Strongly FI-HOLLOW-LIFTING MODULES

Saad A. ALsaadi[1] and Nedal Q. Saaduon[2]

Department of mathematics, College of science, University of AL-Mustansiriya.[1 2]

Saadalsaadi08@ yahoo. com

Abstract

In this paper, we introduce and study the concept of strongly FI-hollow-lifting modules. As proper stronger concept of FI-hollow-lifting modules which is a proper generalization of strongly lifting modules. We say that an R-module M called strongly FI- hollow-lifting module if every fully invariant submodule N of M with M/N is hollow there exists a fully invariant direct summand K of M such that K coessential submodule N in M. Many characterizations and properties of strongly FI-hollow-lifting modules are given and the relation between this type of module and some other known of modules are discussed.

1.Introduction

Recall that an R-module M is a lifting if every submodule N of M contains a direct summand such that K coessential submodule N in M [1].Following (N.Orhan, D.keskin and R.Tribak introduced the concept hollow-lifting modules as a generalization of lifting modules). An R-module M is called Hollow –lifting if every submodule N of M such that M/N is hollow has coessential submodule that is a direct summand of M [2]. On other direction, Y.T.alebi and T.Amoozegar are introduced (strongly) FI-lifting modules as a generalization of lifting module. An R-module M is called (strongly) FI-Lifting if every fully invariant submodule N of M contains a (fully invariant) direct summand such that K coessential submodule N in M [3].Recently, FI-hollow-lifting modules introduced as a proper generalization of Hollow-lifting modules [4]. An R-module M is FI-hollow-lifting if every fully invariant submodule N of M such that M/N is hollow has coessential submodule that is a direct summand of M.Recall that an R-module is strongly lifting module if every submodule N of M contains a stable direct summand such that K coessential submodule N in M [5].

In this paper, we introduce and study the concept of strongly FI-hollow-lifting modules. As a stronger concept of hollow-lifting module and generalization of strongly lifting module. We say that is a strongly FI- hollow-lifting module if every fully invariant submodule N of M with M/N is hollow there exists a fully invariant direct summand K of M such that K coessential submodule N in M. Many characterizations and properties of strongly FI-hollow-lifting modules are given.

Throughout this paper R will denote arbitrary associative ring with identity and all R-modules are unitary left R-module, $N \subseteq M$ will mean N is a submodule of an R-module M. Let M be a module and N be a submodule of M. N is called a small submodule of M (denoted by $N \ll M$) if for any $X \subseteq M$, M=N+X implies X=M. An R- module M is called(FI-) hollow if every proper (fully invariant) submodule is small in M[6] (4). The module M is called local if has a unique maximal submodule N which contains all proper submodules of M. Let K, N be submodules of M such that $K \subseteq N \subseteq M$. Recall that K is called coessential submodule of N in M (briefly $K \subseteq_{ce} N$ in M) if $N/K \ll M/K$. A submodule N of M is called a coclosed submodule of M if N has no proper coessential submodule in M. If N and L are submodules of M, then N is called a supplement of L, if N + L = M and $N \cap L \ll N$. An R-module M is called supplemented module if every submodule of M is supplement.

Recall that a submodule K of M is fully invariant if $g(K) \subseteq K$ for all $g \in End(M)$. An R-module M is called duo if every submodule of M is fully invariant [7]. Moreover, a submodule of an R-module M is a called a stable if $f(N) \subseteq N$ for each homomorphism $f:N\rightarrow M$. An R-module is called fully stable if every submodule of M is stable[8].

2-Strongly FI-Hollow-lifting modules

As a proper stronger concept of hollow-lifting module. We introduce the following concept:

Definition(2-1): An R-module M is strongly FI-Hollow-lifting if for every fully invariant submodule N of M with $\frac{M}{N}$ is hollow, there exists a fully invariant direct summand K of M such that $K \subseteq_{ce} N$ in M.

