The Degree of Best Approximation by the Linear Positive Operator in the Weight Spaces

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ABSTRACT

In this paper we shall use new operators $S_n(f, x), L_n(f, q, r, l, x)$, $n \in N$, $f \in L_p(X)$, in order to find the best approximation of the linear positive operators in a weighted spaces of functions by using weighted averaged modulus of smoothness in $L_p(X)$ spaces.

INTRODUCTION

The Szasz - Mirakyan operators are important in approximation on theory. They have been studied intensively, in connection with different branches of analysis, such as numerical analysis [1]. Let $R_0 = [0, \infty)$ and $f \in C[0, \infty)$. Szasz - Mirakyan operators denoted by $S_n(f, x)$ which are defined by (1.1) is a positive linear operators from the space $C_q$ into $C_p$ proved that $p > q > 0$ and $n > q > \ln(p/q)$.

Let $N_0 = \{0, 1, 2, \ldots\}$, for a fixed, $q \in N_0, x \in R_0 = [0, \infty)$ the weight function $\omega_q(x)$ is given by [2]:

$$\omega_0(x) = 1, \quad \omega_q(x) = (1 + x^q)^{-1} \quad \text{if} \quad q \geq 1$$

Let $C_q = \{f \in C[0, \infty) : \omega_q f \}$ is uniformly continuous and bounded on $[0, \infty)$. Let $m \in N, q \in R_0 = [0, \infty)$ be fixed numbers, and denote
The Degree of Best Approximation by the Linear Positive Operator in the Weight Spaces

Sahib and Zainab

\[ C^m_q = \{ f \in C_q : f^k \in C_q, k = 0,1,2,\ldots,m \} . \]

Let \( B_q = \{ f : f \text{ is real valued continuous functions on } f(x) \text{ on } R_0 = [0,\infty) \} \), such that \( w_q(x) x^k f^{(k)}(x), k = 0,1,2,\ldots,q \)

Are continuous, and bounded on \( R_0 \), and \( f^{(q)}(x) \) is uniformly continuous on \( R_0 \). Then we denote by \( B_q \) the spaces of all continuous and bounded functions \( f \) on \( R_0 \). And we denote by \( L_p(x), (1 \leq p < \infty) \) the spaces of all bounded Measurable functions \( f \) on \( X \), for which \[ \| f \|_p = \left[ \int_a^b |\omega_q(x) f(x)|^p dx \right]^{\frac{1}{p}} < \infty , q \in N_0 \] (1.3)

The modulus of smoothness defined by [3]:
\[ w_1(f,C_q,h)_p = \sup_{0 \leq h \leq x} \| \Delta_n f(x) \|_p, x \in R_0 , q \in N_0 \] (1.4)

Where \( \Delta_n f(x) = f(x+h) - f(x) \) For \( h, x \in R_0, f \in C_q \)

Now let \( r \in N, q \in N, s > 0 \), be fixed numbers. For functions \( f \in B_q \) we defined the operators [4]:
\[ L_n(f,q,r,s,x) = \frac{1}{g(n^s x,r)} \sum_{k=0}^{\infty} \frac{(n^s x)^k}{(k+r)!} \sum_{j=0}^{q} \frac{f^{(j)}(\frac{k+r}{n^s})(x-\frac{k+r}{n^s})^j}{j!}, x \in R_0, n \in N \] (1.5)

Where \( g(t,r) = \sum_{k=0}^{\infty} \frac{t^k}{(k+r)!}, t \in R_0, t \geq 0 \), and \( g(n^s x,r) \neq 0 \) (1.6)

Basic Theorems

We shall give some properties of the above operators which we shall apply in the proofs of the main theorems.

**Lemma 2.1:** [4]

Let \( f \in B_q, q \in N_0, x \in R_0, n,r \in N \) and \( s > 0 \), then
\[ A_n(f,r,s,x) = \frac{1}{g(n^s x,r)} \sum_{k=0}^{\infty} \frac{(n^s x)^k}{(k+r)!} f \left( \frac{k+r}{n^s} \right) \] (2.1)

Clear that \( A_n(f,r,s,x) \) is a linear positive operator.

