n}+\mathbf{k}+1)}$.

After that, Agrawal and Thamer [1] applied the technique of linear combination introduced by May [4] and Rathore [5] for the sequence $B_n(f(t);x)$. Recently, Gupta and Lupas [2] studied some direct results for a sequence of mixed Beta-Szãsz type operator defined as

$$L_n(f(t);x)=\sum_{k=1}^{\infty} b_{n,k}(x)\int_0^{\infty} q_{n,k-1}(t)\,f(t)\;dt+(1+x)^{-n-1}f(0)\,.$$

In this paper, we introduce a new sequence of linear positive operators $M_n(f(t);x)$ of Szãsz-Beta type operators to approximate a function $f(x)$ belongs to the space $C_\alpha[0,\infty)=\{f\in C[0,\infty):|f(t)|\le C(1+t)^\alpha \text{ for some } C>0,\alpha>0\}$, as follows:

$$M_n(f(t);x)=\sum_{k=1}^{\infty} q_{n,k}(x)\int_0^{\infty} b_{n,k-1}(t)\,f(t)\;dt+e^{-nx}f(0)\,, \tag{1}$$

We may also write the operator (1) as $M_n(f(t);x)=\int_0^{\infty} W_n(t,x)\,f(t)\,dt$ where

$W_n(t,x)=\sum_{k=1}^{\infty} q_{n,k}(x)\,b_{n,k-1}(t)+e^{-nx}\delta(t)$, $\delta(t)$ being the Dirac-delta function.

The space $C_\alpha[0,\infty)$ is normed by $\|f\|_{C_\alpha}=\sup_{0\le t<\infty}|f(t)|(1+t)^{-\alpha}$.

Throughout this paper, we assume that C denotes a positive constant not necessarily the same at all occurrence and $[\beta]$ denotes the integer part of β.

2. PRELIMINARY RESULTS

For $f\in C[0,\infty)$ the Szãsz operators is defined as $S_n(f;x)=\sum_{k=1}^{\infty} q_{n,k}(x)\,f\left(\dfrac{k}{n}\right)$, $x\in[0,\infty)$ and for $m\in N^0$ (the set of nonnegative integers), the $m-$th order moment of the Szãsz operators is defined as $\mu_{n,m}(x)=\sum_{k=0}^{\infty} q_{n,k}(x)\left(\dfrac{k}{n}-x\right)^m$.

LEMMA 2.1. [3] For $m\in N^0$, the function $\mu_{n,m}(x)$ defined above, has the following properties:

(i) $\mu_{n,0}(x)=1,\ \mu_{n,1}(x)=0$, and the recurrence relation is

$$n\mu_{n,m+1}(x)=x\left(\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right)\quad , m\geq 1;$$

(ii) $\mu_{n,m}(x)$ is a polynomial in x of degree at most $[m/2]$;

(iii) For every $x\in[0,\infty)$, $\mu_{n,m}(x)=O\left(n^{-[(m+1)/2]}\right)$.

From above lemma, we get

(2) $$\sum_{k=1}^{\infty}q_{n,k}(x)(k-nx)^{2j}=n^{2j}\left(\mu_{n,2j}(x)-(-x)^{2j}e^{-nx}\right)$$
$$=n^{2j}\left\{O(n^{-j})+O(n^{-s})\right\}\quad \text{(for any } s>0\text{)}$$
$$=O(n^{j})\ \text{ (if } s\geq j\text{)}.$$

For $m\in N^0$, the $m-$th order moment $T_{n,m}(x)$ for the operators (1) is defined as:

$$T_{n,m}(x)=M_n\left((t-x)^m;x\right)=\sum_{k=1}^{\infty}q_{n,k}(x)\int_0^{\infty}b_{n,k-1}(t)(t-x)^m\,dt+(-x)^m e^{-nx}.$$

LEMMA 2.2. For the function $T_{n,m}(x)$, we have $T_{n,0}(x)=1$, $T_{n,1}(x)=\dfrac{x}{n-1}$, $T_{n,2}(x)=\dfrac{nx^2+2nx+2x^2}{(n-1)(n-2)}$ and there holds the recurrence relation:

(3) $$(n-m-1)T_{n,m+1}(x)=xT'_{n,m}(x)+\left((2x+1)m+x\right)T_{n,m}(x)$$
$$+mx(x+2)T_{n,m-1}(x),\ n>m+1.$$

Further, we have the following consequences of $T_{n,m}(x)$:

(i) $T_{n,m}(x)$ is a polynomial in x of degree exactly m;

(ii) For every $x\in[0,\infty)$, $T_{n,m}(x)=O\left(n^{-[(m+1)/2]}\right)$.

