Classification of the Projective Line Over Galois Field of Order Sixteen

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الخلاصة

الهدف الرئيسي من هذا البحث هو تصنيف تشكيل هندسي معين يدعى k-set وبالوضع الخاص يدعى خط الإسقاط من الرتبة السادسة عشر $PG(1,q^h), q = 2, h = 4$. المجموعات الجزئية من الخط $PG(1,16)$ والتي هي tetrads, pentads, hexads, heptads, octads تم تصنيفها. الاداوات الأساسية هي النظرية الأساسية لخط الإسقاط الهندسي . يوجد إسقاطي وحيد لخط الإسقاط ينقل ثلاث نقاط إلى إي ثلاث نقاط أخرى . كل هذه k-set تعطي تصحيح لرفع اكبر خطا الشفرة والتي تصحح اكبر عدد ممكن من الأخطاء لطول الشفرة
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ABSTRACT

The main purpose of this paper is to classify certain geometric structure, called k-set, in a particular setting, namely the projective line of order sixteen $\boldsymbol{PG(1,q^h)}$, $\boldsymbol{q=2,h=4}$. The subsets of the line $\boldsymbol{PG(1,16)}$, such the tetrads, the pentads, the hexads, the heptads and the octads are classified. The basic tool is the fundamental theorem of projective geometry; there is a unique projectivity of the projective line transforming three points to any three other points. Each of these k-sets gives rise to an error-correcting code that corrects the maximum possible number of errors for its lengths.

INTRODUCTION

On $PG(1,q)$, a $(k;1)$-arc is just an unordered set of k distinct points simply called a k-set which special case of a $(k;2)$-arc that is a set of k points no three of which are collinear . A 3-set is called a triad, a 4-set a tetrad, a 5-set a pentad, a 6-set a hexad, a 7-set a heptad, an 8-set a octad, a 9-set a nonad. A $(k;2)$-arc in projective plane $PG(2,q)$ is a set of k points no three of which are collinear.

k-sets in $PG(1,q)$ for $q = 2,3,4,5,7,8,9,11,13,17,19$ have been classified; see [1,2,3]. We are looking at the line of order sixteen, as it is the next in the sequence.

We answer the equestion: How many projectively inequivalent k-sets in $PG(1,q)$ are there and what is the stabilizer group of each one?

Associated to any topic in mathematics is its history. Arcs were first introduced by Bose (1947) in connection with designs in statistics. Further development began with Segre in (1954); he showed that every $(q+1)$-arc in $PG(2,q)$ is a conic. An important result is that of Ball, Blokhuis and Mazzocca showed that maximal arcs cannot exist in a plane of odd order. In 1981 Goppa found important applications of curves over finite fields to coding theory. As to geometry over a finite field, it has been thoroughly studied in the major treatise of Hirschfeld 1979-1985 and of Hirschfeld and Thas 1991) and Hirschfeld, G. Korchm$\acute{a}$ros and F. Torres (2007).

The 17 points of $PG(1,16)$ are $P(x_0,x_1)$, $x_i \in \mathbb{F}_{16}$. So

$$PG(1,16) = \{U_0 = P(1,0)\} \cup \{P(x,1) | x \in \mathbb{F}_{16}\}.$$

Each point $P(x_0, x_1)$ with $x_1 \neq 0$ is determined by the non-homogeneous coordinate x_0/x_1; the coordinate for U_0 is ∞. Then, with $\mathbb{F}_{16} \cup \{\infty\}$, each point of $PG(1,16)$ is represented by a single element of $\mathbb{F}_{16} \cup \{\infty\}$. Thus

$$PG(1,16) = \{P(t,1) | t \in \mathbb{F}_{16} \cup \{\infty\}\};$$

Here, $P(\infty, 1) = P(1,0)$. A projectivity $\xi = M(T)$ of $PG(1,16)$ is given by $Y = XT$, where $X = (x_0, x_1)$, $Y = (y_0, y_1)$ and

