Free intersection space (F-space)

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Submitted: 10/01/2011 Accepted: 29/05/2011

ABSTRACT

In this work we introduce the concept of free intersection space with basic definitions and some properties for this space.

INTRODUCTION

A topology on a set X is a collection $T$ of subsets of X, called the open sets, satisfying:

1. Any union of elements of $T$ belongs to $T$,
2. Any finite intersection of elements of $T$ belongs to $T$,
3. $\emptyset$ and $X$ belong to $T$.

We say $(X, T)$ is a topological space [1].

There is another structure called m-structure space defined as follows

Let X be a nonempty set a collection $T$ of subsets of X is called minimal structure on X (m-structure) if $\emptyset$ and X belong to $T$.[2]

Clear that every topological space is m-structure space. The above two structures play as principal structures for many others structures for example a bitopological space is two collections $T_1$ and $T_2$ of subsets of $X$ such that $(X, T_1)$ and $(X, T_2)$ are topological spaces ,a set $A$ called quasi open if $A=W \cup V$ where $W$ belong to $T_1$ and $V$ belong to $T_2$. In all these structures the intersection of any collection of open sets not necessary open set this means in topological space the kernel of a set $A = \cap \{ u : A \subseteq u, u \text{open set} \}$ [3]is not necessary open set ,so the quasi kernel of a set $A = \cap \{ u : A \subseteq u, u \text{quasi open set} \}$ [4] is not necessary quasi open set.
In this work we define a new concept (as we know) called a free intersection space as follows let X be a nonempty set a collection T of subsets of X is called a free intersection space on X (F-space) if:
1. X belong to T
2. Any intersection of element of T belong to T

This idea comes from the fact (in group structure) the intersection of any collection of subgroups is a subgroup and the group itself a subgroup, more than the intersection of any collection of closed sets is closed set. It is clear that the collection of closed sets (in topological space) is F-space and we show that the converse is not true in general.

The property (2) implies that the kernel belongs to T, so the kernel of a set or element will be important to define basic definitions in this structure.

1. Structure of F-space

Definition(1.1) Let X be a non-empty set a subcollection of \( P(X) \) is called free intersection space (F-structure space) simply (F-Space) if \( \mathcal{F} \) satisfies the following:
1. X belong to \( \mathcal{F} \)
2. Any intersection of element in \( \mathcal{F} \) belong to \( \mathcal{F} \).

Definition(1.2) Each member in \( \mathcal{F} \) is called F-open and the complement of F-open is called F-closed.

The following example (1.3.1) shows that the F-Space is not collection of closed sets (for some topological space), by other word the complement of it is sets not topology on X.

Examples (1.3)
1. Let \( X = \{1,2,3,4,5\} \) and \( \mathcal{F} = \{X, \{1\}, \{1,2,3\}, \{1,4\}, \{1,5\}\} \), then \((X, \mathcal{F})\) is F-space.
2. Let G be a group and \( \mathcal{F} \) the set of all subgroups of G, then \((G, \mathcal{F})\) is F-Space called F-space induced by group.
3. Let M be an R-module and \( \mathcal{F} \) be the set of all submodules of M, then \((M, \mathcal{F})\) is an F-space called F-space induced by module.
4. Let \((X,T)\) be a topological space and \( \mathcal{F} \) be the set of all closed sets in X, then \((X, \mathcal{F})\) is an F-space called F-space induced by topology.

Definition(1.4) Let \((X, \mathcal{F})\) be an F-space and A be a subset of X, then:
1. The set of intersection of all F-open which contains A is called F-kernel
of A and is denoted by \( \text{ker}(A) \).

2- The union of all F-closed which is contained in A is called interior closure of A and denoted by \( \text{Ic}(A) \).

3- The union of all F-open which is contained in A is called F-interior of A and denoted by \( \text{F-Int}(A) \).

**Remark (1.5)**

By definition of F-space and F-kernel, the F-kernel of any set is F-open and it is smallest F-open contains A (that is any F-open contains A must contain \( \text{ker}(A) \)).