Proof: By direct computation, we have $T_{n,0}(x)=1$, $T_{n,1}(x)=\dfrac{x}{n-1}$ and $T_{n,2}(x)=\dfrac{nx^2+2nx+2x^2}{(n-1)(n-2)}$. Next, we prove (3). For $x=0$ it clearly holds. For $x\in(0,\infty)$, we have

$$T'_{n,m}(x)=\sum_{k=1}^{\infty}q'_{n,k}(x)\int_0^{\infty}b_{n,k-1}(t)(t-x)^m\,dt-n(-x)^m e^{-nx}-mT_{n,m-1}(x).$$

Using the relations $xq'_{n,k}(x)=(k-nx)\,q_{n,k}(x)$ and $t(1+t)b'_{n,k}(t)=\left(k-(n+1)t\right)b_{n,k}(t)$, we get:

$$xT'_{n,m}(x)=\sum_{k=1}^{\infty}(k-nx)\,q_{n,k}(x)\int_0^{\infty}b_{n,k-1}(t)(t-x)^m\,dt+n(-x)^{m+1}e^{-nx}-mxT_{n,m-1}(x)$$

$$= \sum_{k=1}^{\infty} q_{n,k}(x) \int_0^{\infty} t(1+t)\, b'_{n,k-1}(t)(t-x)^m\, dt + (n+1)T_{n,m+1}(x) - (-x)^{m+1} e^{-nx}$$
$$+ (x+1)T_{n,m}(x) - (x+1)(-x)^m e^{-nx} - mxT_{n,m-1}(x) .$$

By using the identity $t(1+t) = (t-x)^2 + (1+2x)(t-x) + x(1+x)$, we have

$$xT'_{n,m}(x) = \sum_{k=1}^{\infty} q_{n,k}(x) \int_0^{\infty} b'_{n,k-1}(t)(t-x)^{m+2}\, dt + (1+2x)\sum_{k=1}^{\infty} q_{n,k}(x) \int_0^{\infty} b'_{n,k-1}(t)(t-x)^{m+1}\, dt$$
$$+ x(1+x)\sum_{k=1}^{\infty} q_{n,k}(x) \int_0^{\infty} b'_{n,k-1}(t)(t-x)^m\, dt + (n+1)T_{n,m+1}(x)$$
$$+ (1+x)T_{n,m}(x) - mxT_{n,m-1}(x) - (-x)^m e^{-nx} .$$

Integrating by parts, we get

$$xT'_{n,m}(x) = (n-m-1)\,T_{n,m+1}(x) - (m+x+2mx)\,T_{n,m}(x) - mx(x+2)\,T_{n,m-1}(x)$$

from which (3) is immediate.
From the values of $T_{n,0}(x)$ and $T_{n,1}(x)$, it is clear that the consequences (i) and (ii) hold for $m=0$ and $m=1$. By using (3) and the induction on m the proof of consequences (i) and (ii) follows, hence the details are omitted.

■

From the above lemma, we have

(4) $$\sum_{k=1}^{\infty} q_{n,k}(x) \int_0^{\infty} b_{n,k-1}(t)(t-x)^{2r}\, dt = T_{n,2r} - (-x)^{2r} e^{-nx}$$
$$= O(n^{-r}) + O(n^{-s}) \text{ (for any } s>0\text{)}$$
$$= O(n^{-r}) \text{ (if } s \ge r\text{)}.$$

LEMMA 2.3. Let δ and γ be any two positive real numbers and $[a,b] \subset (0,\infty)$. Then, for any $s>0$, we have

$$\left\| \int_{|t-x|\ge\delta} W_n(t,x)\, t^{\gamma}\, dt \right\|_{C[a,b]} = O(n^{-s}) .$$

Making use of Schwarz inequality for integration and then for summation and (4), the proof of the lemma easily follows.