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $s = y_0/y_1$ and $t = x_0/x_1$; then $s = (at + c)/(bt + d)$. If $Q_i = P_i\xi$ for $i = 2,3,4$ and P_i and Q_i have the respective coordinate t_i and s_i, then ξ is given by

$$\frac{(s - s_3)(s_2 - s_4)}{(s - s_4)(s_2 - s_3)} = \frac{(t - t_3)(t_2 - t_4)}{(t - t_4)(t_2 - t_3)}$$

PREVIOUS RESULTS

Definition(1)[3]: A *finite field* is a field with only a finite number of elements. The characteristic of a finite field K is the least positive integer p, and hence a prime, such that

$$pz = \underbrace{z + \cdots + z}_{p} = 0 \text{ For all } z \in K.$$

Definition(2)[3]: The set denoted by $\mathbb{F}_p$, with P prime, consists of the residue classes of the integers modulo P under the natural addition and multiplication.

Definition(3)[3]: Let S and S^* be two spaces of $PG(n, K)$, A *projectivity* $\beta: S \to S^*$ is a bijection given by a matrix T, necessarily non-singular, where $P(X^*) = P(X)\beta$ if $tX^* = XT$, with $t \in K - \{0\}$. Write $\beta = M(T)$; then $\beta = M(\lambda T)$ for any λ in $K - \{0\}$. The group of projectivities of $PG(n, K)$ is denoted by $PGL(n + 1, K)$.

Definition(4)[2]: A group G acts on a set Λ if there is a map $\Lambda \times G \to \Lambda$ such that given g, g' elements in G and 1 its identity, then

1. $x1 = x$,
2. $(x\mathrm{g})\mathrm{g}' = x(\mathrm{g}\mathrm{g}')$ for any x in Λ.

Definition(5)[2]: The orbit of x in Λ under the action of G is the set

$$x\mathrm{G} = \{x\mathrm{g} | \mathrm{g} \in \mathrm{G}\}.$$

Definition(6)[2]: The stabilizer of x in Λ under the action of G is the group

$$\mathrm{G}_x = \{\mathrm{g} \in G | x\mathrm{g} = x\}.$$

Definition(7)[2]: Let the group G act on the set Λ.

1. If $\mathrm{y} = x\mathrm{g}$, for $x, \mathrm{y} \in \Lambda$, then

- $\mathrm{yG} = x\mathrm{G}$;
- $\mathrm{G_y} = \mathrm{g}^{-1}\mathrm{G}_x\mathrm{g}$.

2. $|\mathrm{G}_x| = |G|/|xG|$.

Theorem(8)[3]: There is a unique projectivity of $PG(1,q)$ transforming any three points to any three other points.

Definition(9)[2]: An $[n,k,d]_q$ code C is a subspace of $V(n,q) = (\mathbb{F}_q)^n$, where the dimension of C is dim $C = k$, and the minimum distance is $d(C) = d = \min d(x,y)$.

Definition(10)[2]: For any $[n,k,d]_q$ code we have $d \le n - k + 1$.

RESULTS AND DISCUSSION

1. The Algorithm for Classification of the k -Sets in $PG(1,q)$

On $PG(1{,}16)$, a k-set can be constructed by adding to any $(k-1)$- set one point from the other $q - k + 2$ points. According to the Fundamental Theorem of Projective Geometry, Theorem (2.5), any three distinct points on a line are projectively equivalent; so choose a fixed triad A. A 4-set is formed by adding to A one point from the other $q - 2$ points on $PG(1,q)$; that is, from $PG(1,q) - A = A^c$. A 5-set is formed by adding to any tetrad B one point from the other $q - 3$ points on $PG(1,q)$.The stabilizer group G_B fixes B and splits the other $q - 3$ points into a number of orbits; so, different 5-sets are formed by adding one point from each different orbit. The procedure can be extended to construct $6,7,8,9,\cdots,(\frac{q+1}{2})$-sets, for q is odd and $(\frac{q}{2})$-sets, for q is even in $PG(1,q)$. Let K and K'. be two pentads. To check they are equivalent as following : calculate the transformations between them. By using Theorem (2.8), Two 5-sets K and $K'$are equivalent if $K\beta = K'$ and β is given by a matrix T and $\beta = M(T)$ with $M(\lambda T) = M(T), \lambda \in \mathbb{F}_{16} - \{0\}$.Where T is a non-singular 2×2 matrix. Also can be used to calculate the stabilizer group of each k-set.