Remarks and Examples (2-2):

1-Every strongly FI-hollow-Lifting module is FI-hollow-Lifting but the converse is not true in general. For example Z/2Z $\oplus Z/8Z$=M as Z-module is FI-hollow-Lifting [4].But M is not strongly FI-hollow-lifting module since N==(Z/2Z) $\oplus$ (4Z/8Z) is submodule of M which is not small in M and N does not contain any nonzero fully invariant direct summand of M.

2-Every FI-hollow (resp. hollow) module is strongly FI-hollow-lifting module. In fact, Suppose that M is FI-hollow and let A be fully invariant submodule of M. So A is small A=(0)+A with (0) is a fully invariant direct summand of M and A<< M .So by (pro.(2.5)) then M is strongly FI-hollow-lifting.

3-The class of hollow- Lifting modules and the class of strongly FI-hollow-lifting modules are different. In fact, the Z-module M=(Z/2Z) ⊕ (Z/4Z) is hollow-lifting but M is not strongly FI-hollow-lifting since N= (Z/2Z) ⊕ (2Z/4Z) is a submodule of M which is not small in M and N does not contain any non-zero fully invariant direct summand of M.

4-Every strongly (FI)-lifting is strongly FI-hollow-lifting. But the converse is not true in general.

5-If M Duo module then the following concept are equivalent:

1-M is strongly FI-hollow-lifting module.

2-M is FI-hollow-lifting module.

3-Hollow-lifting modules.

Now, we give some characterizations of strongly FI-hollow-lifting modules.

Theorem (2.3): An R-module M is strongly FI-hollow-lifting if and only if for every fully invariant submodule N of M with $\frac{M}{N}$ hollow, there exists a fully invariant direct summand K of N such that $M = K \oplus K^*$ and $N \cap K^* << K^*$.

Proof: Let N be a fully invariant submodule of M with $\frac{M}{N}$ hollow.Since M is strongly FI-hollow-lifting then there is a fully invariant direct summand K of M such that $K \subseteq_{ce} N$ in M and $M = K \oplus K^*$, where $K^* \subseteq M$. Let $(N \cap K^*) + X = K^*$, where X submodule K^*. So $M = K + K^* = K + (N \cap K^*) + X$. Now, $\frac{M}{K} = \frac{K+(N\cap K^*)}{K} + \frac{X+K}{K}$. But $K \subseteq_{ce} N$ in M and $K + (N \cap K^*) \subseteq N$. Therefore by proposition [1 ,p.20], $K \subseteq_{ce} (K + (N \cap K^*))$ in M and so $M = X + K$. Since $M = K \oplus K^*$ and $K \cap K^* = 0$, $X \subseteq K^*$ thus $K \cap X = 0$ and hence $M = K \oplus X$ this implies $X = K^*$ Thus $N \cap K^* << K^*$.

Conversely, let N be a fully invariant submodule of M such that $\frac{M}{N}$ is hollow, then by our assumption, there exists a fully invariant direct summand K of N such that $M = K \oplus K^*$and $N \cap K^* << K$.* Now, we want to show that $K \subseteq_{ce} N$ in M. Let $\frac{N}{K} + \frac{X}{K} = \frac{M}{K}$ where X is a submodule of M containing K, then $M = N + X$. By modular law, $N = N \cap M = N \cap (K \oplus K^*) = K \oplus (N \cap K^*)$, hence $M = N + X = K + (N \cap K^*) + X$. But $N \cap K^* << K^*$, therefore $N \cap K^* << M$. So M=K+X=X and hence $K \subseteq_{ce} N$ in M. Thus M is strongly FI-hollow-lifting. □

By the same manner of the proof of Theorem (2.3), we can give another characterization of strongly FI-hollow -lifting module.

Proposition(2.4): An R-module M is strongly FI-hollow-lifting if and only if for every fully invariant submodule N of M with $\frac{M}{N}$ hollow, there exists a fully invariant direct summand K of N such that $M = K \oplus K^*$ and $N \cap K^* << M$. □

Recall that an R-module M is lifting if and only if every submodule N of M can written in the form N=A ⊕ S where A is a direct summand of M and S << M[1].We have analogous result for strongly FI-hollow -Lifting modules.

