ON PROPERTIES OF $S^{\infty}$-CONTINUOUS FUNCTIONS
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Abstract
In this work, we study $S^{\infty}$-continuous functions, a function $f : X \rightarrow Y$ is called $S^{\infty}$-continuous function if the inverse image of every semi-open set in $Y$ is semi open in $X$. Several properties of these functions are proved.

1. Introduction and Preliminaries:

Let $(X,T)$ be a topological space, let $A \subseteq X$, closure of $A$ and interior of $A$ are denoted by $clA$, $IntA$ respectively $A$ is called semi-open [3] if $A \subseteq cl/int A$ every open set is semi open but the converse is not necessarily true.

Definition 1.1:
Let $f : (X,T) \rightarrow (Y,\Omega)$ be a function, we say that:

a) $f$ is semi-continuous ($S$-continuous) [2] if the inverse of every open set in $Y$ is semi open in $X$.

b) $f$ is semi-continuous if the inverse image of every semi-open in $Y$ is open in $X$.

c) $f$ is semi-continuous if the inverse image of every semi-open in $Y$ is semi-open in $X$.

2. Certain forms of $S^{\infty}$-continuous functions

In this section, we introduce and study several forms of $S^{\infty}$-continuous functions.

We recall the following definitions.

Definition 2.1:[4]

a) Let $(X,T)$ be a topological space, let $B \subseteq X$, we say that $B$ is semi-closed if $B^{c}$ is semi open in $X$.

b) Let $B \subseteq X$, the semi-closure of $B$ ($scl(B)$) is the intersection of all semi-closed sets in $X$ containing $B$.

c) Let $F \subseteq X$, we say that $F$ is semi-generalized closed in $X$ (sg-closed) if $(F \subseteq O \Rightarrow scl(F) \subseteq O$ ($O$ is semi-open in $X$).

Now we are ready to introduce a weak form of $S^{\infty}$-continuous function which we call $A-S^{\infty}$-continuous function.

Definition 2.2:
Let $f : (X,T) \rightarrow (Y,\Omega)$ be a function, we say that $f$ is $A-S^{\infty}$-continuous function if $F \subseteq f^{-1}(O) \rightarrow scl(F) \subseteq f^{-1}(O)$ ($O$ is semi-open in $Y$, $F$ is sg-closed in $X$) Of courses, if $f$ is $S^{\infty}$-continuous function then $f$ is $A-S^{\infty}$-continuous function.

Example 2.3:
Let $X = \{a,b\}, T = \{\Phi, X, \{a\}\}$, Define $f : X \rightarrow X$ as follows:

$f(a) = b$, $f(b) = a$

Now $A = \{a\}$ is open and hence semi-open, consider $f^{-1}(A) = \{b\}$

Now $\{b\}$ is semi closed in $X$ so the inverse of every semi open in $X$ is semi closed which shows that $f$ is $A-S^{\infty}$-continuous function.

$(F \subseteq f^{-1}(O) \rightarrow scl(F) \subseteq scl(f^{-1}(O) = f^{-1}(O))$,$\quad$But $f$ is not $S^{\infty}$-continuous function.
Because \( \{a\} \) is semi open in \( X \) and
\[
f^{-1}(\{a\}) = \{b\}.
\]
Which is not semi open in \( X \) \( \{b\} \notin cl\{b\} \).
Before, we state the next theorem, we need the following definition.

**Definition 2.4 [3]**
Let \( f : (X, T) \to (Y, \Omega) \) be a function, we say that \( f \) is Contra \( S^{\infty} \)-continuous if the inverse of every semi-open in \( Y \) is semi-closed in \( X \).

**Theorem 2.5:**
Let \( f : (X, T) \to (Y, \Omega) \) be contra \( S^{\infty} \)-continuous function, then \( f \) is \( A-S^{\infty} \)-continuous function.

**Proof:**
Let \( O \) be a semi-open in \( Y \), let \( F \) be \( A \)-closed in \( X \), let \( F \subseteq f^{-1}(O) \), then \( scl(F) \subseteq scl(f^{-1}(O)) = f^{-1}(O) \). Because \( f^{-1}(O) \) is semi-closed in \( X \) which means that \( f \) is \( A-S^{\infty} \)-continuous function.

**Theorem 2.6:**
Let \( f : (X, T) \to (Y, \Omega) \) be a function from a topological space \( (X, T) \) into a topological space \( (Y, \Omega) \). If the semi-open and semi-closed sets of \( (X, T) \) coincide, the \( f \) is \( A-S^{\infty} \)-continuous function if and only if \( f \) is contra \( S^{\infty} \)-continuous function.

