On Fully \((m,n)\)-stable modules relative to an ideal \(A\) of \(R^{n\times m}\)

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**Abstract:**
Let \(R\) be a commutative ring with non-zero identity element. For two fixed positive integers \(m\) and \(n\). A right \(R\)-module \(M\) is called fully \((m,n)\)-stable relative to ideal \(A\) of \(R^{n\times m}\), if \(\theta(N) \subseteq N + M^mA\) for each \(n\)-generated submodule of \(M^m\) and \(R\)-homomorphism \(\theta : N \rightarrow M^m\). In this paper we give some characterization theorems and properties of fully \((m,n)\)-stable modules relative to an ideal \(A\) of \(R^{n\times m}\), which generalize the results of fully stable modules relative to an ideal \(A\) of \(R\).

**Key words:** fully \((m,n)\)-stable modules relative to an ideal \(A\) of \(R^{n\times m}\), \((m,n)\)-multiplication modules and \((m,n)\)-quasi injective modules.

**Introduction:**
Throughout, \(R\) is a commutative ring with non-zero identity and all modules are unitary right \(R\)-module. We use the notation \(R^{m\times n}\) for the set of all \(m\times n\) matrices over \(R\). For \(G \in R^{m\times n}\), \(G^T\) will denote the transpose of \(G\). In general, for an \(R\)-module \(N\), we write \(N^{m\times n}\) for the set of all formal \(m\times n\) matrices whose entries are elements of \(N\). Let \(M\) be a right \(R\)-module and \(N\) be a left \(R\)-module. For \(x \in M^{km}\), \(s \in R^{m\times n}\) and \(y \in N^{nk}\), under the usual multiplication of matrices, \(xs\) (resp. \(sy\)) is a well defined element in \(M^{km}\) (resp. \(N^{nk}\)). If \(X \in M^{km}\), \(S \in R^{m\times n}\) and \(Y \in N^{nk}\), define

- \(\ell_{M^{km}}(S) = \{ u \in M^{km} : us = 0, \forall s \in S \}\)
- \(\gamma_{N^{nk}}(S) = \{ v \in N^{nk} : vs = 0, \forall s \in S \}\)
- \(\ell_{R^{m\times n}}(Y) = \{ s \in R^{m\times n} : sy = 0, \forall y \in Y \}\)
- \(\gamma_{R^{m\times n}}(X) = \{ s \in R^{m\times n} : xs = 0, \forall x \in X \}\)

We will write \(N^n = N^{lxn}\), \(N^n = N^{nxl}\). Fully stable module relative to an ideal have been discussed in [1], an \(R\)-module \(M\) is called fully stable relative to an ideal, if \(\theta(N) \subseteq N + MA\) for each submodule \(N\) of \(M\) and \(R\)-homomorphism \(\theta : N \rightarrow M\). It is an easy matter to see that \(M\) is fully stable relative to an ideal, if and only if \(\theta(xR) \subseteq xR + MA\) for each \(x\) in \(M\) and \(R\)-homomorphism \(\theta : xR \rightarrow M\). An \(R\)-module \(M\) for two fixed positive integers \(m\) and \(n\) is called fully \((m,n)\)-stable relative to an ideal \(A\) of \(R\), if \(\theta(N) \subseteq N + M^nA\) for each \(n\)-generated submodule of \(M^m\) and \(R\)-homomorphism \(\theta : N \rightarrow M^m\). In this paper, for two fixed positive integers \(m\) and \(n\), we introduce the concepts of fully \((m,n)\)-stable modules relative to an ideal \(A\) of \(R^{n\times m}\) and \((m,n)\)-Baer criterion relative to an ideal \(A\) of \(R^{n\times m}\) and we prove that an \(R\)-module \(M\) is fully \((m,n)\)-stable relative to an ideal \(A\) of \(R^{n\times m}\) if and only if \((m,n)\)-Baer criterion relative to an
ideal holds for \( n \)-generated submodules of \( M^m \).

**1. Results:**

**Definition 1.1:** An R-module M is called fully (m,n) -stable relative to an ideal A of \( R^{m\times n} \), if \( \theta(N) \subseteq N + M^n A \) for each \( n \)-generated submodule N of \( M^m \) and \( R \)-homomorphism \( \theta : N \to M^m \). The ring R is fully (m,n) -stable relative to an ideal, if R is fully (m,n) -stable relative to an ideal as R-module.