2. Preliminary to $PG(1{,}16)$

On $PG(1{,}16)$, the projective line over Galois field of order 16, there are 17 points. The points of $PG(1{,}16)$ are the elements of the set

$$\mathbb{F}_{16} \cup \{\infty\}$$
$$= \{\infty, 0, 1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7, \omega^8, \omega^9, \omega^{10}, \omega^{11}, \omega^{12}, \omega^{13}, \omega^{14} \mid 2$$
$$= \omega^4 + \omega + 1 = \omega^8 + \omega^2 + 1 = \omega^{14} + \omega^3 + 1 = \omega^{10} + \omega^5 + 1$$
$$= \omega^{13} + \omega^6 + 1 = \omega^9 + \omega^7 + 1 = \omega^{12} + \omega^{11} + 1 = \omega^{15} + 1 = 0\}.$$

The order of the projective group $PGL(2{,}16)$ is 17.16.15=4080. This is the number of ordered sets of three points.

In the following sections, the k-sets in $PG(1,16)$, $k = 3,\cdots,8$; are classified by giving the projectively inequivalent k -sets with their stabilizer groups.

3. The Triads

Let S be set of all different triads in $PG(1,16)$. Then the order of S is $|S| = 17.16.15 = 4080$. Let $A = \{\infty, 0,1\}$ be a triad which is one of them. By computing the transformations between A and all the other triads, we note that any triad is projectively equivalent to A. This gives the following conclusion.

Theorem(1): On $PG(1,16)$ there is precisely one triad, given with their stabilizer group in Table 1.

Table-1: The triad on $PG(1,16)$

Symbol	The triad	Stabilizer
A	$\{\infty, 0,1\}$	$S_3 = \langle \frac{1}{t+1}, t+1 \rangle$

4.The Tetrads

To construct the tetrad in $PG(1,16)$, it is enough to add one point from each orbit that comes from the action of the projective group of the triad G_A on the complement of A. All orbits of the triad in Table 1 are given in Table 2.

Table-2: Partition of $PG(1,16)$ by the projectivities of triad

A	Partition of A^c
$\{\infty, 0,1\}$	1.$\{\omega, \omega^3, \omega^4, \omega^{11}, \omega^{12}, \omega^{14}\}$ 2.$\{\omega^2, \omega^6, \omega^7, \omega^8, \omega^9, \omega^{13}\}$ 3.$\{ \omega^5, \omega^{10}\}$

The total numbers of all orbits is 3; therefore 3 inequivalent tetrads can be constructed in $PG(1,16)$.

Table 2 gives the following conclusion.

Theorem(2): On $PG(1,16)$, there are precisely three tetrads, given with their stabilizer group in Table 3.

Table-3: Inequivalent tetrads on $PG(1,16)$

Symbol	The tetrad	Stabilizer
B_1	$A \cup \{\omega\}$	$Z_2 \times Z_2 = \langle \frac{t+\omega}{t+1} \rangle \times \langle \frac{t+1}{\omega^{14}t+1} \rangle$
B_2	$A \cup \{\omega^2\}$	$Z_2 \times Z_2 = \langle \frac{t+\omega^2}{t+1} \rangle \times \langle \frac{\omega^2}{t} \rangle$
B_3	$A \cup \{\omega^5\}$	$A_4 = \langle \frac{\omega^5}{t}, \frac{1}{t+1} \rangle$

5. The Pentads

The projective group G_{B_i} splits ${B_i}^c, i = 1,2,3$ into a number of orbits. The pentads are constructed by adding one point from each orbit to the corresponding tetrads. All orbits are listed in Table 4.