**Lemma (1.6)** Let \((X, \tau)\) be an F-space and A be a subset of X, then \( \text{Ic}(A) \) is F-closed.

**Proof:** since \( \text{Ic}(A) = \bigcup \{ G_i : i \in \lambda \} \) such that \( G_i \subseteq A \) and \( G_i \) is F-closed \( \forall \ i \in \lambda \), then \( \text{Ic}(A) = \bigcup \{ (X-U_i) : i \in \lambda \} \) such that \( X-U_i = G_i \) and \( U_i \) is F-open and \( \text{Ic}(A) = X-(\bigcap U_i) \), then \( \bigcap U_i \) is F-open by definition of F-space, then \( \text{Ic}(A) \) is F-closed.

This means \( \text{Ic}(A) \) is largest F-closed set contained in A.

**Theorem (1.7)** Let \((X, \tau)\) be an F-space and A,B are subsets of X, then

1- \( \text{ker}(A) = A \) iff A is F-open.

2- \( \text{Ic}(A) = A \) iff A is F-closed.

3- \( \text{Ic}(X-A) = X-\text{ker}(A) \)

4- \( X-\text{Ic}(A) = \text{ker}(X-A) \)

5- if \( A \subseteq B \), then \( \text{ker}(A) \subseteq \text{ker}(B) \)

6- \( \text{ker}(A \cap B) \subseteq \text{ker}(A) \cap \text{ker}(B) \)

7- \( \text{ker}(A) \cup \text{ker}(B) \subseteq \text{ker}(A \cup B) \)

**Proof:**

1- Since ker(A) is F-open, then A is F-open set.

Conversely: since A \( \subseteq A \) and A is F-open then by remark (1-5) \( \text{ker}(A) \subseteq A \) but \( A \subseteq \text{ker}(A) \) by the same remark, then \( \text{ker}(A) = A \).

2-since \( \text{Ic}(A) \) is F-closed, then A is F-closed.

Conversely: Since A \( \subseteq A \), then A \( \subseteq \text{Ic}(A) \), but \( \text{Ic}(A) \subseteq A \), then \( \text{Ic}(A) = A \).

3- \( \text{X-ker}(A) = X-\bigcap \{ U_i : i \in \lambda, A \subseteq U_i \} \) is F-open

\[ = \bigcup \{ X- U_i : i \in \lambda, A \subseteq U_i \} \text{ is F-open} \]
\[ = \bigcup \{ X- U_i : i \in \lambda, X-A \subseteq X-U_i \} \text{ is F-open} \]
\[ = \text{Ic}(X-A) \]
4- \( \text{X-Ic}(A) = \text{X-} \bigcup \{G_i : i \in \lambda, G_i \subseteq A, G_i \text{ is F-closed}\} \)
   = \bigcap \{\text{X-G}_i : i \in \lambda, G_i \subseteq A, G_i \text{ is F-closed}\}
   = \bigcap \{\text{X-G}_i : i \in \lambda, X-A \subseteq \text{X-G}_i, G_i \text{ is F-closed}\}
   = \ker(X-A)

5- \( \text{Ker}(B) = \bigcap \{U_i : i \in \lambda, U_i \text{ is F-open}\} \), for any \( i \in \lambda \) \( B \subseteq U_i \) (by def. of kernel)
   Now \( A \subseteq B \subseteq U_i \) for any \( i \in \lambda \), then \( A \subseteq \text{Ker}(B) \), but \( \text{Ker}(B) \) is F-open set,
   then \( \text{Ker}(A) \subseteq \text{Ker}(B) \) by (1-5)

6- Since \( A \cap B \subseteq A \) and \( A \cap B \subseteq B \), then by (5) \( \text{Ker}(A \cap B) \subseteq \text{Ker}(A) \) and
   \( \text{Ker}(A \cap B) \subseteq \text{ker}(B) \), therefore \( \text{Ker}(A \cap B) \subseteq \text{ker}(A) \cap \text{Ker}(B) \).

7- Since \( A \subseteq A \cup B \) and \( B \subseteq A \cup B \), then by (5) \( \text{Ker}(A) \subseteq \text{Ker}(A \cup B) \) and
   \( \text{Ker}(B) \subseteq \text{Ker}(A \cup B) \) therefore \( \text{Ker}(A) \cup \text{Ker}(B) \subseteq \text{Ker}(A \cup B) \).

