I-DOOR SPACES

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ABSTRACT

Let (X, τ) be a topological space and let I be an ideal on (X, τ), we call
(X, τ, I) an I-topological space.

In this paper, we study I-door spaces with several properties of these spaces are proved.

المستخلص:

ليكن (X, τ) فضاء تبولوجي وليكن I مثالي في (X, τ)، سوف نسمي الثلاثي (X, τ, I) فضاء تبولوجي-I. في هذا البحث درسنا الفضاءات البابية-I وتم برهان عدة خصائص لهذه الفضاءات.

1- INTRODUCTION

Let (X, τ) be a topological space, then topological is called a door space [3] if every subset of X is either open or closed, and as an examples:

(□ , τ_d) is a door space, where τ_d is the discrete topology on □ . But (□ , τ_u) is not a door space, where τ_u is the usual topology on □ .

In this paper, we will study the I-door spaces, where I is an ideal on the topological space (X, τ).

2- BASIC DEFINITIONS

In this section, we introduce and recall the basic definitions needed in this work.

2-1 Definition [1]:

Let (X, τ) be a topological space. An ideal I on (X, τ) is a non-empty collection of subsets of X, which satisfies the following two properties:

(i) $A \in I$ and $B \subseteq A$ implies to $B \in I$.

(ii) $A \in I$ and $B \in I$ implies to $A \cup B \in I$.

2-2 Remarks and Examples:

(i) Let (X, τ) be a topological space . Let I be the collection of all finite subsets of X, then I is an ideal on (X, τ).

(ii) Let (X, τ, I) be an I-topological space and let $A \in I$, let $B \subseteq X$, then $A \cap B \in I$ (notice that $A \cap B \subseteq A$).

(iii) Let (X, τ) be a topological space and let J be the collection of all compact subsets of X, then J is not an ideal on (X, τ).

2-3 Definition:

Let (X, τ, I) be an I-topological space, w say that (X, τ, I) is an I-door space if every subset of X is either open or closed or belongs to I.

2.4 Remarks and Examples:

(i) Every door space is an I-door space. In fact, let (X, τ) be a door space and let I be the collection of all subsets of X, then (X, τ, I) is an I-door space.

(ii) Let (X, τ, I) be an I-door space, where $I = \varnothing$. Then (X, τ, I) is a door space and we write (X, τ) instead of (X, τ, I).

3- MAIN RESULTS

In this section, we prove several results concerning door spaces and I-door spaces. Before, we state our first result, we recall the following definition.

3-1 Definition:

Let (X, τ) be a topological space, we say that X is extremely disconnected [2] if the closure of every open set is open.

3-2 Theorem:

For a topological space (X, τ), the following conditions are equivalent:

(1) (X, τ) is an extremely disconnected door space.
(2) Every subset A of X is either open or both closed and discrete.

Proof:

(1) $\Rightarrow$ (2). Assume that A is not open and assume that:

A' be the set of all limit points of $A \neq \emptyset$

Since X is a door space, then A is closed

Let $x \in A' \subseteq A$

Let $S = A^c \cup \{x\}$

S cannot be open

But X is a door space, so S is closed

Now, $A^c \cup \{x\} = \overline{A^c} \cup \overline{\{x\}}$

Hence $A^c \subseteq \overline{A^c} \subset A^c \cup \{x\}$

But X is extremely disconnected

So $\overline{A^c}$ is open

So $A^c \subseteq \overline{A^c} \subseteq S$

Thus, $A^c = \overline{A^c}$, which implies that A is open, which is a contradiction, so $A' = \varnothing$, which implies that A is closed and discrete.

(2) $\Rightarrow$ (1). We need to show that X is extremely disconnected, let $A \subseteq X$ be open, then $\overline{A} = A \cup A'$

If $A' = \varnothing$, then $\overline{A} = A$ is open

If $A' \neq \varnothing$, then $(\overline{A})' \neq \varnothing$ and $\overline{A}$ is open

(note that A is closed and discrete if and only if $A' = \varnothing$). $\blacksquare$

3-3 Corollary:

Let (X, τ) be an extremely disconnected topological space and let I be the ideal of all closed and discrete subsets of X, then the following conditions are equivalent:

(1) (X, τ) is door space.

(2) $A \notin \tau \Rightarrow A \in I$.

(3) (X, τ, I) is an I-door space.

3-4 Definition:

Let (X, τ) be a topological space, we say that X is F-door space if every finite subset of X is either open or closed.

3-5 Remarks:

(i) Every door space is F-door space.

(ii) Every F-door space is a $T_{\frac{1}{2}}$-space, [4], where (X, τ) is called $T_{\frac{1}{2}}$ if every singleton is either open or closed).

(iii) Every T_1-space is F-door space, but not necessarily a door space.

Before we state our next result, we recall the following definition:

3-6 Definition, [2]:

Let (X, τ) be a topological space. Let $A \subseteq X$, we say that:

(i) A is regular open if A = int cl A.

(ii) A is regular closed if A = cl int A.

3-7 Theorem:

Let (X, τ) be a topological space, then the following conditions are equivalent:

(1) X is discrete.

(2) X is a regular door space, that is every subset of X is either regular open or regular closed.

Proof:

$(1) \Rightarrow (2)$. Is trivial.

$(2) \Rightarrow (1)$. Let $x \in X$. If $\{x\}$ is not regular open, then $\{x\}$ is regular closed.

So $\{x\} = \mathrm{c\,lint}\,\{x\}$

Since $\mathrm{int}\{x\} \neq \varnothing$

Then $\{x\} = \mathrm{int}\{x\}$

Hence $\{x\}$ is open

This shows that X is discrete. ■

3-8 Theorem:

Let (X, τ) be F-door space. Let I be the ideal of all closed and discrete subsets of X, then the following conditions are equivalent:

(1) (X, τ) is a door space.

(2) (X, τ, I) is an I-door space.

Proof:

(1) $\Rightarrow$ (2). Is trivial.

(2) $\Rightarrow$ (1). If $A \subset X$ is neither open, nor closed

Then $A \in I$

But X is F-door space

Then A is either open or closed. ■

4- REFERENCES

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