LEMMA 2.4. [3] There exist polynomials $Q_{i,j,r}(x)$ independent of n and k such that

$$x^r D^r \; (q_{n,k}(x)) \; = \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i (k-nx)^j Q_{i,j,r}(x) q_{n,k}(x), \text{ where } D=\frac{d}{dx}.$$

3. MAIN RESULTS

Firstly, we show that the derivative $M_n^{(r)}(f(t);x)$ is an approximation process for $f^{(r)}(x)$, $r=1,2,\ldots.$

Theorem 3.1. If $r\in N$, $f\in C_\alpha[0,\infty)$ for some $\alpha>0$ and $f^{(r)}$ exists at a point $x\in(0,\infty)$, then

(5) $$\lim_{n\to\infty} M_n^{(r)}\big(f(t);x\big) = f^{(r)}(x).$$

Further, if $f^{(r)}$ exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty)$, $\eta>0$, then (5) holds uniformly in $[a,b]$.

Proof: By Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^r,$$

where, $\varepsilon(t,x)\to 0$ *as* $t\to x$. Hence

$$M_n^{(r)}\big(f(t);x\big)=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i\,dt+\int_0^\infty W_n^{(r)}(t,x)\,\varepsilon(t,x)\,(t-x)^r\,dt$$
$$:=I_1+I_2.$$

Now, using Lemma 2.2 we get that $M_n(t^m;x)$ is a polynomial in x of degree exactly m, for all $m\in N^0$. Further , we can write it as:

(6) $$M_n(t^m;x)=\frac{(n-m-1)!\,n^m}{(n-1)!}x^m+\frac{(n-m-1)!\,n^{m-1}}{(n-1)!}m(m-1)\,x^{m-1}+O(n^{-2}).$$

Therefore,

$$I_1=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\int_0^\infty W_n^{(r)}(t,x)\,t^j\,dt$$
$$=\frac{f^{(r)}(x)}{r!}\left(\frac{(n-r-1)!}{(n-1)!}n^r r!\right)=f^{(r)}(x)\left(\frac{(n-r-1)!}{(n-1)!}n^r\right)\to f^{(r)}(x) \text{ as } n\to\infty.$$

Next, making use of Lemma 2.4 we have

$$|I_2|\le \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i\frac{|Q_{i,j,r}(x)|}{x^r}\sum_{k=1}^{\infty}q_{n,k}(x)|k-nx|^j\int_0^\infty b_{n,k-1}(t)|\varepsilon(t,x)||t-x|^r\,dt+(nx)^r e^{-nx}|\varepsilon(0,x)|$$

$$:=I_3+I_4.$$

Since $\varepsilon(t,x)\to 0$ as $t\to x$, then for a given $\varepsilon>0$, there exists a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$, whenever $0<|t-x|<\delta$. For $|t-x|\geq\delta$, there exists a constant $C>0$ such that $\left|\varepsilon(t,x)(t-x)^r\right|\leq C|t-x|^{\gamma}$.

Now, since $\sup\limits_{\substack{2i+j\leq r\\ i,j\geq 0}}\dfrac{|Q_{i,j,r}(x)|}{x^r}:=M(x)=C\ \forall x\in(0,\infty)$. Hence,

$$I_3\leq C\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i\sum_{k=1}^{\infty}q_{n,k}(x)|k-nx|^j\left(\int\limits_{|t-x|<\delta}b_{n,k-1}(t)\,\varepsilon|t-x|^r\,dt+\int\limits_{|t-x|\geq\delta}b_{n,k-1}(t)|t-x|^{\gamma}\,dt\right)$$

$$:=I_5+I_6.$$

Now, applying Schwartz inequality for integration and then for summation, (2) and (4) we led to

$$I_5\leq\varepsilon C\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i\sum_{k=1}^{\infty}q_{n,k}(x)|k-nx|^j\left(\int_0^{\infty}b_{n,k-1}(t)\,dt\right)^{1/2}\left(\int_0^{\infty}b_{n,k-1}(t)(t-x)^{2r}\,dt\right)^{1/2}$$

$$\left(\text{since }\int_0^{\infty}b_{n,k-1}(t)\ dt=1\right)$$

$$\leq\varepsilon C\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i\left(\sum_{k=1}^{\infty}q_{n,k}(x)(k-nx)^{2j}\right)^{1/2}\left(\sum_{k=1}^{\infty}q_{n,k}(x)\int_0^{\infty}b_{n,k-1}(t)(t-x)^{2r}\,dt\right)^{1/2}$$

$$\leq\varepsilon C\,O(n^{-r/2})\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^iO(n^{j/2})=\varepsilon\,O(1).$$