**Proof:**
Assume \( f \) is \( A-S^{\infty} \)-continuous function. Let \( A \) be an arbitrary subset of \( (X, T) \) such that \( A \subseteq W \), where \( W \) is semi-open in \( X \), then by hypothesis \( scl(A) \subseteq scl(W) = W \), therefore all subset of \( (X, T) \) are \( sg \)-closed (and hence all are \( sg \)-open) so,
for any \( O \) which is semi-open in \( Y \), \( f^{-1}(O) \) is \( sg \)-closed. \( scl(f^{-1}(O)) \subseteq f^{-1}(O) \).

Therefore \( scl(f^{-1}(O)) = f^{-1}(O) \), i.e. \( f^{-1}(O) \) is semi-closed in \( X \), which means that \( f \) is contra \( S^{\infty} \)-continuous function.

**Corollary 2.7:**
Let \( f : (X, T) \to (Y, \Omega) \) be a function from a topological space \( (X, T) \) into a topological space \( (Y, \Omega) \). If the semi-open and semi-closed sets of \( (X, T) \) coincide, then \( f \) is \( A-S^{\infty} \)-continuous if and only if \( f \) is \( S^{\infty} \)-continuous.

**Proof:**
Let \( f \) be \( A-S^{\infty} \)-continuous function, let \( O \) be semi-open in \( Y \), we will show that \( f \) is \( S^{\infty} \)-continuous in \( X \).

Now \( f^{-1}(O) \) is \( sg \)-closed (Theorem 2.6), \( f^{-1}(O) \subseteq f^{-1}(O) \Rightarrow scl(f^{-1}(O)) \subseteq f^{-1}(O) \).

Which means that \( f^{-1}(O) \) is semi-closed in \( X \). But the semi-open and semi-closed sets in \( X \) coincide, so \( f^{-1}(O) \) is semi-open in \( X \) so \( f \) is \( S^{\infty} \)-continuous.

Another proof, According to theorem 2.6, If \( f \) is \( A-S^{\infty} \)-continuous then \( f \) contra \( S^{\infty} \)-continuous, let \( O \) be semi-open in \( X \), hence \( f^{-1}(O) \) is semi-open in \( X \), so \( f \) is \( S^{\infty} \)-continuous.
**Definition 2.8**: Let \( f : (X, T) \rightarrow (Y, \Omega) \) be a function, we say that \( f \) is perfectly contra-\( S^\infty \)-continuous function if the inverse of every semi-open in \( Y \) is semi-clopen in \( X \) (that is semi-open and semi-closed).

Before, we state the next theorem we need the following definition.

**Definition 2.9**: Let \( f : (X, T) \rightarrow (Y, \Omega) \) be a function we say that \( f \) is \( A \)-semi closed if \( \sin(A) \subset f^{-1}(\text{scl}(A)) \) and \( f^{-1}(\text{scl}(A)) \subset \text{scl}(A) \) for \( A \) is \( sg \)-open subset of \( Y \), \( B \) is semi-closed subset of \( X \).

**Theorem 2.10**: Let \( f : (X, T) \rightarrow (Y, \Omega) \) be \( S^\infty \)-continuous function and \( A \)-semi closed then \( f^{-1}(A) \subset \text{sg-closed} \) whenever \( A \subset \text{sg-closed} \subset Y \).

**Proof**: Let \( A \) be \( sg \)-closed subset of \( Y \), suppose \( f^{-1}(A) \subset \text{O} \) (\( O \) is semi-open in \( X \)).

Now \( O' \subset f^{-1}(A') \) hence \( f(O') \subset A'^c \), then \( f(O') \subset \text{sin}(A') = (\text{scl}(A))^c \) it follows that \( O' \subset (f^{-1}(\text{scl}(A)))^c \) and hence,

\[ (f^{-1}(\text{scl}(A))) \subset O \] since \( f \) is \( S^\infty \)-continuous, then \( (f^{-1}(\text{scl}(A))) \subset \text{semi-closed} \).

Thus we have \( (f^{-1}(A)) \subset \text{scl}(f^{-1}(\text{scl}(A))) = f^{-1}(\text{scl}(A)) \subset O \), This implies that \( f^{-1}(A) \subset \text{sg-} \text{closed} \subset X \).

**References**

[1] Hadi J.Mustafa,contra –B-continuous functions(Kufa University conference 2008)


المستخلص:

\( S^\infty \) تسمى دالة مستمرة - ا.دالة الدالة \( f : X \rightarrow Y \). إذا كانت الصورة النظيرية لأي مجموعة شبه مفتوحة في \( X \) مجموعة شبه مفتوحة في \( Y \). برهنا مجموعة خصائص لهذه الدوال.