**Lemma 2.2:** [5]

Let \( q \in N_0, r \in N \) and \( s > 0 \) be fixed numbers. Then there exists a positive constant \( M_3 = M_3(q,r) \), such that
\[ \sup_{x \in R_0} \{ \omega_q(x) x^k A_n(1/\omega_q(x),r,s,x) \} \leq M_3 , n \in N , k = 0,1,2,\ldots,r \] (2.2)
Lemma 2.3: \([4]\)

For every fixed \( r \in N \) and for all \( n \in N \) and \( x \in R_0 \), then

\[
A_n(t, r, s, x) = x + \frac{1}{n^s(r-1)! g(n^s x, r)}
\]

(2.3)

\[
A_n(t^2, r, s, x) = x^2 + \frac{x}{n^s} \left( 1 + \frac{1}{(r-1)! g(n^s x, r)} + \frac{r}{n^{2s}(r-1)! g(n^s x, r)} \right)
\]

Lemma 2.4: \([5]\)

Fix \( q \in N_0, r \in N \) and \( s > 0 \). Then there exist positive numbers \( \alpha_{q,j} \) depending only on \( j, q \) and \( \beta_{q,j}(r) \) depending only on \( r, j \) and \( q \), \( 0 \leq j \leq q \), such that

\[
A_n(t^q, r, s, x) = \sum_{j=0}^{q} \frac{x^j}{n^{s(q-j)}} \left( \alpha_{q,j} + \frac{\beta_{q,j}(r)}{g(n^s x, r)} \right)
\]

(2.4)

For all \( n \in N \) and \( x \in R_0 \), moreover \( \alpha_{0,0} = 1 \), \( \beta_{0,0}(r) = 0 \) and

\[
\alpha_{q,0} = \beta_{q,0}(r) = 0 \quad \alpha_{q,q} = 1 \quad \beta_{q,0}(r) = \frac{r^{q-1}}{(r-1)!}
\]

for \( q \in N \).

Lemma 2.5: \([5]\)

Fix \( q \in N, r \in N \) and \( s > 0 \). Then for all \( x \in R_0 \) and \( n \in N \)

We have

\[
A_n((x-t)^q, r, s, x) = \sum_{j=1}^{q} \frac{x^{j-1}}{n^{s(q-j+1)}} \left( a_{q,j} + \frac{b_{q,j}}{g(n^s x, r)} \right)
\]

(2.5)

Where \( a_{p,j}, b_{p,j} \) are numbers depending only on the parameters \( r, j \) and \( p \).

3 - Main Results

Theorem 2.1 and 2.2 and corollary in this section show that operators \( L_n(f, q, r, 1, x) \), \( n \in N \), give better degree of approximation of functions \( f \in B_q \), \( q \in N \), than \( S_n \) and the operators examined in (1.1) (1.5), (2.1) and(2.2).

Lemma 3.1:

Let \( q \in N_0, r \in N \) and \( s > 0 \) be fixed numbers. Then there exists a positive constant \( M = M(q, r) \) such that

\[
\left\| A_n(1/\omega_q(t); r, s, x) \right\|_p \leq M \quad \text{where} \quad n \in N, x \in R_0
\]

(3.1)
Proof:
The inequality (3.1) is obvious for \( q = 0 \) by (1.2), (1.3) and (2.3).

Let \( q \in N \), from (1.6) we get \( \frac{1}{g(t,r)} \leq r! \) for \( t \in R_0 \) \hspace{1cm} (3.2)

From (3.2), (1.2), (2.3), (2.4) and (2.1), we have

\[
\left[ \int_{a}^{b} \omega_q(x) A_n(1/\omega_q(t),r,s,x)^{p} \right]^{1/p} = \\
\left[ \int_{a}^{b} \omega_q(x) \left[ 1 + A_n(t^q,r,s,x) \right]^{p} \right]^{1/p} \\
= \left[ \int_{a}^{b} \left( \frac{1}{1+x^q} + \sum_{j=0}^{q} \frac{x^j}{n^{q-j}} \left( \frac{\alpha_{q,j} + \beta_{q,j}(r)}{g(n^x,r)} \right)^p \right) dx \right]^{1/p} \\
\leq \left[ \int_{a}^{b} \left[ 1 + \sum_{j=0}^{q} \frac{x^j}{1+x^q} \left( \alpha_{q,j} + r! \beta_{q,j}(r) \right)^p \right] dx \right]^{1/p} \\
\leq 1 + \left[ \int_{a}^{b} \sum_{j=0}^{q} \frac{x^j}{1+x^q} \left( \alpha_{q,j} + r! \beta_{q,j}(r) \right)^p \right]^{1/p} \\
\leq M(q,r)
\]