**Definition (1.8)** Let \((X, \tau)\) be F-space and \( A \subseteq X, y \in X \),
1- \( y \) is called F-interior element of \( A \) in case \( \text{Ker}(y) \subseteq A \).
2- \( y \) is called F-external element of \( A \) in case \( \text{Ker}(y) \subseteq X-A \).
3- \( y \) is called F-frontier element if \( \text{Ker}(y) \cap A \neq \phi \) and \( \text{Ker}(y) \cap (X-A) \neq \phi \)
   \( (y \) is not F-interior point of \( A \) and not F-external point of \( A \)).

4- \( A \) is called locally F-open set if \( A \) is the union of F-open sets that is
   \( A = \bigcup \{U_i : U_i \text{ is F-open } i \in \lambda\} \).

5- \( \text{Ker}(A) \cap \text{Ker}(X-A) \) is called open frontier of \( A \) and is denoted by \( \text{OFr}(A) \).

**Theorem (1.9)** Let \((X, \tau)\) be F-space and \( A \subseteq X \), the set \( A \) is locally F-open
if and only if \( A = \bigcup \{\text{Ker}(x) : x \in A\} \).

**Proof:** suppose \( A \) is locally F-open then \( A = \bigcup \{U_i : U_i \text{ is F-open } i \in \lambda\} \).
   Let \( x \in A \), then \( x \in U_i \) for some \( i \in \lambda \), that is \( \text{Ker}(x) \subseteq U_i \).
   Now \( A \subseteq \bigcup \{\text{Ker}(x) : x \in A\} \subseteq \bigcup \{U_i : U_i \text{ is F-open } i \in \lambda\} = A \), then
   \( A = \bigcup \{\text{Ker}(x) : x \in A\} \).

**Conversely:** Since \( \text{Ker}(A) \) is F-open set, then \( A \) is locally F-open.

The above theorem means that every element in any locally F-open set is F-interior element (i.e. F-int(A) = A).
**Corollary (1.10)** Every F-open set is locally F-open.

**Proof:** Let A be an F-open set. Since \( \{x\} \subseteq A \), then \( \text{Ker}(x) \subseteq A \), but \( A \subseteq \bigcup \{\text{Ker}(x): x \in A\} \), then \( A = \bigcup \{\text{Ker}(x): x \in A\} \) and by theorem (1-9) A is locally F-open. ■

The converse of corollary (1-10) is not true in general to show that see the following example

**Example (1.11)** Let \( (Z_6, \mathfrak{Z}) \) be F-space induced by the group \( Z_6 \), then \( A = \{[0],[2],[3],[4]\} \) is locally F-open but A not F-open.

**Theorem (1.12)** Let \( (X, \mathfrak{Z}) \) be F-space and \( A \subseteq X \), then \( A \cup \text{OFr}(A) = \text{Ker}(A) \).

**Proof:**
\[
A \cup \text{OFr}(A) = A \cup (\text{Ker}(A) \cap \text{Ker}(X-A)) \\
= (A \cup \text{Ker}(A)) \cap (A \cap \text{Ker}(X-A)) \\
= \text{Ker}(A) \cap X = \text{Ker}(A). 
\]

■

**Definition (1.13)** Let \( (X, \mathfrak{Z}) \) be F-space and \( A \subseteq X \), \( a \in X \)

1. A is called F-dense set in case \( \text{Ker}(A) = X \).
2. a is called center element in case \( \text{Ker}(a) = X \).
3. a is called singular element in case \( \text{Ker}(a) = \{a\} \).

**Remark (1.14)**
1. Every set have center element is F-dense , since \( X = \text{Ker}(a) \subseteq \text{Ker}(A) \), then \( X = \text{Ker}(A) \), but the converse is not true to show that in example (1.11)
2. If A is F-dense set, then \( \text{OFr}(A) = \text{Ker}(X-A) \), since \( \text{Ker}(A) \cap \text{Ker}(X-A) = X \cap \text{Ker}(X-A) = \text{Ker}(X-A) \).
3. If a is center element, then
   a) a is not external element for any non empty set, since \( \text{Ker}(a) \cap A = X \cap A = A \neq \phi \)
   b) a is Frontier element for every non empty set, since \( \text{Ker}(a) \cap A = X \cap A = A \neq \phi \).

And \( \text{Ker}(a) \cap (X-A) = X \cap (X-A) = X-A \neq \phi \).
4. If a is singular element, then a is external element of every non-empty set contains a , since \( \text{Ker}(a) \cap A = \{a\} \cap A = \phi \) if \( a \notin A \).