Again using Schwarz inequality for integration and then for summation, in view of (2) and Lemma 2.3, we have

$$I_6\leq C\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i\sum_{k=1}^{\infty}q_{n,k}(x)|k-nx|^j\int\limits_{|t-x|\geq\delta}b_{n,k-1}(t)|t-x|^{\gamma}\,dt$$

$$\leq C\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i\sum_{k=1}^{\infty}q_{n,k}(x)|k-nx|^j\left(\int_0^{\infty}b_{n,k-1}(t)\,dt\right)^{1/2}\left(\int\limits_{|t-x|\geq\delta}b_{n,k-1}(t)(t-x)^{2\gamma}\,dt\right)^{1/2}$$

$$\leq C\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i\left(\sum_{k=1}^{\infty}q_{n,k}(x)(k-nx)^{2j}\right)^{1/2}\left(\sum_{k=1}^{\infty}q_{n,k}(x)\int\limits_{|t-x|\geq\delta}b_{n,k-1}(t)(t-x)^{2\gamma}\,dt\right)^{1/2}$$

$$\leq O(n^{-s})\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^iO(n^{j/2})\ (\text{for any } s>0)$$

$= O\left(n^{r/2-s}\right) = o(1)$ (for $s > r/2$).

Now, since $\varepsilon > 0$ is arbitrary, it follows that $I_3 = o(1)$. Also, $I_4 \to 0$ as $n \to \infty$ and hence $I_2 = o(1)$, combining the estimates of I_1 and I_2, we obtain (5).

To prove the uniformity assertion, it sufficient to remark that $\delta(\varepsilon)$ in above proof can be chosen to be independent of $x \in [a,b]$ and also that the other estimates holds uniformly in $[a,b]$.

■

Our next theorem is a Voronovaskaja-type asymptotic formula for the operators $M_n^{(r)}(f(t);x)$, $r = 1, 2, \ldots$.

THEOREM 3.2. Let $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$. If $f^{(r+2)}$ exists at a point $x \in (0,\infty)$, then

(7) $$\lim_{n\to 0} n\left(M_n^{(r)}(f(t);x) - f^{(r)}(x)\right) = \frac{r(r+1)}{2} f^{(r)}(x) + ((r+1)x + r) f^{(r+1)}(x) + \frac{1}{2}x(x+2) f^{(r+2)}(x).$$

Further, if $f^{(r+2)}$ exists and is continuous on the interval $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then (7) holds uniformly on $[a,b]$.

Proof: By the Taylor's expansion of $f(t)$, we get

$$M_n^{(r)}(f(t);x) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} M_n^{(r)}\left((t-x)^i;x\right) + M_n^{(r)}\left(\varepsilon(t,x)(t-x)^{r+2};x\right)$$
$$:= I_1 + I_2,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.

By Lemma 2.2 and (6), we have

$$I_1 = \sum_{i=r}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=r}^{i} \binom{i}{j} (-x)^{i-j} M_n^{(r)}(t^j;x)$$

$$= \frac{f^{(r)}(x)}{r!} M_n^{(r)}(t^r;x) + \frac{f^{(r+1)}(x)}{(r+1)!}\left((r+1)(-x) M_n^{(r)}(t^r;x) + M_n^{(r)}(t^{r+1};x)\right)$$
$$+ \frac{f^{(r+2)}(x)}{(r+2)!}\left(\frac{(r+2)(r+1)}{2} x^2 M_n^{(r)}(t^r;x) + (r+2)(-x) M_n^{(r)}(t^{r+1};x) + M_n^{(r)}(t^{r+2};x)\right)$$
$$= f^{(r)}(x)\left(\frac{(n-r-1)!\,n^r}{(n-1)!}\right)$$
$$+ \frac{f^{(r+1)}(x)}{(r+1)}\left\{(r+1)(-x)\left(\frac{(n-r-1)!\,n^r}{(n-1)!} r!\right)\right.$$