Lemma 3.2:
Let \( q \in N \), \( r \in N \) and \( s > 0 \). Then there exists a positive constant \( M \equiv M(f,q,r) \), for all \( f \in B_q \) such that

\[
\| L_n(f,q,r,s,x) \|_p \leq M \hspace{1cm} (3.3)
\]

Proof:
First we suppose that \( f \in B_q \), \( q \in N \) from this using the elementary

inequality, \( (a+b)^k \leq 2^{k-1} \left( a^k + b^k \right) \), \( a,b > 0, k \in N_0 \) we get

\[
| x - t |^k \left| f^{(k)}(t) \right| \leq 2^{k-1} \left| f^{(k)}(t) \right| \left\{ x^k + t^k \right\} \\
\leq M \left( f,q,k \right) \left\{ \frac{1}{\omega_q(t)} + \frac{x^k}{\omega_{q-k}(t)} \right\} \hspace{1cm} k = 0,1,2, \ldots, q,x,t \in R_0
\]

Where \( \omega_q(t) \neq 0 \) and \( \omega_{q-k}(t) \neq 0 \) this implies that
\[
\left[ \int_a^b \omega_q(x) L_n(f,q,r,s,x) \right]^{1/p} dx \leq M(f,q) \left[ \int_a^b \omega_q(x) \right]^{1/p} \ dx \]

\[
M(f,q) = \sum_{k=0}^\infty \left\{ \left( \frac{(n^s x)^k}{(k + r)!} \right) \frac{1}{\omega_q((k + r)/n^s)} + \sum_{j=0}^q \frac{x^j}{\omega_{q-j}((k + r)/n^s)} \right\}^{1/p} \ dx \]

From last equation and in view of (3.1) and (2.3), we get

\[
\left[ \int_a^b \omega_q(x) \right]^{1/p} \ dx \leq M(f,q,r) \]

**Theorem 3.1:**

Let \( q \in N_0, r \in N \) and \( s > 0 \), then there exists a positive constant \( M \equiv M(q,r,s) \) such that for every \( f \in B_{2q+1} \), we have

\[
\| L_n(f,2q+1,r,s,x) - f(x) \|_p \leq M \tau_1(f^{(2q+1)},C_0,\frac{1}{n}) , \quad n \in N, 1 \leq p < \infty
\]

**Proof:**

Suppose that \( f \in B_{2q+1} \), this implies that \( f^{(2q+1)} \in C_0 \), let \( q \in N_0 \) and using the modified of Taylor formula:

\[
f(x) = \sum_{j=0}^{2q+1} f^{(j)}(\frac{k+r}{n^s}) (x - \frac{k+r}{n^s})^j + \frac{(x - \frac{k+r}{n^s})^{2q+1}}{2q!} \]

\[
\left\{ \int_0^1 (1-t)^{2q} \left[ f^{(2q+1)}(\frac{k+r}{n^s} + t(x - \frac{k+r}{n^s})) - f^{(2q+1)}(\frac{k+r}{n^s}) \right] dt \right\}
\]

By definition of the modulus of continuity and (1.6), (1.5), we have

\[
\left[ \int_a^b \omega_{2q+1}(x) \left( L_n(f,2q+1,r,s,x) - f(x) \right) \right]^{1/p} \ dx
\]

383
The Degree of Best Approximation by the Linear Positive Operator in the Weight Spaces

\[ \left[ b \right] \left[ a \right] \omega_{2q}^{a+1}(x) \sum_{k=0}^{\infty} \frac{(n^s)^k}{(k+r)!} \left[ \sum_{j=0}^{2q+1} \frac{f^{(j)}(k+r/n^s)(x - k+r/n^s)^j}{j!} - f(x) \right]^p \int_a^b dx \]

\[ \leq \left[ b \right] \left[ a \right] \omega_{2q}^{a+1}(x) \sum_{k=0}^{\infty} \frac{(n^s)^k}{(k+r)!} \left[ \frac{(x - k+r/n^s)^{2q+1}}{2q!} \right]^p \int_a^b dx \]

\[ \int_0^1 (1-t)^{2q} w_i(f^{(2q+1)}, C_0, t(x - (k+r)/n^s)) \int_0^1 dx \]