Table -4: Partition of $PG(1,16)$ by the projectivities of tetrads

B_i	Partition ${B_i}^c$
B_1	1. $\{\omega^2, \omega^4, \omega^{12}, \omega^{14}\}$ 2. $\{\omega^3, \omega^6, \omega^{10}, \omega^{13}\}$ 3. $\{\omega^5, \omega^7, \omega^9, \omega^{11}\}$ 4. $\{\omega^8\}$
B_2	1. $\{\omega\}$ 2. $\{\omega^3, \omega^7, \omega^{10}, \omega^{14}\}$ 3. $\{\omega^4, \omega^8, \omega^9, \omega^{13}\}$ 4. $\{\omega^5, \omega^6, \omega^{11}, \omega^{12}\}$
B_3	1. $\{\omega^1, \omega^2, \omega^3, \omega^4, \omega^6, \omega^7, \omega^8, \omega^9, \omega^{11}, \omega^{12}, \omega^{13}, \omega^{14}\}$ 2. $\{\omega^{10}\}$

The total numbers of all orbits is 10; therefore 10 pentads can be constructed in $PG(1,16)$. In Table 5 all equivalent pentads with their projective equations are listed.

Table-5: The equivalence of pentads

No.	Equivalent pentads	Projective equation
1	$B_1 \cup \{\omega^3\} \to B_1 \cup \{\omega^5\}$	$(\omega t + \omega)$
2	$B_1 \cup \{\omega^2\} \to B_2 \cup \{\omega\}$	t
3	$B_1 \cup \{\omega^3\} \to B_2 \cup \{\omega^3\}$	ω^3/t
4	$B_1 \cup \{\omega^3\} \to B_2 \cup \{\omega^5\}$	$(\omega^2 t + \omega^2)/(\omega^{12} t + 1)$
5	$B_1 \cup \{\omega^8\} \to B_2 \cup \{\omega^4\}$	$\omega^4/(t+1)$
6	$B_1 \cup \{\omega^3\} \to B_3 \cup \{\omega\}$	$(\omega t + \omega)$

Table 5 gives the following conclusion.

Theorem(3): On $PG(1,16)$, there are precisely four projectively distinct pentads summarized in Table 6.

Table -6: Inequivalent pentads on $PG(1,16)$

Symbol	The pentad	Stabilizer
C_1	$B_1 \cup \{\omega^2\}$	$Z_2 \times Z_2 = \langle \frac{t+\omega}{t+1} \rangle \times \langle \frac{t+1}{\omega^{14}t+1} \rangle$
C_2	$B_1 \cup \{\omega^3\}$	$I = \langle t \rangle$
C_3	$B_1 \cup \{\omega^8\}$	$Z_2 \times Z_2 = \langle \frac{\omega}{t} \rangle \times \langle \frac{t+\omega}{t+1} \rangle$
C_4	$B_3 \cup \{\omega^{10}\}$	G_{60}

Remark(4): The group G_{60} has 15 elements of order 2, 18 elements of order 3, and 26 elements of order 5, and it is non-abelian.

6. The Hexads

The projective group G_{C_i} splits ${C_i}^c$, $i = 1,2,3,4$ into a number of orbits. The hexads are constructed by adding one point from each orbit to the corresponding pentads. All orbits are listed in Table 7.

Table -7: Partition of $PG(1,16)$ by the projectivities of pentads

C_i	Partition ${C_i}^c$
C_1	1. $\{\omega^3, \omega^7, \omega^{10}, \omega^{14}\}$ 2. $\{\omega^4, \omega^8, \omega^9, \omega^{13}\}$ 3. $\{\omega^5, \omega^6, \omega^{11}, \omega^{12}\}$
C_2	G_{C_2} splits ${C_2}^c$ into 12 orbits of single points.
C_3	1. $\{\omega^2, \omega^4, \omega^{12}, \omega^{14}\}$ 2. $\{\omega^3, \omega^6, \omega^{10}, \omega^{13}\}$ 3. $\{\omega^5, \omega^7, \omega^9, \omega^{11}\}$
C_4	$\{\omega, \omega^2, \omega^3, \omega^4, \omega^6, \omega^7, \omega^8, \omega^9, \omega^{11}, \omega^{12}, \omega^{13}, \omega^{14}\}$

The total numbers of all orbits is 19; therefore 19 hexads can be constructed in $PG(1,16)$. In Table 8 all equivalent hexads with their projective equations are listed.