Theorem (2.5): The following statement are equivalent for an R-module M:

1- M is strongly FI- hollow-lifting .

2-Every fully invariant submodule N of M such that M/N is hollow, can be written as N=K ⊕ L with K is a fully invariant direct summand of M and L<< M .

3-Every fully invariant submodule N of M such that M/N hollow , there exists a fully invariant direct summand K of M such that N=K +L and L << M .

Proof: (1⇒2) Let N be a fully invariant submodule of M such that $\frac{M}{N}$ hollow. Since M is strongly FI-hollow-lifting, there exists a fully invariant direct summand K of M such that $K \subseteq_{ce} N$ in M and $= K \oplus K^*$, where $K^* \subseteq M$. By modular law, $N = N \cap M = N \cap (K \oplus K^*) = K \oplus (N \cap K^*)$. We want to show that $N \cap K^* << K^*$. Let $X \subseteq K^*$ with $(N \cap K^*) + X = K^*$, then $N + X = M$. Now, $\frac{M}{K} = \frac{N+K}{K} = \frac{N}{K} + \frac{X+K}{K}$. Since $K \subseteq_{ce} N$ in M, then $M = X + K$. But $M = K \oplus K^*$ and $X \subseteq K^*$, therefore $X = K^*$. Let $L = N \cap K^*$. Thus N=K⊕L with K is a fully invariant direct summand of M and L<<M.

(2⇒3): It is obvious

(3⇒1): let N be a fully invariant submodule of M with $\frac{M}{N}$ hollow. Then by our assumption $N = K + L$, where K is a fully invariant direct summand of M and $L << M$ such that $M = K \oplus K^*$, for some $K^* \subseteq M$. Since K^* is a supplement of K in M, and since $L << M$, then by [11 , p.348] K^* is a supplement of $K + L = N$ in M. So $N \cap K^* << K^*$. Thus by Theorem (2.3), M is strongly FI-hollow-lifting.

Since by [13,Lemma(2.1.6)], every fully invariant direct summand are stable so we can rewrite all results in this paper with " fully invariant direct summand " being replaced by "stable direct summand " for example, we can restate theorem (2.5).

Proposition (2.6): The following statement are equivalent for an R-module M:

1- M is strongly FI- hollow-lifting .

2-Every fully invariant submodule N of M such that M/N is hollow, can be written as N=K ⊕ L with K is a stable direct summand of M and L$<< M$.

3-Every fully invariant submodule N of M such that M/N hollow , there exists a stable direct summand K of M such that N=K +L and L $<< M$. □

It is well-known that, if M is lifting module then every coclosed submodule of M is a direct summand[1]. For strongly FI-hollow-lifting we have the following.

Proposition (2.7): Let M be strongly FI-hollow-lifting module. Then every fully invariant coclosed submodule K of M with $\frac{M}{K}$ hollow is a direct summand of M.

Proof: Let K be a fully invariant coclosed submodule of M such that $\frac{M}{K}$ is hollow. Since M is strongly FI-hollow-lifting, then there is a fully invariant direct summand N of M such that N $\subseteq_{ce} K$ in M Since K is a coclosed submodule of M , then N=K. So K is a direct summand of M.
□

Following [4], a finite direct sum of FI- hollow-lifting module is FI-hollow-lifting. But we can see that a direct sum of strongly FI-hollow-lifting need not be strongly FI-hollow-lifting For example, Z-module Z_p and Z_{p3} are strongly FI-hollow-lifting (where p is a prime number). Since Z_p and Z_{p3} are hollow (see (Remarks (2.2), (2)).Then $Z_p \oplus Z_{p3}$ is not strongly FI-hollow-lifting as Z-module. Now, we give a condition under which a direct sum of strongly FI-hollow-lifting modules is strongly FI-hollow-lifting.

