On Generalized Simple Singular P-injective Rings

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Abstract

A ring R is called GSSP-ring , if for any maximal essential right ideal M of R and any $b \in M$ then bR/bM is p-injective. In this paper we give conditions under which *GSSP-ring* are strongly regular. Finally, some new characteristic properties of *GSSP-ring* are given.

1. Introduction:

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary right R-modules.

A right R-modules M is said to be p-injective if, for any principle right ideal p of R, any right R-homomorphism $f: P \to M$, there exists y in M such that $f(b) = yb$ for all $b \in P$. This concept was introduced by Ming [4] . We Recall that : **(1)** R is called strongly regular if for every a in R, there exists an element b in R such that $a = a^2b$. see [6], **(2)** A ring R is said to be ERT-ring if for every essential right ideal of R is a two sided ideal. See [5], **(3)** An ideal I of the ring R is essential if I has a non-zero intersection with every non-zero ideal of R, **(4)** Let R be a ring such that every maximal right ideal is a two sided ideal, then R is called a quasi-duo ring, see [7] **(5)** A ring R is called reduced if, R contains no non-zero nilpotent element, see [2], **(6)** For any element a in R, $r(a)$ and $I(a)$ denote the right and left annihilator of a respectively, see [2], **(7)** $J(R), Z(R)$ will stand respectively for the Jacobson radical, the left singular ideal. See [1]

2. GSSP-rings:

In this section, some of the definitions and basic properties of *GSSP-ring* are given and we introduce a generalization of such rings. Following [4], a ring is said to be *SSPI-rings*, if and only if every simple singular R-module is P-injective.

Definition 2-1:

A ring R is called a *GSSP-ring* (generalized simple singular P-injective) if, for any maximal essential right ideal M of R, any $b \in M$, bR/bM is P-injective.

Following [2] a ring R is said to be abelean if each idempotent element of R is central. Next, we give the following lemma which play the key role in several of our proofs.

Lemma 2-2:

Let R be abelean ring, for any maximal right ideal M of R, and for any $a \in M$, if $r(a) \subset M$, then M is an essential right ideal of R.

Proof:

Let $0 \neq a \in M$, and let $r(a) \subset M$. Suppose that M is not essential, then M is direct summand, and hence there exists $0 \neq e = e^2$ in R such that $M = r(e)$. Since $a \in M = r(e)$, then $ea = 0$. Since R is abelean, then $ae = 0$, and $e \in r(a) \subset M = r(e)$. Therefore $e = 0$, a contradiction. Thus M is essential .

Now, we introduce the following theorem.

Theorem 2-3:

Let R be abelean *GSSP-ring*, then any right ideal of R is idempotent.

Proof:

Let I be a right ideal of R and let $a \in I$. If $RaR + r(a) \neq R$. Let M be a maximal right ideal containing $RaR + r(a)$. Then by lemma (2-2), M is essential right ideal of R . If $aR = aM$, then $a = ac$ for some c in M and this implies $a(1-c) = 0$. So, $(1-c) \in r(a) \subset M$, whence $1 \in M$, a contradiction. $M \neq R$. If $aR \neq aM$, the right R-homomorphism $g: R/M \to aR/aM$ defined by $g(b+M) = ab + aM$, for all b in R implies $R/M \cong aR/aM$. Defined $f: aR \to R/M$ as a right R-homomorphism by $f(ax) = x + M$, for all x in R, then f is a well define right R-homomorphism. Indeed, let $x_1, x_2 \in R$ with $ax_1 = ax_2$ implies $(x_1 - x_2) \in r(a) \subset M$, thus $x_1 + M = x_2 + M$. Hence $f(ax_1) = x_1 + M = x_2 + M = f(ax_2)$. Since R/M is P-injective, then there exists c in R such that $f(ac) = (c+M)ax = cax + M$, yields $1 + M = f(a) = da + M$, for some d in R .Whence $1 \in M$, a contradiction. Thus $RaR + r(a) = R$. In particular $xay + c = 1$, for some x, y in R and c in $r(a)$, so we have $a = axay + ac = axay + 0$. Therefore $a = axay \in I^2$. This prove $I = I^2$.

Theorem 2-4:

Let R be *ERT* and *GSSP-ring* such that the right annihilator of any element in R is essential. Then:

(1) R is reduced.

(2) $J(R) = 0$.

Proof 1:

Let $0 \neq a \in R$ such that $a^2 = 0$ and let M be a maximal right ideal containing $r(a)$. If $aR = aM$, then $a = ac$ for some c in M this implies $(1-c) \in r(a) \subset M$, whence $1 \in M$, a contradiction. Now, since $R/M \cong aR/aM$, then R/M is P-injective.