We observe that

\[ w_i(f^{(2q+1)}, C_0, t(x - (k+r)/n^s)) \leq (1+t(x - (k+r)/n^s))w_i(f^{(2q+1)}, C_0, 1/n^s) \]

From this, using the elementary inequality

\[ (a+b)^k \leq 2^{k-1} (a^k + b^k) \quad a, b > 0, \quad k \in \mathbb{N}_0 \]

We have

\[ w_i(f^{(2q+1)}, C_0, t(x - (k+r)/n^s)) \]

\[ \leq (1+t(x - (k+r)/n^s))w_i(f^{(2q+1)}, C_0, 1/n^s) \]

\[ \leq \left[ b \right] \left[ a \right] \omega_{2q}^{a+1}(x) \sum_{k=0}^{\infty} \frac{(n^s)^k}{(k+r)!} \left[ (x - k+r/n^s)^{2q+2} n^s + (x - k+r/n^s)^{2q+1} \right]^p \int_a^b dx \]

\[ \left[ b \right] \left[ a \right] \omega_{2q+1}(x) \left[ n^s A_n \left( (x-t)^{2q+2}, r, s, x \right) + 2^{2q} A_n \left( (x-t)^{2q+2}, r, s, x \right) \right] \]

\[ w_i \left( f^{(2q+1)}, C_0, 1/n^s \right) \int_a^b dx \]

384
\[
\leq \left[ \int_a^b \omega_{2q+1}(x) \left[ n^s A_n \left( (x-t)^{2q+2}, r, s, x \right) + 2^{2q} A_n \left( t^{2q+1}, r, s, x \right) + 2^q x^{2q+1} \right] \right]^{1/p}
\]

By (2.4), (1.2), (3.1) and (3.2) we get that

\[
\leq \left[ \int_a^b \omega_{2q+1}(x) \left( L_n (f, 2q+1, r, s, x) - f(x) \right) \right]^{1/p}
\]

\[
\leq \left[ \int_a^b \omega_{2q+1}(x) \left( \frac{x^{2q+1}}{n^s(n^s+1)} \left( a_{2q+1,j} + \frac{b_{2q+1,j}}{n^s(r,x)} \right) + 2^q x^{2q+1} \right) \right]^{1/p}
\]

\[
2^q \sum_{j=0}^{2q+1} \frac{x^{2q+1}}{n^s(n^s+1)} \left( a_{2q+1,j} + \frac{b_{2q+1,j}}{n^s(r,x)} \right) + 2^q x^{2q+1} \right) \right]^{1/p}
\]

\[
\leq \left[ \int_a^b \omega_{2q+1}(x) \left( \frac{x^{2q+1}}{n^s(n^s+1)} \right)^p \right]^{1/p} \left[ \int_a^b \left( a_{2q+1,j} + r!(b_{2q+1,j}) \right) \right]^{1/p}
\]

\[
2^q \sum_{j=1}^{2q+1} \left( a_{2q+1,j} + r!(b_{2q+1,j}) \right) \right]^{1/p}
\]

\[
\leq \left[ \int_a^b \omega_{2q+1}(x) \left( \frac{x^{2q+1}}{n^s(n^s+1)} \right)^p \right]^{1/p} \left[ \int_a^b \left( a_{2q+1,j} + r!(b_{2q+1,j}) \right) \right]^{1/p}
\]

\[
\leq M_1(q,s) \left[ \int_a^b \sum_{j=1}^{2q+1} \left( a_{2q+1,j} + r!(b_{2q+1,j}) \right) \right]^{1/p} \left[ \int_a^b \left( a_{2q+1,j} + r!(b_{2q+1,j}) \right) \right]^{1/p}
\]

\[
2^q \sum_{j=1}^{2q+1} \left( a_{2q+1,j} + r!(b_{2q+1,j}) \right) \right]^{1/p}
\]

\[
\leq M_2(q,s) \left[ \int_a^b \omega_{2q+1}(x) \left( \frac{x^{2q+1}}{n^s(n^s+1)} \right)^p \right]^{1/p} \left[ \int_a^b \left( a_{2q+1,j} + r!(b_{2q+1,j}) \right) \right]^{1/p}
\]

\[
\leq M_3(q,s,r) \cdot \tau_1 \left( f^{(2q+1)}, C_n, \frac{1}{n^s} \right)
\]