Proposition (2.8): Let $M = \oplus_{i=1}^{n} M_i$ where M_i is fully invariant submodule of M. If M_i is strongly FI-hollow-lifting, then M is strongly FI-hollow-lifting.

proof :suppose $M = \oplus_{i=1}^{n} M_i$ where M_i is fully invariant submodule of M. Let N be fully invariant submodule of M such that $\frac{M}{N}$ is hollow module. Since $\frac{M_1+N}{N} + \ldots + \frac{M_n+N}{N} = \frac{M}{N}$ there exist $i \in \{1,..n\}$ such that $\frac{M}{N} = \frac{M_i+N}{N} \cong \frac{M_i}{M_i \cap N}$. Thus

$\frac{Mi}{Mi\cap N}$ is hollow since N is fully invariant submodule of M then $N=\oplus_{i=1}^{n} N\cap M_i$ and $N\cap M_i$ fully invariant of M_i. Now, since M_i is strongly FI-hollow-lifting then $N\cap M_i = B_i \oplus S_i$ where B_i is fully invariant direct summand of M_i and $S_i \ll M_i$. Now since B_i is fully invariant of M_i for all $1\leq i \leq n$ and M_i is fully invariant submodule of M then B_i fully invariant submodule of M. Also B_i is a direct summand of M_i and M_i a direct summand of M then B_i a direct summand of M. New, let $B=\oplus_{i=1}^{n} B_i$ and $S=\oplus_{i=1}^{n} S_i$. But $S=\oplus_{i=1}^{n} S_i \ll M$ since (a finite sum of small is a small)[9], then $N=B\oplus S$ where B is fully invariant direct summand of M and $S \ll M$. Hence M is strongly FI-hollow-lifting. □

Proposition (2.9): Let M be a strongly FI- hollow-lifting module. If $M = X + Y$, where Y is a fully invariant direct summand of M and X is a fully invariant submodule of M with $\frac{M}{X\cap Y}$ is hollow, then Y contains a supplement submodule of X in M.

Proof: Let M be a strongly FI-hollow-lifting and M=X+Y, where Y is a fully invariant direct summand of M. Since the intersection of two fully invariant submodule is fully invariant [3]. Since M is strongly FI-hollow-lifting, then by Theorem (2.4). New, $X\cap Y = N\oplus S$, where N is a fully invariant direct summansd of M and $S<<M$. But Y is a fully invariant direct summand of M and $S\subseteq Y$, therefore by[9] $S<<Y$. Let N^* be a submodule of M such that $M=N\oplus N^*$.Thus by modular low, $Y = Y\cap M = Y\cap(N\oplus N^*)= N\oplus (Y\cap N^*)$. Let $N_1=Y\cap N^*$, this implies that $M = X+Y = X+N+N_1= X+N_1$. We want to show that N_1 is a supplement of X in M. By modular law, $X\cap Y = X\cap Y\cap (N\oplus N_1)=N\oplus (X\cap N_1)$.

Let $\pi 1 : N\oplus N_1 \rightarrow N_1$ be the natural projection map. So we have, $X\cap N_1 = \pi 1(N \oplus (X\cap N_1)) = \pi 1(X\cap Y) = \pi 1(N\oplus S) = \pi 1(S)$.Since $S << Y= N\oplus N_1$,then by[10], $\pi 1(S) << N_1$and hence $X\cap N_1 << N_1$. Thus N_1 is a supplement of X in M and N_1 is contained in Y. □

Let M be an R-module. Recall that an R-module P is called Projective if for any epimorphism φ : M→ N and for any homomorphism f: $P\rightarrow$N there is homomorphism h: $P\rightarrow M$ such that f= φh. Also an R-module P is called projective cover of M if, P is projective and there exists an epimorphism $\varphi : P\rightarrow M$ with $ker\varphi << P$[10].

It well known that is not be every module has Projective cover. We give a conditions under a quotient of strongly FI-hollow-lifting module to have Projective cover.