Table-8: The equivalence of hexads

No.	Equivalent hexads	Projective equation
1	$C_1 \cup \{\omega^3\} \to C_2 \cup \{\omega^2\}$	t
2	$C_1 \cup \{\omega^4\} \to C_3 \cup \{\omega^2\}$	$(\omega t)/(t+\omega^4)$
3	$C_1 \cup \{\omega^5\} \to C_2 \cup \{\omega^4\}$	$(\omega^5)/(t+\omega)$
4	$C_2 \cup \{\omega^5\} \to C_3 \cup \{\omega^3\}$	$(t+\omega)/\omega$
5	$C_2 \cup \{\omega^7\} \to C_4 \cup \{\omega\}$	$\omega^4/(t+\omega^3)$
6	$C_2 \cup \{\omega^5\} \to C_2 \cup \{\omega^8\}$	$(\omega^{14}t+1)$
7	$C_2 \cup \{\omega^9\} \to C_3 \cup \{\omega^5\}$	$(\omega t+\omega)$
8	$C_2 \cup \{\omega^9\} \to C_2 \cup \{\omega^{10}\}$	$\omega^4/(t+\omega)$
9	$C_1 \cup \{\omega^3\} \to C_2 \cup \{\omega^{12}\}$	t/ω
10	$C_1 \cup \{\omega^4\} \to C_2 \cup \{\omega^{14}\}$	t/ω
11	$C_1 \cup \{\omega^5\} \to C_2 \cup \{\omega^6\}$	$(\omega^{13}t+1)$

Table 8 gives the following conclusion.

Theorem(5): On $PG(1,16)$, there are precisely eight projectively distinct hexads summarized in Table 9.

Table -9: Inequivalent hexads on $PG(1,16)$

Symbol	The hexad	Stabilizer
E_1	$C_1 \cup \{\omega^3\}$	$Z_2 = \langle \omega^3/t \rangle$
E_2	$C_1 \cup \{\omega^4\}$	$Z_2 = \langle (t+1)/(\omega^{14}t+1) \rangle$
E_3	$C_1 \cup \{\omega^5\}$	$Z_2 = \langle (\omega t+\omega^3)/(t+\omega) \rangle$
E_4	$C_2 \cup \{\omega^5\}$	$Z_2 = \langle (t+\omega^3)/(t+1) \rangle$
E_5	$C_2 \cup \{\omega^7\}$	$Z_5 = \langle (t)/(t+\omega^3) \rangle$
E_6	$C_2 \cup \{\omega^9\}$	$Z_2 = \langle (t+\omega^3)/(\omega^6 t+1) \rangle$
E_7	$C_2 \cup \{\omega^{11}\}$	$S_3 = \langle \frac{1}{t+1}, \frac{\omega t+\omega^4}{t+\omega} \rangle$
E_8	$C_2 \cup \{\omega^{13}\}$	$S_3 = \langle \frac{t+\omega}{t+\omega^2}, \frac{\omega}{t} \rangle$

7.The Heptads

The projective group G_{E_i} splits ${E_i}^c$, $i = 1, \cdots, 8$ into a number of orbits. The heptads are constructed by adding one point from each orbit to the corresponding hexads. All orbits are listed in Table 10.