**Theorem 3.2:**
Let \( q \in N_0, r \in N \) and \( s > 0 \), then there exists a positive constant \( M \equiv M(q,r,s) \) such that for every \( f \in B_{2q+2} \), we have got

\[
\left\| L_n (f, 2q+2, r, s, x) - f(x) \right\|_p \leq \frac{M(q,r,s)}{n^s} \left\| f^{(2q+2)} \right\|_p, n \in N.
\]
Proof:
Suppose that \( f \in B_{2q+2} \), this implies that \( f^{(2q+2)} \in C_q \), let \( q \in \mathbb{N}_0 \)
Using the modified Taylor formula.
\[
f(x) = \sum_{j=0}^{2q+2} f^{(j)} \left( \frac{k+r}{n^s} \right) \left( x - \frac{k+r}{n^s} \right)^j + \frac{(x - \frac{k+r}{n^s})^{2q+2}}{(2q+1)!}
\]
\[
\int_0^1 (1-t)^{2q+2} \left[ f^{(2q+2)} \left( \frac{k+r}{n^s} + t(x - \frac{k+r}{n^s}) \right) - f^{(2q+2)} \left( \frac{k+r}{n^s} \right) \right] dt
\]
By definition of the modulus of smoothness of continuity and (1.5) and (1.6), we have that
\[
\|L_n(f, 2q+2, r, s, x) - f(x)\|_p = \left[ \int_a^b \omega_{2q+2}(x) \left( L_n(f, 2q+2, r, s, x) - f(x) \right)^p dx \right]^{\frac{1}{p}}
\]
\[
\leq \left[ \int_a^b \frac{\omega_{2q+2}(x)}{g(n^s x, r)} \sum_{k=0}^{\infty} \frac{(n^s x)^k}{(k+r)!} \left[ \sum_{j=0}^{2q+2} f^{(j)} \left( \frac{k+r}{n^s} \right) \left( x - \frac{k+r}{n^s} \right)^j \right] \left( x - \frac{k+r}{n^s} \right)^j - f(x) \right]^p dx \right]^{\frac{1}{p}}
\]
From this, applying (1.2), (2.5) and (3.2), we immediately obtain

$$\leq 2 \left\| f^{(2q+2)} \right\|_p \left[ \int_a^b \left| \omega_{2q+2}(x) A_s \left( (x-t)^{2q+2}, r, s, x \right) \right|^p dx \right]^{\frac{1}{p}}$$

$$= 2 \left\| f^{(2q+2)} \right\|_p \left[ \int_a^b \left| \omega_{2q+2}(x) \sum_{j=1}^{2q+2} \frac{x^{j-1}}{n^{x(j+2)}} \left( a_{q+2,j} + b_{q+2,j} \right) \right|^p dx \right]^{\frac{1}{p}}$$
The Degree of Best Approximation by the Linear Positive Operator in the Weight Spaces

Sahib and Zainab

\[ \frac{2M(q,r,s)}{n^r} \cdot C(p) \cdot \left\| f^{(2q+2)} \right\|_p \]

\[ \leq \frac{M}{n^r} \cdot \left\| f^{(2q+2)} \right\|_p \]

Corollary:
For every fixed \( q \in N, r \in N, s > 0 \) and \( f \in B_q \), we have
\[ \lim_{n \to \infty} \left\| L_n(f,q,r,s,x) - f(x) \right\|_p = 0 \]

In Conclusion
1) We find the best approximation of functions \( f \in B_q, q \in N \) in weighted space by using the linear operator \( L_n(f,q,r,1,x) \), \( n \in N \).
2) We show that the operator \( L_n(f,q,r,1,x) \), \( n \in N \) give a better degree of approximation of \( f \in B_q, q \in N \), than \( S_n(f,x) \).

REFERENCE