Proposition(2.10): Let M be a projective strongly FI-hollow-lifting module then For every fully invariant submodule N of M such that $\frac{M}{N}$ is hollow, $\frac{M}{N}$ has a projective cover.

Proof: Let N be a fully invariant submodule N of M such that $\frac{M}{N}$ is hollow. Since M is strongly FI-hollow-lifting module, then by Theorem (2.3), there exists a submodule K of N such that $M=K\oplus K^*$, for some $K^*\subseteq M$ and $N\cap K^*<<K^*$.

Now, consider the following two short exact sequences:

$$0 \rightarrow N \overset{i_1}{\rightarrow} N+K^* \overset{T_1}{\rightarrow} \frac{N+K^*}{N} \rightarrow 0$$

$$0 \rightarrow N\cap K^* \overset{i_2}{\rightarrow} K^* \overset{T_2}{\rightarrow} \frac{K^*}{N\cap K^*} \rightarrow 0$$

Where i_1, i_2 are the inclusion maps and T_1, T_2 are the natural epimorphism. By the (second isomorphism theorem), $\frac{M}{N} = \frac{N+K^*}{N} \cong \frac{K^*}{N\cap K^*}$. Since M is a projective and K^* is a fully invariant direct summand of M, then K^* is a projective. But $ker\, T_2 = N\cap K^* << K^*$, therefore K^* is a projective cover of $\frac{K^*}{N\cap K^*}$. Since $\frac{M}{N} \cong \frac{K^*}{N\cap K^*}$, thus $\frac{M}{N}$ has a projective cover. □

We assert that every strongly FI-hollow-lifting module is FI-hollow-lifting but the converse is not true in general.

Proposition(2.11):

If an R-module M is FI-hollow-lifting SS-module then M is strongly FI-hollow-lifting

Proof: Let N be a fully invariant sub module of M with $\frac{M}{N}$ is hollow.Since M is FI-hollow-lifting there exits direct summand D of M where D $\subseteq_{ce}$N. But M is SS-module hence D is fully invariant sub module. So M is strongly FI-hollow- lifting. □

Remark: The concepts of strongly FI-hollow-lifting modules and SS-modules are different. For example, Z as Z-module is SS-module which is not strongly FI-hollow-lifting. In other hand M= $Z_{p^\infty} \oplus Z_{p^\infty}$ is strongly FI- hollow-lifting[12],

which is not SS-module [13, remark and example (2.2.9)].

Corollary(2.12):

If an R-module M is FI-hollow-lifting fully stable then M is strongly FI-hollow-lifting

Corollary(2.13): If an R-module M is FI-hollow-lifting indecomposable module then M is strongly FI-hollow-lifting.

Corollary(2.14): A commutative ring R is FI-hollow-lifting then R is strongly FI-hollow-lifting.

Lemma(2.15): If R-module M is strongly FI-hollow-lifting, then every fully invariant submodule N of M with $\frac{M}{N}$ hollow has supplement K^* in M and N$\cap$ K^* is a direct summand in N.

Proof : Suppose that M is a strongly FI- hollow-lifting module and N is a fully invariant submodule of M such that $\frac{M}{N}$ is hollow. Then there is a fully invariant direct summand submodu K of N in M such that $K \subseteq_{ce} N$ in M and M=K⊕K^*, for some $K^* \subseteq M$. By modular law, N=N∩M= $N \cap (K \oplus K^*) = K \oplus (N \cap K^*)$. One can easily show that $M = N + K^*$. We want to show that $N \cap K^* << K^*$. Let $(N \cap K^*) + X = K^*$, where $X \subseteq K^*$ So $M = K + K^*$=$K + (N \cap K^*) + X$. This implies that $M = N + X$ and $\frac{M}{K} = \frac{N+X}{K} = \frac{N}{K} + \frac{X+K}{K}$. Since $K \subseteq_{ce} N$ in M, then M=$X + K$. But $M = K \oplus K^*$ and $X \subseteq K^*$, therefore $X = K^*$ and hence $N \cap K^* << K^*$. Thus N has a supplement fully invariant K^* in M and N∩ K^* is a direct summand in N. □

Finally we obtained another characterization of strongly FI-hollow-lifting.