Table-10: Partition of $PG(1,16)$ by the projectivities of hexads

E_i	Partition ${E_i}^c$
E_1	1. $\{\omega^4, \omega^{14}\}$ 2. $\{\omega^5, \omega^{13}\}$ 3. $\{\omega^6, \omega^{12}\}$ 4. $\{\omega^7, \omega^{11}\}$ 5. $\{\omega^8, \omega^{10}\}$ 6. $\{\omega^9\}$
E_2	1. $\{\omega^3, \omega^6\}$ 2. $\{\omega^5, \omega^9\}$ 3. $\{\omega^7, \omega^{11}\}$ 4. $\{\omega^8\}$ 5. $\{\omega^{10}, \omega^{13}\}$ 6. $\{\omega^{12}, \omega^{14}\}$
E_3	1. $\{\omega^3, \omega^{13}\}$ 2. $\{\omega^4, \omega^{11}\}$ 3. $\{\omega^6, \omega^8\}$ 4. $\{\omega^7, \omega^{14}\}$ 5. $\{\omega^9\}$ 6. $\{\omega^{10}, \omega^{12}\}$
E_4	1. $\{\omega^2, \omega^{13}\}$ 2. $\{\omega^4, \omega^6\}$ 3. $\{\omega^7, \omega^{10}\}$ 4. $\{\omega^8, \omega^{11}\}$ 5. $\{\omega^9\}$ 6. $\{\omega^{12}, \omega^{14}\}$
E_5	1. $\{\omega^2, \omega^4, \omega^6, \omega^{11}, \omega^{12}\}$ 2. $\{\omega^5, \omega^8, \omega^9, \omega^{10}, \omega^{13}\}$ 3. $\{\omega^{14}\}$
E_6	1. $\{\omega^2, \omega^4\}$ 2. $\{\omega^5, \omega^{14}\}$ 3. $\{\omega^6\}$ 4. $\{\omega^7, \omega^{13}\}$ 5. $\{\omega^8, \omega^{10}\}$ 6. $\{\omega^{11}, \omega^{12}\}$
E_7	1. $\{\omega^2, \omega^6, \omega^7\}$ 2. $\{\omega^4, \omega^8, \omega^9, \omega^{12}, \omega^{13}, \omega^{14}\}$ 3. $\{\omega^5, \omega^{10}\}$
E_8	1. $\{\omega^2, \omega^5, \omega^6, \omega^{10}, \omega^{11}, \omega^{14}\}$ 2. $\{\omega^4, \omega^{12}\}$ 3. $\{\omega^7, \omega^8, \omega^9\}$

There are 39 different orbits; therefore 39 heptads can be constructed in $(1,16)$. The projectively distinct heptads with their stabilizer groups are given in the following theorem.

Theorem(6): On $PG(1,16)$, there are 10 projectively distinct heptads summarized in Table 11.

Table -11: Inequivalent heptads on $PG(1,16)$

Symbol	The heptad	Stabilizer
F_1	$E_1 \cup \{\omega^4\}$	$Z_2 = \langle \omega^4/t \rangle$
F_2	$E_1 \cup \{\omega^5\}$	$I = \langle t \rangle$
F_3	$E_1 \cup \{\omega^6\}$	$Z_3 = \langle (t)/(\omega^{14}t + \omega^5) \rangle$
F_4	$E_1 \cup \{\omega^7\}$	$Z_2 = \langle (t + \omega^2)/(t + 1) \rangle$
F_5	$E_1 \cup \{\omega^8\}$	$Z_2 = \langle (t + \omega)/(\omega^{11}t + 1) \rangle$
F_6	$E_1 \cup \{\omega^9\}$	$Z_2 = \langle \omega^3/t \rangle$
F_7	$E_2 \cup \{\omega^5\}$	$Z_2 = \langle t + \omega \rangle$
F_8	$E_2 \cup \{\omega^8\}$	$D_5 = \langle \frac{t}{\omega^{14}t + \omega^3}, t + 1 \rangle$
F_9	$E_4 \cup \{\omega^7\}$	$Z_2 = \langle (t + \omega^5)/(\omega^{14}t + 1) \rangle$
F_{10}	$E_4 \cup \{\omega^8\}$	$Z_3 = \langle (t + \omega^8)/(t + \omega^3) \rangle$

8. The Octads

The projective group G_{F_i} splits ${F_i}^c, i = 1, \cdots, 10$ into a number of orbits. The octads are constructed by adding one point from each orbit to the corresponding heptads. All orbits are listed in Table 12.