It is clear that M is fully (1,1)-stable relative to ideal, if and only if M is fully stable relative to ideal.

It is an easy matter to see that an R-module M is fully (m,n)-stable relative to ideal, if and only if it is fully (m,q)-stable relative to ideal for all \( 1 \leq q \leq n \) , if and only if it is fully (p,n)-stable relative to ideal for all \( 1 \leq p \leq m \), if and only if it is fully (p,q)-stable relative to ideal for all \( 1 \leq p \leq m \) and \( 1 \leq q \leq n \).

In [2], an R-module M is called fully-stable, if \( \theta(N) \subseteq N \) for each cyclic submodule N of M and \( R \)-homomorphism \( \theta : N \to M \). An R-module M is called fully (m,n) -stable, if \( \theta(N) \subseteq N \) for each \( n \)-generated submodule N of \( M^m \) and \( R \)-homomorphism \( \theta : N \to M^m \) [3]. It is clear that every fully (m,n)-stable module M is a fully (m,n)-stable relative to non-zero ideal A of R for this follows from the fact \( \theta(N) \subseteq N + M^n A \).

An R-module M is fully (m,n)-stable relative to an ideal A of \( R^{m\times n} \), if and only if for each \( \theta : N(\sum_{i=1}^{n} \alpha_i R) \to M^m \) (where \( \alpha_i \in M^m \) ) and each \( w \in N \), there exists \( t = (t_1, \ldots, t_n) \in R^n \) such that \( \theta(w) = \sum_{i=1}^{n} \alpha_i t_i + AM^m = (\alpha_1, \ldots, \alpha_n) t^T + M^mA \), if \( r = (r_1, \ldots, r_n) \in R^n \), then \( \theta((\alpha_1, \ldots, \alpha_n) r^T) + M^mA = (\alpha_1, \ldots, \alpha_n) t^T + M^mA \).

**Proposition 1.2:** An R-module M is fully (m,n)-stable relative to an ideal A of \( R^{m\times n} \), if and only if any two \( m \)-element subsets \( \{\alpha_1, \ldots, \alpha_m\} \) and \( \{\beta_1, \ldots, \beta_m\} \) of \( M^n \) such that \( \beta_j \in \sum_{i=1}^{n} \alpha_i R + M^mA \), \( \forall j = 1, \ldots, m \) implies \( \gamma_{\alpha_1} \{\alpha_1, \ldots, \alpha_m\} \subset \gamma_{\beta_j} \{\beta_1, \ldots, \beta_m\} \). Define \( f : \sum_{i=1}^{n} \alpha_i R \to M^m \) by \( f(\sum_{i=1}^{n} \alpha_i r_i) = \sum_{i=1}^{n} \beta_j r_i \).

Let \( \alpha_i = (a_{i1}, a_{i2}, \ldots, a_{im}) \). If \( \sum_{i=1}^{n} \alpha_i r_i = 0 \), then \( \sum_{i=1}^{n} a_{ij} r_i = 0, j = 1, \ldots, m \) implies \( \alpha_i r_i^T = 0 \) where \( r = (r_1, \ldots, r_n) \) and hence \( r^T \in \gamma_{\alpha_1} \{\alpha_1, \ldots, \alpha_m\} \). By assumption \( \beta_j r_i^T = 0, j = 1, \ldots, m \) so \( \sum_{i=1}^{n} \beta_j r_i = 0 \). This show that f is well defined. It is an easy matter to see that f is R-homomorphism. Fully (m,n)-stability relative to an ideal A of \( R^{m\times n} \) implies that there exists \( t = (t_1, \ldots, t_n) \in R^n \) and \( w \in M^mA \) such that \( f(\sum_{i=1}^{n} \alpha_i r_i) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \alpha_i r_i \right) t_k + w = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \alpha_i r_i t_k \right) + w \) for each \( \sum_{i=1}^{n} \alpha_i r_i \in \sum_{i=1}^{n} \alpha_i R \).