Theorem (2.16): An R-module M is strongly FI-hollow-lifting if and only if for every fully invariant submodule N of M with $\frac{M}{N}$ hollow, there exists an idempotent e ∈ End(M) such that e(M) is fully invariant submodule of M with e(M)⊆ N and (I-e)(N)<< (I-e)(M).

Proof: Let N be a fully invariant submodule of M such that $\frac{\mathrm{M}}{\mathrm{N}}$ hollow. Since M strongly is FI-hollow-lifting, then by Proposition(2.4), there is a decomposition M=X ⊕K such that X ⊆ N with X is fully invariant submodule of M and N ∩ X $<< M$. Now, let e : M = X ⊕K →X be a Projection mapping. Thus, it easy check that e is an idempotent and e(X) = X ⊆ N. Also (I-e)(M)=K. since N ∩ X $<< M$ and K is a direct summand of M, then N ∩ K $<< K$ [9].Since X ⊆ N, then e(M) ⊆ N. Now, (I-e)(M) = {(I-e)(m), m∈M} = {(I-e) (a+b), where a ∈ X , b ∈ K} ={(I-e)(a+b)=a+b-a=b}= K.

We want show that (I-e)(N)= N ∩(I-e)(M). Let x ∈(I-e)(N), then there is n ∈ N such that x=(I-e)(n)=n-e(n). Thus x ∈ N and x ∈(I-e)(M). So x ∈ N ∩(I-e)(M). Hence, (I-e)(N)⊆N ∩(I-e)(M). Let d ∈ N ∩ (I-e)(M), then d ∈ N and d ∈(I-e)(M). There is y ∈ M such that d =(I-e)(y) = y-e(y). Thus d+e(y)=y ∈ N, then d ∈(I-e)(N). So (I-e)(N)=N ∩(I-e)(M)= N ∩ K

<< K. Hence, (I-e)(N) << (I-e)(M).

Conversely, let N be a fully invariant submodule of M such that $\frac{M}{N}$ is hollow. By our assumption there exists an idempotent e ∈ End(M) such that e(M) is fully invariant submodule of M with e(M) ⊆ N and (I-e)(N)<< (I-e)(M). We Claim that M=e(M) ⊕(I-e)(M). To show that, let m ∈ M then m = m+e(m)-e(m)=e(m)+m-e(m)=e(m)+ (I-e)(m). Thus M=e(M) + (I- e)(M). Now, let w ∈ e(M) ∩ (I-e)(M) , then w = e(m_1) and w=(I-e)(m_2), for some m_1, m_2 ∈ M. So e(w)=e(m_1)=e((I-e)(m_2))=e(m_2)-e(m_2) = 0, then e(e(m_1)) = e(m_1) = 0, hence w = 0. Thus M=e(M) ⊕ (I-e)(M). Clearly, N ∩ (I- e)(M)=(I-e)(N).Since (I-e)(N) << (I-e)(M), then N ∩ (I-e)(M) << (I-e)(M), thus M is strongly FI-hollow-lifting. □

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اقوى مقاسات رفع مجوفه من النوع- FI

سعد عبد الكاظم الساعدي و نضال قاسم سعدون

الخلاصة

في هذا البحث، تم تقديم مفهوم مقاسات الرفع المجوف القوي من النمط FI . نقول عن M انه مقاس رفع مجوّف القوي من النمط FI إذا كان لكُلّ مقاس جزئي ثابت N من M بحيث ان $\frac{M}{N}$ مقاس مجوف، يوجد حد مباشر ثابت K من M بحيث $K \subseteq_{ce} N$ في M . تم أعطاء عدد من التشخيصات والخواص المختلفة للمقاسات الرفع المجوف القوي من النمط FI . وناقشنا العلاقة بين هذا الصنف من المقاسات وبعض المقاسات الاخرى المعرفه .