$$+\left(\frac{(n-r-2)!n^{r+1}}{(n-1)!}(r+1)!x+\frac{(n-r-2)!n^{r}}{(n-1)!}(r+1)r\,r!\right)\Bigg\}$$

$$+\frac{f^{(r+2)}(x)}{(r+2)!}\left\{\frac{(r+1)(r+2)}{2}x^2\left(\frac{(n-r-1)!n^{r}}{(n-1)!}r!\right)\right.$$

$$+(r+2)(-x)\left(\left(\frac{(n-r-2)!n^{r+1}}{(n-1)!}\right)(r+1)!x+\left(\frac{(n-r-2)!n^{r}}{(n-1)!}\right)(r+1)!r\right)$$

$$\left.+\left(\frac{(n-r-3)!n^{r+2}}{(n-1)!}.\frac{(r+2)!}{2}x^2+\frac{(n-r-3)!n^{r+1}}{(n-1)!}(r+2)(r+1)(r+1)!x\right)\right\}+O(n^{-2}).$$

Hence in order to prove (7) it suffices to show that $nI_2 \to 0$ as $n\to\infty$, which follows on proceeding along the lines of proof of $I_2 \to 0$ as $n\to\infty$ in Theorem 3.1. The uniformity assertion follows as in the proof of Theorem 3.1.
■

Finally, we present a theorem which gives as an estimate of the degree of approximation by $M_n^{(r)}(.;x)$ for smooth functions.

THEOREM 3.3. Let $f\in C_\alpha[0,\infty)$ for some $\alpha>0$ and $r\le q\le r+2$. If $f^{(q)}$ exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty),\eta>0$, then for sufficiently large n,

$$\left\|M_n^{(r)}\big(f(t);x\big)-f^{(r)}(x)\right\|_{C[a,b]}\le C_1 n^{-1}\sum_{i=r}^{q}\left\|f^{(i)}\right\|_{C[a,b]}$$
$$+C_2\,n^{-1/2}\omega_{f^{(q)}}\left(n^{-1/2}\right)+O(n^{-2})$$

where C_1,C_2 are constants independent of f and n, $\omega_f(\delta)$ is the modulus of continuity of f on $(a-\eta,b+\eta)$, and $\|.\|_{C[a,b]}$ denotes the sup-norm on $[a,b]$.

Proof. By Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)+h(t,x)(1-\chi(t)),$$

where ξ lies between t,x, and $\chi(t)$ is the characteristic function of the interval $(a-\eta,b+\eta)$. Now,

$$M_n^{(r)}\big(f(t);x\big)-f^{(r)}(x)=\left(\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i\,dt-f^{(r)}(x)\right)$$
$$+\int_0^\infty W_n^{(r)}(t,x)\left\{\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)\right\}dt+\int_0^\infty W_n^{(r)}(t,x)h(t,x)(1-\chi(t))\,dt$$
$$:=I_1+I_2+I_3.$$

By using Lemma 2.2 and (6), we get

$$I_1 = \sum_{i=r}^{q} \frac{f^{(i)}(x)}{i!} \sum_{j=r}^{i} \binom{i}{j} (-x)^{i-j} \frac{d^r}{dx^r} \left(\frac{(n-j-1)!\, n^j}{(n-1)!} x^j + \frac{(n-j-1)!\, n^{j-1}}{(n-1)!} j(j-1) x^{j-1} + O(n^{-2}) \right) - f^{(r)}(x).$$

Consequently,

$$\|I_1\|_{C[a,b]} \le C_1 n^{-1} \left(\sum_{i=r}^{q} \left\| f^{(i)} \right\|_{C[a,b]} \right) + O(n^{-2}), \text{ uniformly on } [a,b].$$

To estimate I_2 we proceed as follows:

$$\begin{aligned}
|I_2| &\le \int_0^\infty \left| W_n^{(r)}(t,x) \right| \left\{ \frac{\left| f^{(q)}(\xi) - f^{(q)}(x) \right|}{q!} |t-x|^q \chi(t) \right\} dt \\
&\le \frac{\omega_{f^{(q)}}(\delta)}{q!} \int_0^\infty \left| W_n^{(r)}(t,x) \right| \left(1 + \frac{|t-x|}{\delta} \right) |t-x|^q \, dt \\
&\le \frac{\omega_{f^{(q)}}(\delta)}{q!} \left[\sum_{k=1}^{\infty} \left| q_{n,k}^{(r)}(x) \right| \int_0^\infty b_{n,k-1}(t) \left(|t-x|^q + \delta^{-1} |t-x|^{q+1} \right) dt \right. \\
&\qquad \left. + (-n)^r e^{-nx} (x^q + \delta^{-1} x^{q+1}) \right], \delta > 0.
\end{aligned}$$