Let \( r_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n \) where 1 in the i th position and 0
otherwise. \( \beta_i = f(\alpha_i) = \sum_{k=1}^{n} \alpha_i t_k + w \)
which is contradiction. Conversely assume that there exists \( n \)-generated submodule of \( M^m \) and R-homomorphism \( \theta : \sum_{i=1}^{n} \alpha_i R \to M^m \) such
that \( \theta (\sum_{i=1}^{n} \alpha_i R) \not\in \sum_{i=1}^{n} \alpha_i R + M^n A \). Then
there exists an element \( \beta (\sum_{i=1}^{n} \alpha_i r_i) \in \sum_{i=1}^{n} \alpha_i R \) such that \( \theta (\beta \alpha R) \not\in \sum_{i=1}^{n} \alpha_i R + M^n A \). Take \( \beta_j = \beta_j, j = 1, \ldots, m \), then we have \( m \)-element subset \( \{ \theta (\beta), \ldots, \theta (\beta) \} \), such that \( \theta (\beta) \not\in \sum_{i=1}^{n} \alpha_i R + M^n A \). 
\( \text{Let } \eta = (t_1, t_2, \ldots, t_m) \in \gamma_{R^n} \gamma_{R^n} \{ \alpha_1, \ldots, \alpha_m \} \text{ then } \alpha_j \eta = 0, \text{ i.e. } \sum_{i=1}^{n} a_{ij} t_i = 0, \forall j = 1, \ldots, m \), \( \alpha_j = (a_{1j}, a_{2j}, \ldots, a_{nj}) \) and \( \{ \theta (\beta), \ldots, \theta (\beta) \} \eta \)
\( \sum_{k=1}^{n} \theta (\beta) t_k = \sum_{k=1}^{n} \theta (\sum_{i=1}^{n} \alpha_i r_i) t_k = \sum_{k=1}^{n} (\theta (\sum_{i=1}^{n} \alpha_i r_i) t_k = 0, \text{ hence } \gamma_{R^n} \gamma_{R^n} \{ \alpha_1, \ldots, \alpha_m \} \subseteq \gamma_{R^n} \gamma_{R^n} \{ \theta (\beta), \ldots, \theta (\beta) \}, \text{ thus } \gamma_{R^n} \gamma_{R^n} \{ \alpha_1, \ldots, \alpha_m \} \subseteq \gamma_{R^n} \gamma_{R^n} \{ \theta (\beta), \ldots, \theta (\beta) \} \}
which is a contradiction. Thus \( M \) is fully \( (m,n) \)-stable module relative to ideal 

**Corollary 1.3:** Let \( M \) be fully \( (m,n) \)-stable module relative to an ideal \( A \) of \( R^{m_m} \), then for any two m-element subsets \( \{ \alpha_1, \ldots, \alpha_m \} \) and \( \{ \beta_1, \ldots, \beta_m \} \) of \( M^n \), \( \gamma_{R^n} \gamma_{R^n} \{ \alpha_1, \ldots, \alpha_m \} \subseteq \gamma_{R^n} \gamma_{R^n} \{ \beta_1, \ldots, \beta_m \} \) implies \( \alpha_1 R + \cdots + \alpha_m R + M^n A = \beta_1 R + \cdots + \beta_m R + M^n A \). 

**Corollary 1.4:** [1] Let \( M \) be a fully stable module relative to an ideal \( A \) of \( R \), then for each \( x, y \in M \), \( y \not\in (x) \), \( \gamma_{R^n} (x) = \gamma_{R^n} (y) \) implies \( (x) + AM = (y) + AM \) 

A submodule \( N \) of an \( R \)-module \( M \) satisfies Baer criterion relative to an ideal \( A \) of \( R \), if for every \( R \)-homo morphism \( f : N \to M \), there exists an element \( r \in R \) such that \( f(n) - rn \in AM \) for each \( n \in N \). An \( R \)-module \( M \) is said to satisfy Baer criterion relative to \( A \), if each submodule of \( M \) satisfies Baer criterion relative to \( A \) and it is proved that an \( R \)-module \( M \) satisfies Baer criterion relative to \( A \) for cyclic submodules, if and only if \( M \) is fully stable relative to \( A \) [1]. 