Defined $f:aR\to R/M$ by $f(ar)=r+M$, for every r in R. Now, we show that f is a well defined right R-homomorphism. Indeed if $ar_1=ar_2$ for every r_1,r_2 in R. Then $a(r_1-r_2)=0$, therefore $(r_1-r_2)\in r(a)\subset M$ and hence $r_1+M=r_2+M$. Since R/M is P-injective, then there exists y in R such that $f(ar)=(y+M)ar$, yields $1+M=f(a)=ya+M$, for some y in R, so $(1-ya)\in M$, but $ya\in r(a)$ is a right annihilator and hence it is essential. Since R is *ERT*, therefore $r(a)$ is a two sided ideal so $ya\in M$, thus $1\in M$, a contradiction. Therefore $a=0$, whence R is reduced .

Proof 2:

Let $a\in J(R)$, If $aR+r(a)\neq R$. Then there exists a maximal right ideal M containing $aR+r(a)$. Since $a\in M$ and $r(a)\subset M$, then by Theorem (2-4)(1) and lemma (2-2), then M is essential. If $aR=aM$, then $a=ab$ for some b in M this implies $(1-b)\in r(a)\subset M$, so $1\in M$, a contradiction. If $aR=aM$, the right R-homomorphism $g:R/M\to aR/aM$ defined by $g(b+M)=ab+aM$, for all b in R implies that $R/M\cong aR/aM$.

Define $f:aR\to R/M$ as a right R–homomorphism by $f(ax)=x+M$ for all x in R, since R is reduced (Theorem 2-4)(1) then clearly f is a well define R-homomorphism, so there exists y in R such that $f(ax)=(y+M)ax$. Thus $1+M=f(a)=ya+M$, but $a\in J\subset M$, so $1\in M$, a contradiction.

Therefore $aR+r(a)=R$. In particular $ar+d=1$, for $d\in r(a)$, this implies $a=a^2r$, since $a\in J$, then there exists an invertible element u in R such that $(1-ar)u=1$, so $(a-a^2r)u=a$, yields $a=0$. This proves that $J(R)=0$.

The following theorem gives the condition of being right *GSSP-ring* are strongly regular.

Theorem

2-5: Let R be an abelean ring and right quasi-duo ring. If R is *GSSP-ring*, then R is strongly regular.

Proof:

Assume that $0\neq a\in R$ such that $a^2=0$. Then there exists the maximal right ideal M of R such that $aR+r(a)\subset M$. Observe that M must be an essential right ideal of R. (lemma 2-2) . By similar method of proof used in Theorem (2-4)(2), we get $aR+r(a)=R$. In particular $ay+d=1$ for some y in R,d in $r(a)$, thus we have $a^2y=a$. Therefore R is strongly regular ring.

Before closing this section, we present the following result.

Proposition 2-6 :

If R is a quasi-duo, *GSSP-ring*, then $Z(R)=(0)$.

Proof :

If $Z(R)\neq(0)$, there exists a non-zero element a in $Z(R)$ with $a^2=0$. We want to prove that $aR+r(a)=R$. If $aR+r(a)\neq R$. Let M be a maximal right ideal of R containing $aR+r(a)$. Since $a\in Z(R)$, then $r(a)$ is essential right ideal and by lemma (2-2) M is essential maximal right ideal of R. If $aR=aM$, then $a=ab$ for some b in M and $(1-b)\in r(a)\subset M$, whence $1\in M$, a contradiction. $M\neq R$. If $aR\neq aM$, the right R-homomorphism $g:R/M\to aR/aM$ defined by $g(b+M)=ab+aM$ for all $b\in R$ implies that $R/M\cong aR/aM$, since aR/aM is P-injective, then R/M is P-injective.

Consider the canonical mapping $f:aR\to R/M$, then there exists a in R such that $f(a)=1+M=ba+M$ implies $(1-ba)\in M$, $ba\in M$ (because M is two sided ideal), then $1\in M$, a contradiction . Hence $aR+r(a)=R$.

In particular $1+ar+d$, r in R,d in $r(a)$, so $a=a^2r+ad$. Therefore $Z(R)=(0)$.

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الملخص

يقال للحلقة R بأنها من النمط $GSSP$ ، إذا كان لأي مثالي أعظم أساسي أيمن M في R ولأي $b \in M$ يكون bR/bM موديل غامر من النمط P . في هذا البحث تم إعطاء شروطاً أخرى لكي تكون كل حلقة من النمط $GSSP$ حلقة منتظمة بقوة . أخيرا حصلنا على خواص جديدة للحلقات من النمط $GSSP$.