Now, for $s = 0, 1, 2, \ldots$, using Schwartz inequality for integration and then for summation, (2) and (4), we have

$$\begin{aligned}
(8) \qquad \sum_{k=1}^{\infty} q_{n,k}(x) |k-nx|^j \int_0^\infty b_{n,k-1}(t) |t-x|^s dt &\le \sum_{k=1}^{\infty} q_{n,k}(x) |k-nx|^j \left\{ \left(\int_0^\infty b_{n,k-1}(t) dt \right)^{1/2} \right. \\
&\qquad \left. \times \left(\int_0^\infty q_{n,k-1}(t)(t-x)^{2s} dt \right)^{1/2} \right\} \\
&\le \left(\sum_{k=1}^{\infty} q_{n,k}(x)(k-nx)^{2j} \right)^{1/2} \left(\sum_{k=1}^{\infty} q_{n,k}(x) \int_0^\infty b_{n,k-1}(t)(t-x)^{2s} dt \right)^{1/2} \\
&= O(n^{j/2}) O(n^{-s/2}) \\
&= O(n^{(j-s)/2}), \text{ uniformly on } [a,b].
\end{aligned}$$

Therefore, by Lemma 2.4 and (8), we get

$$\begin{aligned}
(9) \qquad \sum_{k=1}^{\infty} \left| q_{n,k}^{(r)}(x) \right| \left| \int_0^\infty b_{n,k-1}(t) |t-x|^s dt \right. &\le \sum_{k=1}^{\infty} \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i |k-nx|^j \frac{|Q_{i,j,r}(x)|}{x^r} q_{n,k}(x) \\
&\qquad \times \int_0^\infty b_{n,k-1}(t) \, |t-x|^s dt
\end{aligned}$$

$$\leq \left(\sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \sup_{x\in[a,b]} \frac{|Q_{i,j,r}(x)|}{x^r} \right) \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left(\sum_{k=1}^{\infty} q_{n,k}(x)|k-nx|^j \int_0^\infty b_{n,k-1}(t)|t-x|^s\,dt \right)$$

$$= C \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i O(n^{(j-s)/2}) = O(n^{(r-s)/2}) \text{ ,uniformly on } [a,b] \text{ .}$$

$$\left(\text{since } \sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \sup_{x\in[a,b]} \frac{|Q_{i,j,r}(x)|}{x^r} := M(x) \text{ but fixed }\right)$$

Choosing $\delta = n^{-1/2}$ and applying (9), we are led to

$$\|I_2\|_{C[a,b]} \leq \frac{\omega_{f^{(q)}}(n^{-1/2})}{q!}\left[O(n^{(r-q)/2}) + n^{1/2}O(n^{(r-q-1)/2}) + O(n^{-m})\right], \text{ (for any } m>0\text{)}$$

$$\leq C_2\, n^{-(r-q)/2}\omega_{f^{(q)}}(n^{-1/2}).$$

Since $t \in [0,\infty)\setminus(a-\eta, b+\eta)$, we can choose $\delta > 0$ in such a way that $|t-x| \geq \delta$ for all $x \in [a,b]$. Thus, by Lemmas 2.3 and 2.4 , we obtain

$$|I_3| \leq \sum_{k=1}^{\infty} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i |k-nx|^j \frac{|Q_{i,j,r}(x)|}{x^r} q_{n,k}(x) \int_{|t-x|\geq\delta} b_{n,k-1}(t)|h(t,x)|\,dt + (-n)^r e^{-nx}|h(0,x)|.$$

For $|t-x| \geq \delta$, we can find a constant C such that $|h(t,x)| \leq C|t-x|^\alpha$. Hence, using Schwarz inequality for integration and then for summation ,(2), (4), it easily follows that $I_3 = O(n^{-s})$ for any $s > 0$, uniformly on $[a,b]$.

Combining the estimates of $I_1, I_2\, I_3$, the required result is immediate.

■

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