Table -12: Partition of $PG(1,16)$ by the projectivities of heptads

F_i	Partition ${F_i}^c$
F_1	1.$\{\omega^5, \omega^{14}\}$ 2. $\{\omega^6, \omega^{13}\}$ 3. $\{\omega^7, \omega^{12}\}$ 4. $\{\omega^8, \omega^{11}\}$ 5. $\{\omega^9, \omega^{10}\}$
F_2	G_{F_2} splits ${F_2}^c$ into 10 orbits of single points.
F_3	1. $\{\omega^4, \omega^8, \omega^{10}\}$ 2. $\{\omega^5, \omega^9, \omega^{12}\}$ 3. $\{\omega^7, \omega^{13}, \omega^{14}\}$ 4. $\{\omega^{11}\}$
F_4	1. $\{\omega^4, \omega^9\}$ 2. $\{\omega^5, \omega^6\}$ 3. $\{\omega^8, \omega^{13}\}$ 4. $\{\omega^{10}, \omega^{14}\}$ 5. $\{\omega^{11}, \omega^{12}\}$
F_5	1. $\{\omega^4, \omega^{11}\}$ 2. $\{\omega^5, \omega^9\}$ 3. $\{\omega^6, \omega^{12}\}$ 4. $\{\omega^7, \omega^{13}\}$ 5. $\{\omega^{10}, \omega^{14}\}$
F_6	1. $\{\omega^4, \omega^{14}\}$ 2. $\{\omega^5, \omega^{13}\}$ 3. $\{\omega^6, \omega^{12}\}$ 4. $\{\omega^7, \omega^{11}\}$ 5. $\{\omega^8, \omega^{10}\}$
F_7	1. $\{\omega^3, \omega^9\}$ 2. $\{\omega^6, \omega^{11}\}$ 3. $\{\omega^7, \omega^{14}\}$ 4. $\{\omega^8, \omega^{10}\}$ 5. $\{\omega^{12}, \omega^{13}\}$
F_8	$\{\omega^3, \omega^5, \omega^6, \omega^7, \omega^9, \omega^{10}, \omega^{11}, \omega^{12}, \omega^{13}, \omega^{14}\}$
F_9	1. $\{\omega^2, \omega^{12}\}$ 2. $\{\omega^4, \omega^9\}$ 3. $\{\omega^6, \omega^{14}\}$ 4. $\{\omega^8, \omega^{10}\}$ 5. $\{\omega^{11}, \omega^{13}\}$
F_{10}	1. $\{\omega^2, \omega^9, \omega^{11}\}$ 2. $\{\omega^4, \omega^{10}, \omega^{13}\}$ 3. $\{\omega^6, \omega^{12}, \omega^{14}\}$ 4. $\{\omega^7\}$

There are 49 different orbits; therefore 49 octads can be constructed in $PG(1,16)$. The projectively distinct octads with their stabilizer groups are given in the following theorem.

Theorem(7): On $PG(1,16)$, there are 11 projectively distinct octads summarized in Table 13.