**Definition 1.5:** For a fixed positive integers \( n \) and \( m \), we say that an \( R \)-module \( M \) satisfies \( (m,n) \)-Baer criterion relative to an ideal \( A \) of \( R \), if for any \( n \)-generated submodule \( N \) of \( M^n \) and any \( R \)-homomorphism \( \theta : N \to M^n \) there exists \( t \in R \) such that \( \theta (x) - xt \in M^n A \) for each \( x \in N \). 

It is clear that if \( M \) satisfies \( (m,n) \)-Baer criterion relative to an ideal \( A \) then \( M \) satisfies \( (p,q) \)-Baer criterion relative to an ideal \( A \), \( \forall \ 1 \leq p \leq m \) and \( 1 \leq q \leq n \). 

**Proposition 1.6:** Let \( A \) be an ideal of \( R^{m_m} \) and \( M \) be an \( R \)-module such that \( \gamma_{R^n} (N \cap K) = \gamma_{R^n} (N) + \gamma_{R^n} (K) \) for each two \( n \)-generated submodule of \( M^n \). If \( M \) satisfies \( (m,1) \)-Baer criterion relative to \( A \). Then \( M \) satisfies \( (m,n) \)-Baer criterion relative to \( A \) for each \( n \geq 1 \). 

**Proof:** Let \( L = x_1 R + x_2 R + \ldots + x_n R \) be \( n \)-generated submodule of \( M^n \) and \( f : L \to M^n \) an \( R \)-homo morphism. We use induction on \( n \). It is clear that \( M \) satisfies \( (m,n) \)-Baer criterion, if \( n = 1 \). Suppose that \( M \) satisfies \( (m,n) \)-Baer criterion for all \( k \)-generated submodule of \( M^n \), for \( k \leq n - 1 \). Write \( N = x_1 R, K = x_2 R + \ldots + x_n R \), then for each \( w_1 \in N \) and \( w_2 \in K \), \( f|N (w_1) = w_1 r, f|K (w_2) = w_2 s \) for some \( r, s \in R \). It is clear
r - s ∈ γ_r (N ⊕ K) = γ_r (N) + γ_r (K). Suppose that r-s = u+v with u ∈ γ_r (N), v ∈ γ_r (K) and let t = r - u = s +v. Then for any w = w_1+w_2 ∈ L with w_1 ∈ N and w_2 ∈ K, f(w) - wt = f(w_1) + f(w_2) - (w_1 + w_2)t = f(w_1)-w_1t + f(w_2) - w_2t = f(w_1) - w_1(r - u) + f(w_2) - w_2(s+v) = f(w_1) - w_1r + w_1u + f(w_2) - w_2s - w_2v = f(w_1) - w_1r + f(w_2) - w_2s ∈ M^n A.

**Proposition 1.7:** Let M be an R-module and A be an ideal of R. Then M satisfies (m,n)-Baer criterion relative to an ideal A, if and only if \( \ell_{M^n} \gamma_{R^n} (\alpha_1 R, \ldots, \alpha_n R) \subseteq \alpha_1 R + \ldots + \alpha_n R + M^n A \) for any n-element subset \{ \alpha_1, \ldots, \alpha_n \} of \( M^n \).