**2. The continuity**

In this section X and Y means F-space
**Definition (2.1)** A function \( f: X \to Y \) is called F-continuous at a point \( a \) in \( X \) if \( f(Ker(a)) \subseteq Ker(f(a)) \) and \( f \) is called F-continuous on \( X \) if it is F-continuous on each \( a \) in \( X \).

**Theorem (2.2)** Let \( f:X \to Y \) be a function, then \( f \) is F-continuous if and only if for each locally F-open \( V \) in \( Y \) \( f^{-1}(V) \) is locally F-open in \( X \).

**Proof:** Suppose \( f \) is F-continuous. Let \( V \) be locally F-open in \( Y \) and \( a \in f^{-1}(V) \), then \( f(a) \in V \). Since \( V \) is locally F-open, then \( Ker(f(a)) \subseteq V \). Now \( f \) is F-continuous, then \( f(Ker(a)) \subseteq Ker(f(a)) \subseteq f^{-1}(V) \), but \( Ker(a) \subseteq f^{-1}(f(Ker(a))) \), thus \( Ker(a) \subseteq f^{-1}(V) \), therefore \( f^{-1}(V) \) is locally F-open.

Conversely: Let \( a \in X \), then \( f(a) \in Y \), since \( Ker(f(a)) \) is locally F-open, then \( f^{-1}(Ker(f(a))) \) is locally F-open. Now \( a \in f^{-1}(Ker(f(a))) \) is locally F-open, then \( Ker(a) \subseteq f^{-1}(Ker(f(a))) \), thus \( f(Ker(a)) \subseteq f(f^{-1}(Ker(f(a)))) \subseteq Ker(f(a)) \), therefore \( f \) is F-continuous function.

**Definition (2.3)** A function \( f: X \to Y \) is called strongly F-continuous in case for each F-open \( V \) in \( Y \), \( f^{-1}(V) \) is F-open in \( X \).

**Theorem (2.4)** Every strongly F-continuous function is F-continuous function.

**Proof:** Let \( f:X \to Y \) be strongly F-continuous and let \( V \) be locally F-open in \( Y \), then \( V = \bigcup \{ U_i : i \in \lambda \} \) where \( U_i \) is F-open for any \( i \in \lambda \). Now \( f^{-1}(V) = f^{-1}(U_i) = \bigcup f^{-1}(U_i) \), since \( f \) is strongly F-continuous, then \( f^{-1}(U_i) \) is F-open in \( X \) for any \( i \in \lambda \), thus \( f^{-1}(V) \) is locally F-open in \( X \), therefore \( f \) is F-continuous.

The converse of theorem (2.4) is not true to show that see the following example.

**Example (2.5)** Let \( X = \mathbb{Z}_6 \) and \( Y = \mathbb{Z}_4 \) and \( f:X \to Y \) defined by \( f([0]) = [0], f([1]) = [1], f([2]) = [2], f([3]) = [0], f([4]) = [2], f([5]) = [1] \), then \( f \) is F-continuous but not strongly F-continuous.

**Theorem (2.6)** A function \( f:X \to Y \) is strongly F-continuous iff for every F-closed \( G \) in \( Y \), \( f^{-1}(G) \) is F-closed in \( X \).

**Proof:** Let \( f \) be strongly F-continuous function and \( G \) is F-closed in \( Y \) then \( G = Y - U \) such that \( U \) is F-open in \( Y \), thus \( f^{-1}(G) = f^{-1}(Y - U) = X - f^{-1}(U) \). Since \( f \) is strongly F-continuous then \( f^{-1}(U) \) is F-open in \( X \) therefore \( f^{-1}(G) = X - f^{-1}(U) \) is F-closed in \( X \).
Conversely: Let $U$ be $F$-open in $Y$, thus $f^{-1}(U) = f^{-1}(Y-G)$ such that $G$ is $F$-closed, then $Y-U$ is $F$-closed in $Y$, therefore $f^{-1}(Y-U)$ is $F$-closed in $X$. hence $f^{-1}(Y-U)= X - f^{-1}(U)$ , $f^{-1}(U)$ is $F$-open in $X$ therefore $f$ is strongly $F$-continuous. ■

Example(2.7) Every homomorphism group ($R$-module) is strongly $F$-continuous.

REFERENCES


