A Solution of Second Kind Volterra Integral Equations Using Third Order Non-Polynomial Spline Function

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Received 1 April, 2014
Accepted 14 May, 2014

Abstract:
In this paper, third order non-polynomial spline function is used to solve second kind Volterra integral equations. Numerical examples are presented to illustrate the applications of this method, and to compare the computed results with other known methods.

Keywords: Volterra integral equation, non-polynomial spline function, cubic spline function.

Introduction:
Many problems of mathematical physics can be started in the form of integral equations. These equations also occur as reformulations of other mathematical problems such as partial differential equations and ordinary differential equations. Numerical simulation in engineering science and in applied mathematics has become a powerful tool to model the physical phenomena, particularly when analytical solutions are not available then very difficult to obtain. Therefore, the study of integral equations and methods for solving them are very useful in application. In recent years, there has been a growing interest in the Volterra integral equations arising in various fields of physics and engineering [1], Lima, P. and Diogo, T. in(1997) [2] presented an extrapolation method to find numerical solution of VIE's with weakly singular kernel .Rashidinia, J. and Zarebnia, M. in(2008) [3] used sinc function method to find the numerical solution of linear VIE's of the second kind. Bizar ,J. and Eslami , M. in(2011) [4] presented Homotopy Perturbation and Taylar series method for solving VIE's of second kind . Maleknejad, K. Hashmizadeh, E.andEzzati, R. in(2012)[5] studied a new approach to find the numerical solution of VIE's by using Bernsteins Approximation.


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Chebyshev polynomial for solving Volterra second kind.

**Third order Non-polynomial Spline Function:**

Let the linear Volterra integral equation (VIE) of the second kind be in the form

\[ u(x) = f(x) + \int_a^x k(x,t)u(t)dt, \quad a \leq x \leq b \quad (1) \]

Consider the partition \( \Delta = \{ t_0, t_1, t_2, ..., t_n \} \subset \mathbb{R} \). Let \( S(\Delta) \) denote the set of piecewise continuous polynomials on subinterval \( I_i = [t_i, t_{i+1}] \) of partition \( \Delta \). In this work, third order non-polynomial spline function will be used for finding approximate solution of VIE’s of the second kind. Consider the grid point \( t_i \) on the interval \([a,b]\) as follows:

\[
a = t_0 < t_1 < t_2 < ... < t_n = b \quad (2)
\]

\[
t_i = t_0 + ih, \quad i = 0, 1, ..., n \quad (3)
\]

\[
h = \frac{b-a}{n} \quad (4)
\]

where \( n \) is a positive integer. The suggested third order non-polynomial spline function is:

\[ P_i(t) = a_i \cos k(t - t_i) \]
\[ + b_i \sin k(t - t_i) \]
\[ + c_i (t - t_i) + d_i (t - t_i)^2 + e_i (t - t_i)^3 + m_i \quad (5) \]

where \( a_i, b_i, c_i, d_i, e_i \) and \( m_i \) are constants to be determined, and \( k \) is the frequency of the trigonometric functions which will be used to raise the accuracy of the method.

Let \( u(t) \) be the exact solution of equation (1) and \( S(t) \) be an approximate to \( u = u(t) \) obtained by the segment \( P_i(t) \).

The following relations must be satisfied:

\[ P_i(t_i) = a_i + m_i = u(t_i) \approx S_i(t_i) \]
\[ P_i'(t_i) = b_i + c_i = u'(t_i) \approx S_i'(t_i) \]
\[ P_i''(t_i) = -k^2a_i + 2d_i = u''(t_i) \approx S_i''(t_i) \]

\[ p^{(m)}_i(t_i) = -k^3b_i + 6e_i = \]
\[ u^{(m)}(t_i) \approx S_i^{(m)}(t_i) \]
\[ P_i^{(m)}(t_i) = k^4a_i = u^{(m)}(t_i) \approx S_i^{(m)}(t_i) \]

then we can obtain the values of \( a_i, b_i, c_i, d_i, e_i \) and \( m_i \) as follows:

\[ a_i = \frac{1}{k^2} u''(t_i) \approx S_i''(t_i) \quad (6) \]
\[ b_i = \frac{1}{k^2} u'''(t_i) \approx S_i'''(t_i) \quad (7) \]
\[ c_i = u'(t_i) - k b_i \approx S_i'(t_i) - k b_i \quad (8) \]
\[ d_i = \frac{1}{k^2} u''(t_i) + k^2 a_i \approx \]
\[ 1/2[S_i''(t_i) + k^2 a_i] \quad (9) \]
\[ e_i = d_i = \frac{1}{6}[u'''(t_i) + k^2 b_i] \approx \]
\[ 1/6[S_i'''(t_i) + k^3 b_i] \quad (10) \]
\[ m_i = u(t_i) - a_i \approx S_i(t_i) - a_i \quad (11) \]

for \( i=0,1,...,n \).

**Method of Solution:**

To solve the linear VIE’s of the second kind eq.(1), we differentiate it five times with respect to \( x \) and evaluate them at \( x=a \):

\[ u_0 = u(a) = f(a) \quad (12) \]
\[ u_0' = u'(a) = f'(a) + k(a,a)u(a) \quad (13) \]
\[ u_0'' = u''(a) = f''(a) + \left( \frac{\partial k(x,t)}{\partial x} \right)_{t=x} \]
\[ u(a) + \left( \frac{dk(x,t)}{dx} \right)_{x=a} u(a) + \]
\[ k(a,a)u(a) \quad (14) \]
\[ u_0''' = u'''(a) = f'''(a) + \left( \frac{\partial^2 k(x,t)}{\partial x^2} \right)_{t=x} \]
\[ u(a) + \left( \frac{d}{dx} \left( \frac{\partial k(x,t)}{\partial x} \right)_{t=x} \right)_{x=a} u'(a) + \]
\[ \frac{dk(x,t)}{dx} \left|_{x=a} \right. u(a) + \]
\[ 2 \left( \frac{dk(x,t)}{dx} \right)_{x=a} u'(a) + \right)_{x=a} u''(a) \quad (15) \]
\[ u_0^{(4)} = u^{(4)}(a) \]
\[ = f^4(a) + \left[ \frac{\partial^3 k(x,t)}{\partial x^3} \right]_{t=x=a} u(a) + \left[ \frac{d}{dx} \left( \frac{\partial^2 k(x,t)}{\partial x^2} \right) \right]_{t=x=a} u(a) \]
\[ + \left( \frac{\partial^2 k(x,t)}{\partial x^2} \right)_{t=x=a} u'(a) + \frac{d^2}{dx^2} \left[ \frac{\partial k(x,t)}{\partial x} \right]_{t=x=a} u''(a) \]
\[ + 2 \left[ \frac{d}{dx} \left[ \frac{\partial k(x,t)}{\partial x} \right] \right]_{t=x=a} u'(a) + \left( \frac{\partial k(x,t)}{\partial x} \right)_{t=x=a} u''(a) \]
\[ + \left( \frac{d^3 k(x,t)}{dx^3} \right)_{t=x=a} u(a) + 3 \left( \frac{d^2 k(x,t)}{dx^2} \right)_{t=x=a} u'(a) + 3 \left( \frac{dk(x,t)}{dx} \right)_{t=x=a} u''(a) \]
\[ + k(a,a)u'''(a) \]  
\[ \ldots (16) \]

\[ u_0^{(5)} = u^{(5)}(a) = f^5(a) + \left[ \frac{\partial^4 k(x,t)}{\partial x^4} \right]_{t=x=a} u(a) + \left[ \frac{d}{dx} \left( \frac{\partial^3 k(x,t)}{\partial x^3} \right) \right]_{t=x=a} u(a) + \]
\[ \left( \frac{\partial^3 k(x,t)}{\partial x^3} \right)_{t=x=a} u'(a) + \frac{d^2}{dx^2} \left[ \frac{\partial^2 k(x,t)}{\partial x^2} \right]_{t=x=a} u''(a) + \]
\[ 2 \left[ \frac{d}{dx} \left[ \frac{\partial^2 k(x,t)}{\partial x^2} \right] \right]_{t=x=a} u'(a) + \left( \frac{\partial^2 k(x,t)}{\partial x^2} \right)_{t=x=a} u''(a) + \]
\[ \frac{d^3}{dx^3} \left[ \frac{\partial k(x,t)}{\partial x} \right]_{t=x=a} u''(a) + \left( \frac{\partial k(x,t)}{\partial x} \right)_{t=x=a} u'''(a) + \frac{d^4 k(x,t)}{dx^4} \right]_{t=x=a} u(a) + \]
\[ 4 \left[ \frac{d^3 k(x,t)}{dx^3} \right]_{t=x=a} u'(a) + 6 \left[ \frac{d^2 k(x,t)}{dx^2} \right]_{t=x=a} u''(a) + 4 \left( \frac{dk(x,t)}{dx} \right)_{t=x=a} u'''(a) + \]
\[ k(a,a)u''''(a) \]  
\[ \ldots (17) \]