**Proof:** First assume that (m,n)-Baer criterion relative to an ideal A holds for n-generated submodule of \( M^m \), let \( \alpha_i = (a_{i1}, a_{i2}, \ldots, a_{im}) \), for each \( i = 1, \ldots, n \) and \( \beta = (\beta_1, \ldots, \beta_n) \in \ell_{M^n} \gamma_{R^n} (\alpha_1 R + \ldots + \alpha_n R) \). Define \( \theta: \alpha_1 R, \ldots, \alpha_n R \rightarrow M^n \) by \( \theta (\sum_{i=1}^{n} \alpha_i r_i) = \sum_{i=1}^{n} \beta_i r_i \). If \( \sum_{i=1}^{n} \alpha_i r_i = 0 \), then \( \sum_{i=1}^{n} a_{ij} r_i = 0 \), \( j = 1, \ldots, m \), this implies that \( \alpha_i r_i = 0 \) where \( r = (r_1, \ldots, r_n) \) and hence \( r^T \in \gamma_{R^n} (\alpha_1 R, \ldots, \alpha_n R) \). By assumption \( \beta_i, r_i^T = 0 \), \( \forall i = 1, \ldots, n \) so \( \sum_{i=1}^{n} \beta_i r_i = 0 \). This show that \( \theta \) is well defined. It is an easy matter to see that \( \theta \) is R-homomorphism. By assumption there exists \( t \in R \) such that \( \theta (\sum_{i=1}^{n} \alpha_i r_i) - (\sum_{i=1}^{n} \alpha_i r_i)t \in M^n A \) for each \( \sum_{i=1}^{n} \alpha_i r_i \in \sum_{i=1}^{n} \alpha_i R \). Let \( r_1 = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n \) where 1 in the \( i \)th position and 0 otherwise. \( \beta_i - \alpha_i t = \theta (\alpha_i) - \alpha_i t \in AM^n \) thus \( \beta_i \in \sum_{i=1}^{n} \alpha_i R + AM^n \) which is contradiction. This implies that \( \ell_{M^n} \gamma_{R^n} (\alpha_1 R + \ldots + \alpha_n R) \supseteq \alpha_1 R + \ldots + \alpha_n R + M^n A \). Conversely, assume that \( \ell_{M^n} \gamma_{R^n} (\alpha_1 R + \ldots + \alpha_n R) \subseteq \alpha_1 R + \ldots + \alpha_n R + M^n A \), for each \{ \alpha_1, \ldots, \alpha_n \} of \( M^n \). Then for each R-homomorphism \( f: \alpha_1 R + \ldots + \alpha_n R \rightarrow M^n \) and \( s = (s_1, \ldots, s_n) \in \gamma_{R^n} (\alpha_1 R + \ldots + \alpha_n R) \), \( \sum_{k=1}^{n} (\sum_{i=1}^{n} \alpha_i r_i) s_k = 0 \) for each \( \sum_{i=1}^{n} \alpha_i r_i \in \sum_{i=1}^{n} \alpha_i R \), hence \( \sum_{i=1}^{n} f (\sum_{i=1}^{n} \alpha_i r_i) s_k = 0 \), thus \( f (\sum_{i=1}^{n} \alpha_i r_i) \in \ell_{M^n} \gamma_{R^n} (\alpha_1 R + \ldots + \alpha_n R) \).

**Corollary 1.8:** An R-module M is fully (m,n)-stable relative to an ideal A of \( R^{nm} \), if and only if \( \ell_{M^n} \gamma_{R^n} (\alpha_1 R + \ldots + \alpha_n R) \subseteq \alpha_1 R + \ldots + \alpha_n R + M^n A \) for any n-element subset \{ \alpha_1, \ldots, \alpha_n \} of \( M^n \).

We can summarize the above results in the following theorem.

**Theorem 1.9:** The following statements are equivalent for an R-module M and an ideal A of R.
1. M is fully (m,n)-stable relative to A.
2. For any two m-element subsets \{ \alpha_1, \ldots, \alpha_m \} and \{ \beta_1, \ldots, \beta_m \} of \( M^n \), if \( \beta_j \notin \sum_{i=1}^{n} \alpha_i R + M^n A \), for each \( j = 1, \ldots, m \) implies \( \gamma_{R^n} \{ \alpha_1, \ldots, \alpha_m \} \nsubseteq \gamma_{R^n} \{ \beta_1, \ldots, \beta_m \} \).
3. (m,n)-Baer criterion relative to A for n-generated submodules of M^n.
4. \( \ell_{M^n} \gamma_{R^n} (\alpha_1 R + \ldots + \alpha_n R) \subseteq \alpha_i R + \ldots + \alpha_n R + M^n A \) for any n-element subset \( \{ \alpha_1, \ldots, \alpha_n \} \) of M^n.

**Corollary 1.10:** [1] The following statements are equivalent for an R-module M and an ideal A of R.
1. M is fully-stable relative to A.
2. For each x, y in M, \( y \in \gamma_R (x) \Rightarrow y + MA = (y) + MA \).
3. M satisfies Baer criterion to A for each cyclic submodule.
4. For each x in M, \( 1_M (\gamma_R (x)) \subseteq (x) + \text{AM} \).