Table 13: Inequivalent octads on $PG(1,16)$

Symbol	The octad	Stabilizer
H_1	$F_1 \cup \{\omega^5\}$	$Z_2 = \langle \omega^5/t \rangle$
H_2	$F_1 \cup \{\omega^6\}$	$Z_2 = \langle (t+1)/(\omega^{14}t+1) \rangle$
H_3	$F_1 \cup \{\omega^7\}$	$I = \langle t \rangle$
H_4	$F_1 \cup \{\omega^8\}$	$I = \langle t \rangle$
H_5	$F_1 \cup \{\omega^9\}$	$Z_2 = \langle (t+\omega^3)/(\omega^6 t+1) \rangle$
H_6	$F_2 \cup \{\omega^6\}$	$Z_2 = \langle (t+\omega^6)/(\omega^{12}t+1) \rangle$
H_7	$F_2 \cup \{\omega^8\}$	$Z_2 = \langle (t+\omega^2)/(\omega^{10}t+1) \rangle$
H_8	$F_2 \cup \{\omega^{10}\}$	$I = \langle t \rangle$
H_9	$F_2 \cup \{\omega^{13}\}$	$Z_2 \times Z_2 \times Z_2 = \langle \frac{t+\omega^3}{t+1} \rangle \times \langle \frac{t+\omega^2}{\omega^{14}t+1} \rangle \times \langle \frac{t+\omega}{\omega^{13}t+1} \rangle$
H_{10}	$F_3 \cup \{\omega^{11}\}$	$S_3 = \langle \frac{t}{\omega^{14}t+\omega^5}, \frac{t+\omega^{11}}{t+1} \rangle$
H_{11}	$F_9 \cup \{\omega^8\}$	$S_3 = \langle \frac{t+\omega^8}{t+\omega^3}, \frac{t+1}{\omega^{10}t+1} \rangle$

9.MDS Code of Dimension Two

From Definition (10), an $[n,k,d]_q$ −code is maximum distance separable (MDS) when $d = n - k + 1$. So, if $k = 2$ and $d = n - 1$ of an $[n,k,d]_q$ −code, the code C converts to a set K of n points on the projective line $PG(1,q)$. In Table 14, the MDS codes corresponding to the n-sets in $PG(1,16)$ and the parameter e of errors corrected are given.

Table-14: MDS code over $PG(1,16)$

n-set	: MDS code	e
Triad	$[3,2,2]_{16}$	0
Tetrad	$[4,2,3]_{16}$	1
Pentad	$[5,2,4]_{16}$	1
Hexad	$[6,2,5]_{16}$	2
Heptad	$[7,2,6]_{16}$	2
Octad	$[8,2,7]_{16}$	3
Nonad	$[9,2,8]_{16}$	3

If C has minimum distance d, then it can detect $d-1$ errors and correct $e = \lfloor (d-1)/2 \rfloor$ errors, where $\lfloor m \rfloor$ denotes the integer part of m:

d	1	2	3	4	5	6	7	8
e	0	0	1	1	2	2	3	3

REFERENCES

[1] Al-seraji N.A.M, " The Geometry Of The Plane Of Order Seventeen And Its ApplicationTo Error-correcting Codes" Ph.D. Thesis, University of Sussex, UK, (2010)

[2] Hirschfeld J.W.P, Korchmáros G and Torres F," *Algebraic Curves Over a Finite field"* Oxford University Press, Oxford, (2007)

[3] Hirschfeld J.W.P," *Projective Geometries Over Finite Fields* " S*econd Edition*, Oxford University Press, Oxford, (1998)

[4] Al-seraji N.A.M, "On Almost Maximum Distance Separable codes" journal of Baghdad, College of Science, University of Baghdad, No. 3 Vol. 54. (2013)

[5] Al-zangana E.B, " The Geometry Of The Plane Of Order Nineteen and Its Application To Error-Correcting Codes" Ph.D. Thesis,University of Sussex, UK,(2011)

[6] A.H. Ali, "Classification of Arcs in Galois Plane of Order Thirteen", Ph.D. Thesis, University of Sussex, (1993)

[7] R.C. Bose. Mathematical theory of the symmetrical factorial design. Sankhya, 8:107-166, (1947)

[8] B. Segre. Sulle ovali nei piani lineari finiti. Atti Accad. Naz. Lincei Rend., 17:1-2,(1954)

[9] V.D. Goppa. Codes on algebraic curves. Soviet Math. Dokl., 24:170-172, (1981)

[10] J.W.P. Hirschfeld, D.R. Hughes, and J.A. Thas, editors. Advances in Finite Geometries and Designs. Oxford University Press, Oxford, (1991)