Therefore, we approximate the solution of equation (1) using equation (5) in the following algorithm (VIENPS):

Algorithm (VIENPS)

To find the approximate solution of eq.(1), first we select positive integer n, and perform the following steps:

**Step 1:** Set \( h=(b-a)/n \); \( t_i = t_0 + ih, i = 0,1, \ldots, n \) \( t_0 = a, t_n = b \). \( u_0 = f(a) \)

**Step 2:** Evaluate \( a_0, b_0, c_0, d_0, e_0 \) and \( m_0 \) by substituting (12)-(17) in equations(6)-(11).

**Step 3:** Calculate \( P_i(t_i) \) using step 2 and equation (5) for \( i=0 \).

**Step 4:** Approximate \( u_1 \approx P_1(t_1) \).

**Step 5:** For \( i=1 \) to \( n-1 \) do the following steps:

**Step 6:** Evaluate \( a_i, b_i, c_i, d_i, e_i \) and \( m_i \) using equations (6)-(11) and replacing \( u(t_i) \) and its derivatives by \( P_i(t_i) \) and its derivatives.

**Step 7:** Calculate \( P_i(t) \) using step 6 and equation (5).

**Step 8:** Approximate \( u_{i+1} = P_i(t_{i+1}) \).

Numerical Examples:

Example (1): Consider the VIE of the second kind [10]:

\[ u(x) = x^3 + \int_0^x -3x-tu(t)dt \]  
\[ 0 \leq x \leq 1 \]

with exact solution \( u(x) = 3x(1 - e^{-x}) \). Results have been shown in Table 1, where \( P_i(x) \) denote the approximate solution by the proposed method and \( \text{err} = |u(x) - P_i(x)| \).
Table 1: Computed Absolute Error of Example (1) and The Result Obtained in [10]

<table>
<thead>
<tr>
<th>x</th>
<th>Exact Solution u(x)</th>
<th>P_1(x)</th>
<th>Error</th>
<th>error obtain in [10]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.106213163030966</td>
<td>0.106213158495688</td>
<td>4.535278860795522e-09</td>
<td>1.1600483e-002</td>
</tr>
<tr>
<td>0.2</td>
<td>0.225812709291563</td>
<td>0.225812415451626</td>
<td>2.94775368409855e-07</td>
<td>2.8068994e-002</td>
</tr>
<tr>
<td>0.3</td>
<td>0.36036539107344</td>
<td>0.36036029707884</td>
<td>3.409399463250096e-06</td>
<td>2.2232608e-002</td>
</tr>
<tr>
<td>0.4</td>
<td>0.511612377368213</td>
<td>0.51159298966171</td>
<td>1.9448402171072e-05</td>
<td>1.0103823e-002</td>
</tr>
<tr>
<td>0.5</td>
<td>0.68150888898327</td>
<td>0.68143357054314</td>
<td>7.531054091380884e-05</td>
<td>1.7283779e-002</td>
</tr>
<tr>
<td>0.6</td>
<td>0.872229243985166</td>
<td>0.872001001385275</td>
<td>2.282425998107094e-04</td>
<td>6.5041788e-003</td>
</tr>
<tr>
<td>0.7</td>
<td>1.086202425097018</td>
<td>1.085618339803819</td>
<td>5.840854477256549e-05</td>
<td>8.3481474e-003</td>
</tr>
<tr>
<td>0.8</td>
<td>1.326139582081997</td>
<td>1.324818967967492</td>
<td>1.320618339803819</td>
<td>5.7238171e-003</td>
</tr>
<tr>
<td>0.9</td>
<td>1.59580668010478</td>
<td>1.592350411094480</td>
<td>2.71638950296438e-03</td>
<td>1.187893e-003</td>
</tr>
<tr>
<td>1.0</td>
<td>1.896361676584567</td>
<td>1.891176122008416</td>
<td>5.18554477256549e-03</td>
<td>1.830649e-002</td>
</tr>
</tbody>
</table>

‖err‖_∞ = 5.18554477256549e-03, 2.8608994e-002

Example 2: Consider the VIE of the second kind [3]:

\[
u(x) = 1 - x + \frac{x^2}{2} + \int_0^x (t - x) u(t) dt
\]

With exact solution \(u(x) = 1 - \sin(x)\). Results have been shown in Table 2, where \(P_1(x)\) denote the approximate solution by the proposed method.

Table 2: Computed Absolute Error of Example (2) and The Result Obtained in [3]

<table>
<thead>
<tr>
<th>x</th>
<th>Exact Solution u(x)</th>
<th>P_1(x)</th>
<th>Error</th>
<th>error obtain in [3]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>9.00166583351718e-01</td>
<td>9.00166583351718e-01</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>0.2</td>
<td>8.01330669204388e-01</td>
<td>8.01330669204388e-01</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>0.3</td>
<td>7.04479793386604e-01</td>
<td>7.04479793386604e-01</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>0.4</td>
<td>6.105816576913494e-01</td>
<td>6.105816576913494e-01</td>
<td>2.22046409250313e-16</td>
<td>-</td>
</tr>
<tr>
<td>0.5</td>
<td>5.205744613957970e-01</td>
<td>5.205744613957970e-01</td>
<td>1.110223024625157e-16</td>
<td>-</td>
</tr>
<tr>
<td>0.6</td>
<td>4.35357526049646e-01</td>
<td>4.35357526049646e-01</td>
<td>1.110223024625157e-16</td>
<td>-</td>
</tr>
<tr>
<td>0.7</td>
<td>3.557823127623090e-01</td>
<td>3.557823127623090e-01</td>
<td>1.110223024625157e-16</td>
<td>-</td>
</tr>
<tr>
<td>0.8</td>
<td>2.826439091004772e-01</td>
<td>2.826439091004772e-01</td>
<td>2.22046409250313e-16</td>
<td>-</td>
</tr>
<tr>
<td>0.9</td>
<td>2.16730903725166e-01</td>
<td>2.16730903725166e-01</td>
<td>3.306690973875470e-16</td>
<td>-</td>
</tr>
<tr>
<td>1.0</td>
<td>1.585290151921035e-01</td>
<td>1.585290151921035e-01</td>
<td>4.44089208950626e-16</td>
<td>-</td>
</tr>
</tbody>
</table>

‖err‖_∞ = 4.44089208950626e-16, 3.6208210e-04

Conclusion:

In this paper, non-polynomial spline function method for solving Volterra integral equations of the second kind is presented successfully. This idea based on the use of the VIE's and its derivatives. So it is necessary to mention that this approach can be used when \(f(x)\) and \(k(x,t)\) are analytic. The proposed scheme is simple and computationally attractive and their accuracy are high and we can execute this method in a computer simply. The numerical examples support this claim, and fig. (1,2) are plotted to show the comparison between the exact and approximate solution of these examples.
References:
حل لمعادلات فولتيرا التكاملية من النوع الثاني باستخدام دالة الثلمة الغير متعددة الحدود من الدرجة الثالثة

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الخلاصة:
في هذا البحث تم استخدام دالة الثلمة الغير متعددة الحدود من الدرجة الثالثة لإيجاد حل عددي تقريبي لمعادلات فولتيرا التكاملية من النوع الثاني. تم إعطاء أمثلة عدديات لتوضيح تطبيق الطريقة، كما تم مقارنة النتائج مع طرق أخرى معروفة.

الكلمات المفتاحية: معادلة فولتيرا التكاملية، دالة الثلمة الغير متعددة الحدود، دالة الثلمة من الدرجة الثالثة.