Recall that an R-module M is (m,n)-multiplication module if each n-generated submodule of M^n is of the form \( M_n I \) for some ideal I of \( R^{\alpha m} \).

**Proposition 1.11:** Let M be an (m,n)-multiplication R-module. Then M is fully (m,n)-stable module if and only if M is fully (m,n)-stable relative to each non-zero ideal of \( R^{\alpha m} \).

**Proof:** \( \Rightarrow \) It is clear.
\( \Leftarrow \) Let N be any n-generated submodule of M^n and f : N → M^n be any R-homomorphism. If N = \{0\}, then it is clear that M is fully (m,n)-stable relative to ideal. Let N \( \neq \{0\} \), and since M is an (m,n)-multiplication module, then M = M_n I, for some non-zero ideal I of \( R^{\alpha m} \). By hypothesis f(N) \( \subseteq N + IM_n = N + N = N \). Hence, M is fully (m,n)-stable module.

**Corollary 1.12:** [1] Let M be multiplication R-module. Then M is fully stable module if and only if M is fully stable relative to each non-zero ideal of R.

Recall that an R-module M is (m,n)-quasi-injective in each R-homomorphism from an n-generated submodule of M^n to M extends to one from M^n to M [4].

The following theorem follows from Theorem (2.14) in [5] and Proposition (1.11).

**Theorem 1.13:** Let M be an (m,n)-multiplication R-module. Then M is (m,n)-quasi injective if and only if M is fully (m,n)-stable relative to each non-zero ideal of \( R^{\alpha m} \).

Now we introduce the concept of (m,n)-quasi injective module relative to an ideal A of \( R^{\alpha m} \).

**Definition 1.14:** An R-module M is called (m,n)-quasi injective relative to an ideal A of \( R^{\alpha m} \) if for every R-homomorphism \( g : N → M^n \) where N is n-generated submodule of M^n and R-homomorphs f : N → M there exists R-homomorphism \( h : M^n → M \) such that \( fg(x) - h(x) \in \alpha M A \) for each x in N.

**Proposition 1.15:** If M is a fully (m,n)-stable R-module relative to an ideal A of \( R^{\alpha m} \), then M is (m,n)-quasi injective relative to A.

**Proof:**
Let \( N = \alpha_1 R + \ldots + \alpha_n R \) be n-generated submodule of M^n where \( \alpha_i \in M^n \) and f : N → M^n be any R-homomorphism. Since M is a fully (m,n)-stable module relative to A, then f(\( \alpha_1 R + \ldots + \alpha_n R \)) \( \subseteq \alpha_i R + \ldots + \alpha_n R + MA \), thus there exist \( s = (s_1, \ldots, s_n) \) \( \in R^n \) and \( w \in \alpha M A \). Let \( r_i = (0, \ldots, 1_i, \ldots, 0) \) such that \( f(\sum_{i=1}^n \alpha_i) \) = \( \sum_{i=1}^n \alpha_i \) + w. Define \( g : M^n → M \) by \( g(\alpha_i) = \alpha_i s^T \), it is clear that g is a well defined R-homomorphism. Now \( f(\sum_{i=1}^n \alpha_i) - g(\sum_{i=1}^n \alpha_i) = (\sum_{i=1}^n \alpha_i)s + w - (\sum_{i=1}^n \alpha_i)s = w \in \alpha M A \) and since for
each \( y \in \alpha, R + \ldots + \alpha_n R \), \( y = \sum_{i=1}^{n} \alpha_i t_i \)
for some \( t= (t_1, \ldots, t_n) \in R \), \( f(y) - g(y) = f(\sum_{i=1}^{n} \alpha_i) - g(\sum_{i=1}^{n} \alpha_i) \).

The following theorem follows from Theorem (1.13) and Proposition (1.115).

**Theorem 1.16:** If \( M \) is \((m,n)\)-quasi injective \( R \)-module then \( M \) is \((m,n)\)-quasi injective relative to an ideal \( A \) of \( R \).

The following theorem follows from Theorem (1.13) and Proposition (1.115).

**Theorem 1.16:** If \( M \) is \((m,n)\)-quasi injective \( R \)-module then \( M \) is \((m,n)\)-quasi injective relative to an ideal \( A \) of \( R \).

References: