A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

Dr. Jinan Abbas Naser Al-obedy - Assistant professor
Technical College of Management-Baghdad

Abstract

A comparison of double informative and non-informative priors assumed for the parameter of Rayleigh distribution is considered. Three different sets of double priors are included, for a single unknown parameter of Rayleigh distribution. We have assumed three double priors: the square root inverted gamma (SRIG) - the natural conjugate family of priors distribution, the square root inverted gamma – the non-informative distribution, and the natural conjugate family of priors - the non-informative distribution as double priors.

The data is generating from three cases from Rayleigh distribution for different sample sizes (small, medium, and large). And Bayes estimators for the parameter is derived under a squared error loss function and weighted squared error loss function) in the cases of the three different sets of prior distributions. Simulations is employed to obtain results. And determine the best estimator according to the smallest value of mean squared error and weighted mean squared error. We found that the best estimation for the parameter (θ) for all sample sizes (n), when the double prior distribution for θ is SRIG - the natural conjugate family of priors distribution with values (a=5, b=0.5, α =8, β =0.5) and (a=8, b=1, α =5, β =1) for the true value of θ (θ = 0.5 and 1) respectively. Also, we obtained the best estimation for θ when the double prior distribution for θ is the natural conjugate family of priors-non-informative distribution with values(α =0.5, β =5, c=1) for the true value of θ (θ = 1.5).

Key words/ Rayleigh distribution, Bayes method double informative and non-informative priors, the posterior distribution, the squared error loss function, the weighted squared error loss function.
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

Introduction

The Rayleigh distribution is a member of continuous probability distributions, and it’s a suitable model for life testing experiments and clinical studies. We mention some of studies in a brief manner: Hendi et al and others [6] (2007) used Bayes estimation. They obtained bayes estimates for the unknown parameter of the Rayleigh distribution based on upper record values. They derived Bayes' estimators by assuming the quasi-density prior as the prior distribution for the parameter. And he derived Bayes' estimators under two different loss functions which are squared error loss function and linex loss function. Dey [4] (2009) used Bayes estimation to obtained the estimators for the parameter and reliability function of Rayleigh distribution. He derived Bayes' estimators by assuming the natural conjugate family of priors as the prior distribution for the parameter. And he derived Bayes' estimators under two different loss functions which are squared error loss function and linex loss function. Al Mayali and Al-Shaibani [2] (2013) used different estimation methods to estimate the unknown parameter of Rayleigh distribution, which is maximum likelihood estimation, Bayes estimation, shrinkage estimation, and Bayesian shrinkage estimation. They used simulation to obtain the results and compared between these results to find best method.

Pak et al and others [8] (2014) proposed different procedures for estimating the parameter of Rayleigh distribution under progressive type-II censoring when the available observations are described by means of fuzzy information. They conducted simulation study to assess the performance of these procedures. Dey et al and others [5] (2015) used Bayes shrinkage estimation to obtain the estimators for the parameter of Rayleigh distribution; they derived Bayes' estimators by assuming the natural conjugate family of priors as the prior distribution for the parameter of Rayleigh distribution. They derived Bayes' estimators for the parameter of Rayleigh distribution under the assumption of general entropy loss function for progressive Type-II censored data.

The aim of research

The Aim of this paper is to obtain bayes Estimators for the parameter of the Rayleigh distribution under different double informative and non-informative priors. A few studies present in double informative priors, we mention it: Haq and Aslam [1], they used double prior selection for discrete case in the case of Poisson distribution. Radha and Vekatesan (2015), they studied double prior selection for continuous case in the case of Maxwell distribution [9]. They have assumed generalized uniform-inverted Gamma distribution as double priors.

Here we study double prior selection for continuous case in the case of the Rayleigh distribution. We have assumed the square root inverted gamma (SRIG) - the natural conjugate family of priors distribution, the square root inverted gamma – the non-informative distribution, and the natural conjugate family of priors - the non-informative distribution as double priors.
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

We try to find best method to estimate parameter of Rayleigh distribution. According to the smallest value of Mean Square Errors (MSE) and Mean Weighted Square Errors (MWSE) were calculated to compare the methods of estimation. Several cases from Rayleigh distribution for data generating, of different samples sizes (small, medium, and large). The results were obtained by using simulation technique.

1. The Rayleigh Distribution

Let us consider \( t_1, t_2, ..., t_n \) is a random sample of \( n \) independent observations from a Rayleigh distribution having the probability density function (pdf) defined [3]:
\[
f(t; \theta) = \frac{t}{\theta^2} \exp\left(-\frac{t^2}{2\theta^2}\right), \quad t \geq 0, \theta > 0
\]  \( ... (1) \)

where \( \theta \) is a scale parameter. And the cumulative distribution function (cdf) is given as:
\[
F(t; \theta) = 1 - \exp\left(-\frac{t^2}{2\theta^2}\right), \quad t \geq 0, \theta > 0
\]  \( ... (2) \)

Also, the Reliability function is
\[
R(t) = 1 - F(t) = \int_{t}^{\infty} f(u)du = \exp\left(-\frac{t^2}{2\theta^2}\right), \quad t \geq 0, \theta > 0
\]  \( ... (3) \)

Where \( R(t) \) is probability of surviving at least till age \( t \), and the failure rate function (or hazard function) at mission time \( t \) is given by
\[
h(t) = \frac{t}{\theta^2}, \quad t \geq 0, \theta > 0
\]  \( ... (4) \)

3. Bayes Estimation Method

In this section, we used bayes estimation to derived Bayes’ estimators for the parameter of Rayleigh distribution, under different double informative priors. Let \( t_1, t_2, ..., t_n \) be a random sample of size \( n \) with probability density function given in equation (1) and likelihood function given by equation (5) [3]:
\[
L(t|\theta) = \prod_{i=1}^{n} f(t_i; \theta) = \prod_{i=1}^{n} \frac{t_i}{\theta^2} \exp\left(-\frac{\sum_{i=1}^{n} t_i^2}{2\theta^2}\right)
\]  \( ... (5) \)

In this paper the posterior distributions for the unknown parameter \( \theta \) are derived using the following three types of double informative priors, and then get bayes estimation.
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

Here we have assumed the square root inverted gamma [7] - the natural conjugate family of priors distribution, the square root inverted gamma – the non-informative distribution, and the natural conjugate family of priors - the non-informative distribution as double priors, to get bayes estimation for the parameter of Rayleigh distribution.

3.1 The posterior distribution using different double priors

In this section, we derive the posterior distributions .It is assumed that \( \theta \) follows three types of prior distributions with pdf as given in table -1:

<table>
<thead>
<tr>
<th>Prior distribution</th>
<th>( f_i(\theta) ), ( i = 1, 2, 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta \sim \text{SRIG distribution (a,b)} )</td>
<td>( f_1(\theta) = \frac{2b^a}{\Gamma(a)} \theta^{-(2a+1)} \exp(-\frac{b}{\theta^2}) ) for ( a, b, \theta &gt; 0 )</td>
</tr>
<tr>
<td>( \theta \sim \text{natural conjugate family of priors distribution (} \alpha, \beta) )</td>
<td>( f_2(\theta) = \frac{1}{\theta^{\alpha+1}} \exp[-\frac{\beta}{\theta^{2\alpha}}] ) for ( \theta, \alpha, \beta &gt; 0 )</td>
</tr>
<tr>
<td>( \theta \sim \text{non-informative distribution (c)} )</td>
<td>( f_3(\theta) = \frac{1}{\theta^c} ) for ( \theta, c &gt; 0 )</td>
</tr>
</tbody>
</table>

And their double prior's distributions with pdf as given in table -2:

<table>
<thead>
<tr>
<th>Double prior dist(^a).</th>
<th>( P_i(\theta) ), ( i = 1, 2, 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta \sim \text{SRIG (a,b)-} \theta \sim \text{the natural conjugate family of priors (} \alpha, \beta) )</td>
<td>( P_1(\theta) = f_1(\theta) f_2(\theta) ) ( P_1(\theta) \alpha[\frac{2b^a}{\Gamma(a)}] \theta^{-(2a+\alpha+2)} \exp[-\frac{1}{\theta^2} (b+\beta)] ) for ( a, b, \alpha, \beta, \theta &gt; 0 )</td>
</tr>
<tr>
<td>( \theta \sim \text{SRIG (a,b)-} \theta \sim \text{non-informative distribution (c)} )</td>
<td>( P_2(\theta) = f_1(\theta) f_3(\theta) ) ( P_2(\theta) \alpha[\frac{2b^a}{\Gamma(a)}] \theta^{-(2a+c+1)} \exp[-\frac{b}{\theta^2}] ) for ( a, b, c, \theta &gt; 0 )</td>
</tr>
</tbody>
</table>
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

\[ \theta \sim \text{the natural conjugate family of priors } (\alpha, \beta) - 0 \sim \text{non-informative distribution } (c) \]

\[ P_3(\theta) = f_2(\theta) f_3(\theta) \]

\[ P_3(\theta) = \theta^{-(\alpha+c+1)} \exp\left[\frac{-\beta}{2\theta^2}\right] \text{ for } \alpha, \beta, c, \theta > 0 \]

Then the posterior distribution of \( \theta \) for the given data \( t = (t_1, t_2, ..., t_n) \) is given by:

\[
P(\theta | t) = \frac{L(t | \theta) P(\theta)}{\int_0^\infty L(t | \theta) P(\theta) d\theta} \quad \text{... (6)}
\]

Substituting the equation (5) and for each \( P(\theta) \) as shown in table -2 in equation (6), we get the posterior distributions for the unknown parameter \( \theta \) are derived using the following three types of double priors (for more details see Appendix-A).

Table -3: The posterior distributions \( P(\theta | t) \) for the unknown parameter \( \theta \) are derived using the following three types of double prior distributions.

| Double prior dist\( ^a \). | The posterior distribution \( P(\theta | t) \) |
|-------------------------------|-----------------------------------------|
| \( 0 \sim \text{SRIG } (a,b) \)- \( 0 \sim \text{the natural conjugate family of priors } (\alpha, \beta) \) | \( P_1(\theta | t) \sim \text{SRIG dist}^a \) with \( a_{(\text{new})} = (a + n + 0.5\alpha + 0.5b + \beta) \)
\[
P_1(\theta | t) = \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)} \theta^{2(\alpha+n+0.5\alpha+0.5)-1} \exp\left[\frac{-1}{\theta^2} \left(0.5\sum_{i=1}^{n} t_i^2 + b + \beta\right)\right]
\]
| \( a, b, \alpha, \beta, a, \theta \geq 0 \) |
| \( 0 \sim \text{SRIG } (a,b) \)- \( 0 \sim \text{non-informative distribution } (c) \) | \( P_2(\theta | t) \sim \text{SRIG dist}^a \) with \( a_{(\text{new})} = (a + n + 0.5c) \)
\[
P_2(\theta | t) = \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b)^{(a+n+0.5c)}}{\Gamma(a+n+0.5c)} \theta^{2(\alpha+n+0.5c+1)-1} \exp\left[\frac{-1}{\theta^2} \left(0.5\sum_{i=1}^{n} t_i^2 + b\right)\right]
\]
| \( a, b, c, a, \theta \geq 0 \) |
| \( 0 \sim \text{the natural conjugate family of priors } (\alpha, \beta) \)- \( 0 \sim \text{non-informative distribution } (c) \) | \( P_3(\theta | t) \sim \text{SRIG dist}^a \) with \( a_{(\text{new})} = (n + 0.5\alpha + 0.5c) \)
\[
P_3(\theta | t) = \frac{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c)}}{\Gamma(n+0.5\alpha+0.5c)} \theta^{2(\alpha+n+0.5\alpha+0.5c+1)-1} \exp\left[\frac{-1}{\theta^2} \left(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta\right)\right]
\]
| \( a, b, \alpha, \beta, c, n, \theta \geq 0 \) |
3.2 Bayes’ Estimators

The objective of this section, to find Bayesian estimators of the scale parameter of Rayleigh distribution under two loss functions and three different of double prior distributions:

- The squared error loss function \( L_1(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \).

- The weighted squared error loss function \( L_2(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\theta} \).

Where \( \hat{\theta} \) is an estimator for \( \theta \) was considered with three different of double prior distributions, and under two loss functions. Following is the derivation of these estimators:

**First: The squared error loss function**

To find Bayesian estimators of the scale parameter of Rayleigh distribution under the squared error loss function and three different of double prior distributions:

\[
L_1(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2
\]

After simplified steps, we get Bayes estimator of \( \theta \) denoted by \( \hat{\theta}_{SE} \) for the above prior as follows

\[
\hat{\theta}_{SE} = E(\theta \mid t) = \int_0^\infty \theta P(\theta \mid t) d\theta
\]

So, the following results are the derivations of these estimators under the squared error loss function with three different double priors (for more details see Appendix-B).
Table -4: The estimators (\( \hat{\theta}_{SE} \)) under the squared error loss function

<table>
<thead>
<tr>
<th>Double Prior distribution</th>
<th>( \hat{\theta}_{SE} = E(\theta \mid t) = \int_0^\infty \theta P(\theta \mid t) d\theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta \sim SRIG (a,b) ) - ( \theta \sim ) the natural conjugate family of priors (( \alpha, \beta ))</td>
<td>( \hat{\theta}<em>{SE1} = \frac{\Gamma(a + n + 0.5\alpha)}{\Gamma(a + n + 0.5\alpha + 0.5)} \frac{(0.5\sum</em>{i=1}^n \theta_i^2 + b + \beta)^{0.5}}{n, \alpha, \beta, b, a &gt; 0} )</td>
</tr>
<tr>
<td>( \theta \sim SRIG (a,b) ) - ( \theta \sim ) non-informative distribution (c)</td>
<td>( \hat{\theta}<em>{SE2} = \frac{\Gamma(a + n + 0.5c - 0.5)}{\Gamma(a + n + 0.5c)} \frac{(0.5\sum</em>{i=1}^n \theta_i^2 + b)^{0.5}}{n, c, b, a &gt; 0} )</td>
</tr>
<tr>
<td>( \theta \sim ) the natural conjugate family of priors (( \alpha, \beta )) - ( \theta \sim ) non-informative distribution (c)</td>
<td>( \hat{\theta}<em>{SE3} = \frac{\Gamma(n + 0.5\alpha + 0.5c - 0.5)}{\Gamma(n + 0.5\alpha + 0.5c)} \frac{(0.5\sum</em>{i=1}^n \theta_i^2 + 0.5\beta)^{0.5}}{\alpha, \beta, c, n &gt; 0} )</td>
</tr>
</tbody>
</table>

Where \( \Gamma(.) \) is a gamma function.

Second: The weighted squared error loss function

To find Bayesian estimators of the scale parameter of Rayleigh distribution under the weighted squared error loss function and three different double priors:

\[
L_w(\theta, \theta) = \frac{(\hat{\theta} - \theta)^2}{\theta}
\]

... (9)

After simplified steps, we get Bayes estimator of \( \theta \) denoted by \( \hat{\theta}_{WSE} \) for the above prior as follows:

\[
\hat{\theta}_{WSE} = \frac{1}{E(\frac{1}{\theta} \mid t)} = \frac{1}{\int_0^\infty \frac{1}{\theta} P(\theta \mid t) d\theta}
\]

... (10)

So, the following results are the derivations of these estimators under the weighted squared error loss function with three different double priors (for more details see Appendix-C).
Table -5: The estimators $\hat{\theta}_{WSE}$ under the weighted squared error loss with three different double priors.

<table>
<thead>
<tr>
<th>Double Prior distribution</th>
<th>$\hat{\theta}_{WSE} = \frac{1}{E(\frac{1}{\theta})} = \frac{1}{\int_0^\infty \frac{1}{\theta} P(\theta) , d\theta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \sim$ SRIG (a,b)- $0 \sim$ the natural conjugate family of priors (α, β)</td>
<td>$\hat{\theta}<em>{WSE} = \frac{\Gamma(a + n + 0.5\alpha + 0.5) (0.5\sum</em>{i=1}^n t_i^2 + b + \beta)^{0.5}}{\Gamma(a + n + 0.5\alpha + 1)} \quad n, \alpha, \beta, b, a &gt; 0$</td>
</tr>
<tr>
<td>$0 \sim$ SRIG (a,b)- $0 \sim$ non-informative distribution (c)</td>
<td>$\hat{\theta}<em>{WSE} = \frac{\Gamma(a + n + 0.5c) (0.5\sum</em>{i=1}^n t_i^2 + b)^{0.5}}{\Gamma(a + n + 0.5c + 0.5)} \quad n, c, b, a &gt; 0$</td>
</tr>
<tr>
<td>$0 \sim$ the natural conjugate family of priors (α, β)- $0 \sim$ non-informative distribution (c)</td>
<td>$\hat{\theta}<em>{WSE} = \frac{\Gamma(n + 0.5\alpha + 0.5c)}{\Gamma(n + 0.5\alpha + 0.5c + 0.5)} (0.5\sum</em>{i=1}^n t_i^2 + 0.5\beta)^{0.5} \quad \alpha, \beta, c, n &gt; 0$</td>
</tr>
</tbody>
</table>

4. Simulation Study

In this study, we have generated random samples from Rayleigh distribution and compared the performance of Bayes estimators based on them. So we have considered several steps to perform simulation study as follows:

1. We have chosen sample size $n = 15, 25, 50$ and $100$ to represent small, moderate and large sample size.
2. We generated data from Rayleigh distribution for the scale parameter; we have considered randomly several values for the parameter of Rayleigh distribution $\theta = 0.5, 1, 1.5$.
3. We used randomly the values for the parameters of the square root inverted gamma (SRIG) distribution with (a, b) - the natural conjugate family of priors distribution with (α, β) as double prior distribution for $\theta$, when $(a, b, \alpha, \beta) = (5,0.5, 8,0.5),(8,0.5, 5,0.5),(5,1,8,1)$ and $(8,1,5,1)$. 
4. We used randomly the values for the parameters of the square root inverted gamma (SRIG) distribution with (a, b) – the non-informative distribution with (c) as double prior distribution for $\theta$, when (a, b, c) = (5, 0.5, 2), (5, 0.5, 3), (5, 0.5, 5), (5, 1, 2), (5, 1, 3) and (5, 1.5).

5. We used randomly the values for the parameters of the natural conjugate family of priors distribution with $(\alpha, \beta)$ - the non-informative distribution with (c) as double prior distribution for $\theta$, when $(\alpha, \beta, c) = (0.5, 5, 0.5), (0.5, 5, 1), (0.5, 8, 0.5)$ and $(0.5, 8, 1)$.

6. The number of replication used was ($r = 1000$) for each sample size ($n$).
   We obtained estimators for scale parameter in table -4, it means the estimators $(\hat{\theta}_{SE})$ under the squared error loss function with three different double priors.

   And the estimators in table -5, it means the estimators $(\hat{\theta}_{WSE})$ under the weighted squared error loss function with three different double.

   The simulation program was written by using MATLAB-R2008a program.
   After the parameter $\theta$ was estimated, Mean Square Errors (MSE) and Mean weighted squared Errors (MWSE) were calculated to compare the methods of estimation, where:

   $\text{MSE} = \frac{1}{r} \sum_{r=1}^{1000} (\hat{\theta}(r) - \theta)^2$  \hspace{1cm} \ldots(11)

   $\text{MWSE} = \frac{1}{r} \sum_{r=1}^{1000} [(\hat{\theta}(r) - \theta)^2 / \theta]$ \hspace{1cm} \ldots(12)

   See appendix-D, for the programs algorithm. The results of the simulation study are summarized and tabulated in tables (4-1 to 4-3). In each row of tables (4-1) – (4-3), we have four estimated values for $\theta (\hat{\theta})$ with MSE for all samples sizes ($n$) and values ((a, b), (a, $\beta$), c) respectively. Also the results of the simulation study are summarized and tabulated in tables (4-4to4-6). In each row of tables (4-4)-(4-6), we have four estimated values for $\theta (\hat{\theta})$ with MWSE for all sample size ($n$) and values ((a, b), (a, $\beta$), c) respectively.

   The Bayes estimators under three types of double prior distribution. So our criteria is the best estimator that gives the smallest value of (MSE) and (MWSE). We list the results in the following tables (4-1 to 4-6).
### Table 4-1: Shows the values for $\hat{\theta}$ under square error loss function (MSE).

| Method | parameters | $\theta$ | $a$ | $b$ | $\alpha$ | $\beta$ | $E(\theta | \hat{\theta})$ | MSE(\theta) |
|--------|------------|---------|-----|-----|---------|--------|----------------|--------------|
| Bayes  | $\theta$ a b c | 0.5  | 0.5 | 8  | 0.5 | 0.49521 | 0.49756 | 0.49676 | 0.49826 | 0.00276 | 0.00188 | 0.00103 | 0.00065 |
|        |            | 0.5  | 5  | 0.5 | 0.5 | 0.51751 | 0.51083 | 0.50571 | 0.50189 | 0.00321 | 0.00210 | 0.00109 | 0.00057 |
|        |            | 0.5  | 1  | 8  | 1   | 0.54571 | 0.53102 | 0.51689 | 0.50782 | 0.00433 | 0.00261 | 0.00124 | 0.00060 |
|        |            | 0.5  | 1  | 5  | 1   | 0.58316 | 0.54518 | 0.52419 | 0.51151 | 0.00706 | 0.00377 | 0.00157 | 0.00068 |
| Bayes  | $\theta$ a b c | 0.5  | 0.5 | 2  | 0.5 | 0.51728 | 0.51115 | 0.50589 | 0.50191 | 0.00404 | 0.00243 | 0.00117 | 0.00059 |
|        |            | 0.5  | 5  | 0.5 | 3   | 0.50926 | 0.50666 | 0.50342 | 0.50067 | 0.00371 | 0.00230 | 0.00114 | 0.00059 |
|        |            | 0.5  | 5  | 0.5 | 5   | 0.49438 | 0.49722 | 0.49858 | 0.48822 | 0.00345 | 0.00218 | 0.00111 | 0.00058 |
|        |            | 0.5  | 1  | 2  | 1   | 0.54746 | 0.53029 | 0.51588 | 0.50684 | 0.00559 | 0.00305 | 0.00134 | 0.00062 |
|        |            | 0.5  | 1  | 3  | 1   | 0.53988 | 0.52522 | 0.51306 | 0.50559 | 0.00476 | 0.00273 | 0.00125 | 0.00060 |
|        |            | 0.5  | 1  | 5  | 5   | 0.52312 | 0.51549 | 0.50813 | 0.50311 | 0.00358 | 0.00225 | 0.00113 | 0.00058 |

### Table 4-2: Shows the values for $\hat{\theta}$ under square error loss function (MSE).

| Method | parameters | $\theta$ | $a$ | $b$ | $\alpha$ | $\beta$ | $E(\hat{\theta} | \theta)$ | MSE(\theta) |
|--------|------------|---------|-----|-----|---------|--------|----------------|--------------|
| Bayes  | $\theta$ a b c | 0.5  | 0.5 | 8  | 0.5 | 0.509605 | 0.50375 | 0.50803 | 0.50003 | 0.00252 | 0.00108 | 0.00054 | 0.00027 |
|        |            | 0.5  | 5  | 0.5 | 5   | 0.53172 | 0.53182 | 0.52756 | 0.52331 | 0.00358 | 0.00212 | 0.00109 | 0.00059 |
|        |            | 0.5  | 1  | 8  | 1   | 0.52389 | 0.52157 | 0.51743 | 0.50947 | 0.00476 | 0.00273 | 0.00125 | 0.00060 |
|        |            | 0.5  | 1  | 5  | 5   | 0.50111 | 0.50608 | 0.49775 | 0.49916 | 0.00443 | 0.00293 | 0.00140 | 0.00068 |
| Bayes  | $\theta$ a b c | 0.5  | 0.5 | 2  | 0.5 | 0.509605 | 0.50375 | 0.50803 | 0.50003 | 0.00252 | 0.00108 | 0.00054 | 0.00027 |
|        |            | 0.5  | 5  | 0.5 | 3   | 0.53172 | 0.53182 | 0.52756 | 0.52331 | 0.00358 | 0.00212 | 0.00109 | 0.00059 |
|        |            | 0.5  | 5  | 0.5 | 5   | 0.52389 | 0.52157 | 0.51743 | 0.50947 | 0.00476 | 0.00273 | 0.00125 | 0.00060 |
|        |            | 0.5  | 1  | 2  | 1   | 0.52389 | 0.52157 | 0.51743 | 0.50947 | 0.00476 | 0.00273 | 0.00125 | 0.00060 |
|        |            | 0.5  | 1  | 3  | 1   | 0.52389 | 0.52157 | 0.51743 | 0.50947 | 0.00476 | 0.00273 | 0.00125 | 0.00060 |
|        |            | 0.5  | 1  | 5  | 5   | 0.50111 | 0.50608 | 0.49775 | 0.49916 | 0.00443 | 0.00293 | 0.00140 | 0.00068 |

The tables show the comparison of Bayes estimators for the parameter of the Rayleigh distribution with simulation.
Table 4-3: Shows the values for $\hat{\theta}$ under square error loss function (MSE).

<table>
<thead>
<tr>
<th>Method</th>
<th>parameters</th>
<th>Estimate for $\hat{\theta}$ ($\hat{\theta}$)</th>
<th>MSE(\theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\beta$ when double prior is ($\beta$, $\alpha, \theta$)</td>
<td>$15$ $25$ $50$ $100$ $15$ $25$ $50$ $100$</td>
</tr>
<tr>
<td>Bayes</td>
<td>$\theta$</td>
<td>$0.5$ $0.5$ $0.5$</td>
<td>$1.5039$</td>
</tr>
<tr>
<td></td>
<td>$\alpha$</td>
<td>$0.5$ $0.5$ $0.5$</td>
<td>$1.5542$</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>$1$ $0.5$ $1$</td>
<td>$1.322$</td>
</tr>
<tr>
<td></td>
<td>$\gamma$</td>
<td>$1$ $0.5$ $1$</td>
<td>$1.3739$</td>
</tr>
</tbody>
</table>

Table 4-4: Shows the values for $\hat{\theta}$ under weighted square error loss function (MWSE).

<table>
<thead>
<tr>
<th>Method</th>
<th>parameters</th>
<th>Estimate for $\hat{\theta}$ ($\hat{\theta}$)</th>
<th>MWSE(\theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\beta$ when double prior is ($\beta$, $\alpha, \theta$)</td>
<td>$15$ $25$ $50$ $100$ $15$ $25$ $50$ $100$</td>
</tr>
<tr>
<td>Bayes</td>
<td>$\theta$</td>
<td>$0.5$ $0.5$ $0.5$</td>
<td>$0.4889$</td>
</tr>
<tr>
<td></td>
<td>$\alpha$</td>
<td>$0.5$ $0.5$ $0.5$</td>
<td>$0.5089$</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>$1$ $0.5$ $1$</td>
<td>$0.5363$</td>
</tr>
<tr>
<td></td>
<td>$\gamma$</td>
<td>$1$ $0.5$ $1$</td>
<td>$0.5032$</td>
</tr>
</tbody>
</table>

| Bayes  | $\theta$   | $0.5$ $0.5$ $0.5$ | $0.5016$ | $0.5018$ | $0.5008$ | $0.4994$ | $0.00704$ | $0.00443$ | $0.00224$ | $0.00117$ |
|        | $\alpha$   | $0.5$ $0.5$ $0.5$ | $0.4872$ | $0.4827$ | $0.4962$ | $0.4947$ | $0.00696$ | $0.00437$ | $0.00222$ | $0.00117$ |
|        | $\beta$    | $1$ $0.5$ $1$ | $0.5398$ | $0.5252$ | $0.5106$ | $0.5059$ | $0.00852$ | $0.00546$ | $0.00251$ | $0.00121$ |
|        | $\gamma$   | $1$ $0.5$ $1$ | $0.5308$ | $0.5208$ | $0.5107$ | $0.5043$ | $0.00819$ | $0.00493$ | $0.00238$ | $0.00118$ |
|        | $\delta$   | $1$ $0.5$ $1$ | $0.5157$ | $0.5108$ | $0.5057$ | $0.5018$ | $0.00643$ | $0.00420$ | $0.00218$ | $0.00114$ |

A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation
## A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

Table 4-5: Shows the values for $\hat{\theta}$ under weighted square error loss function (MWSE).

| Method | $\theta$ | $a$ | $b$ | $c$ | $d$ | $(P(\hat{\theta}|\theta))$ when double prior is (SRIG - the natural conjugate family of priors) dist. | Sample Size | $\theta$ | Sample Size | $\theta$ |
|--------|---------|-----|-----|-----|-----|-----------------------------------------------|-------------|---------|-------------|---------|
| Bayes  | $\hat{\theta}$ | 5   | 0.5 | 8   | 0.5 | 0.85452, 0.926, 0.96039, 0.97727               | 15          | 25      | 50          | 100     |
|        |         | 8   | 0.5 | 5   | 0.5 | 0.91222, 0.94985, 0.9736, 0.98431               | 15          | 25      | 50          | 100     |
|        |         | 5   | 1   | 8   | 1   | 0.91241, 0.94388, 0.96971, 0.98215               | 15          | 25      | 50          | 100     |
|        |         | 8   | 1   | 5   | 1   | 0.9482, 0.96515, 0.95315, 0.95921                | 15          | 25      | 50          | 100     |
| Bayes  | $\hat{\theta}$ | 5   | 0.5 | 2   | 2   | 0.95661, 0.97508, 0.98733, 0.99146               | 15          | 25      | 50          | 100     |
|        |         | 5   | 0.5 | 3   | 2   | 0.94264, 0.96061, 0.98259, 0.98953               | 15          | 25      | 50          | 100     |
|        |         | 5   | 0.5 | 5   | 2   | 0.91665, 0.94894, 0.97332, 0.98425               | 15          | 25      | 50          | 100     |
|        |         | 5   | 1   | 2   | 2   | 0.97554, 0.98471, 0.99223, 0.99364               | 15          | 25      | 50          | 100     |
|        |         | 5   | 1   | 3   | 2   | 0.95834, 0.97564, 0.98747, 0.9915                | 15          | 25      | 50          | 100     |
|        |         | 5   | 1   | 5   | 2   | 0.93172, 0.95822, 0.97816, 0.98609               | 15          | 25      | 50          | 100     |

Table 4-6: Shows the values for $\hat{\theta}$ under weighted square error loss function (MWSE).

| Method | $\theta$ | $a$ | $b$ | $c$ | $d$ | $(P(\hat{\theta}|\theta))$ when double prior is (the non-informative) dist. | Sample Size | $\theta$ | Sample Size | $\theta$ |
|--------|---------|-----|-----|-----|-----|-----------------------------------------------|-------------|---------|-------------|---------|
| Bayes  | $\hat{\theta}$ | 5   | 0.5 | 0.5 | 0.5 | 1.0671, 1.0427, 1.0217, 1.0088               | 15          | 25      | 50          | 100     |
|        |         | 0.5 | 1   | 0.5 | 0.5 | 1.0584, 1.0376, 1.0192, 1.0076               | 15          | 25      | 50          | 100     |
|        |         | 0.5 | 0.5 | 0.5 | 0.5 | 1.1227, 1.1071, 1.0853, 1.0602               | 15          | 25      | 50          | 100     |
|        |         | 0.5 | 0.5 | 0.5 | 0.5 | 1.1037, 1.0658, 1.0337, 1.0149               | 15          | 25      | 50          | 100     |
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

5. Discussion
We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values for MSE as shown in tables (4-1 to 4-3) .
As we see in table (4-1), when the true value of \( \theta (\theta = 0.5) \) and the double prior distribution for \( \theta \) is
- SRIG - the natural conjugate family of priors distribution with \((a=5, b=0.5, \alpha=8, \beta=0.5)\).
- SRIG - the non-informative distribution with \((a=5, b=0.5, c=5)\).
- The natural conjugate family of priors-non-informative distribution with \((\alpha=0.5, \beta=5, c=1)\).
And in tables (4-2) and (4-3), when the true value of \( \theta (\theta = 1 & 1.5) \) and the double prior distribution for \( \theta \) is
- SRIG - the natural conjugate family of priors distribution with \((a=8, b=1, \alpha=5, \beta=1)\).
- SRIG - the non-informative distribution with \((a=5, b=1, c=2)\).
- The natural conjugate family of priors-non-informative distribution with \((\alpha=0.5, \beta=5, c=1)\).
Also, we obtained a good estimation according to the smallest values of MWSE for all samples sizes (n) comparative to the other estimated values for MWSE as shown in tables (4-4 to 4-6) .
In table (4-4), when the true value of \( \theta (\theta = 0.5) \) and the double prior distribution for \( \theta \) is
- SRIG - the natural conjugate family of priors distribution with \((a=5, b=0.5, \alpha=8, \beta=0.5)\).
- SRIG - the non-informative distribution with \((a=5, b=1, c=5)\).
- The natural conjugate family of priors-non-informative distribution with \((\alpha=0.5, \beta=5, c=1)\).
And in tables (4-5), when the true value of \( \theta (\theta = 1) \) and the double prior distribution for \( \theta \) is
- SRIG - the natural conjugate family of priors distribution with \((a=8, b=1, \alpha=5, \beta=1)\).
- SRIG - the non-informative distribution with \((a=5, b=1, c=2)\).
- The natural conjugate family of priors-non-informative distribution with \((\alpha=0.5, \beta=5, c=1)\).
And in tables (4-6), when the true value of \( \theta (\theta = 1.5) \) and the double prior distribution for \( \theta \) is
- SRIG - the natural conjugate family of priors distribution with \((a=8, b=1, \alpha=5, \beta=1)\).
- SRIG - the non-informative distribution with \((a=5, b=1, c=2)\).
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

- The natural conjugate family of priors-non-informative distribution with \((\alpha=0.5, \beta=5, c=1)\).

For more details see Appendix-E.

6. Conclusion

When we compared the estimated values for \(\theta\) the parameter of the Rayleigh distribution by using Bayes with respect to Mean Square Errors (MSE) and Mean weighted squared Errors (MWSE) respect to the smallest values of them. In general, we found best estimation for the parameter \(\theta\) for all sample sizes \((n)\), when the double prior distribution for \(\theta\) is SRIG - the natural conjugate family of priors distribution with values \((a=5, b=0.5, \alpha=8, \beta=0.5)\) and \((a=8, b=1, \alpha=5, \beta=1)\) for the true value of \(\theta\) \((\theta=0.5\text{ and }1)\) respectively. Also, we obtained the best estimation when the double prior distribution for \(\theta\) is the natural conjugate family of priors-non-informative distribution with values \((\alpha=0.5, \beta=5, c=1)\) for the true value of \(\theta\) \((\theta=1.5)\).

References

Appendix-A: The posterior distribution by using different double Priors.

1. The posterior distribution using SRIG - natural conjugate family of priors distribution as double prior:

To find the posterior distribution using the square root inverted gamma distribution (SRIG) - the natural conjugate family of priors distribution we follow these steps:

When the prior distribution of $\theta$ take SRIG distribution, then the p.d.f is given by:

$$f_1(\theta) = \frac{2b}{\Gamma} e^{(-2a+1)} \exp\left(-\frac{b}{\theta^2}\right) \quad \text{for} \quad a, b, \theta > 0 \quad \text{(A.1)}$$

Again, when prior distribution of $\theta$ take the natural conjugate family of priors distribution, then the p.d.f is given by:

$$f_2(\theta) = \frac{1}{\theta^{\alpha+1}} \exp\left(-\frac{\beta}{\theta}\right) \quad \text{for} \quad \theta, \alpha, \beta > 0 \quad \text{(A.2)}$$

We define the double prior for $\theta$ by combining these two priors as follows:

$$P_1(\theta) = f_1(\theta) \cdot f_2(\theta) \quad \text{(A.3)}$$

$$P_2(\theta) = \left(\frac{2b}{\Gamma}\right) e^{(-2a+1)} \exp\left(-\frac{b}{\theta^2}\right) \left[\frac{1}{\theta^{\alpha+1}} \exp\left(-\frac{\beta}{\theta}\right)\right] \quad \text{(A.4)}$$

Then the posterior distribution of $\theta$ for the given the data $t = (t_1, t_2, \ldots, t_n)$ is given by:

$$P(\theta \mid t) = \frac{L(t \mid \theta) \cdot P(\theta)}{\int L(t \mid \theta) \cdot P(\theta) \, d\theta} \quad \text{(A.5)}$$

Substituting the equation (5) and (A.5) in equation (A.6), we get:

$$P_1(\theta \mid t) = \prod_{i=1}^{n} \left[\frac{1}{\theta^{\alpha+1}} \exp\left(-\frac{\beta}{\theta}\right)\right] \exp\left(-\frac{\Sigma_{i=1}^{n} t_i^2}{2\theta^2}\right) \exp\left(-\frac{1}{\theta} \left(t_i^2 + b + \beta\right)\right)$$

$$P_2(\theta \mid t) = \prod_{i=1}^{n} \left[\frac{1}{\theta^{\alpha+1}} \exp\left(-\frac{\beta}{\theta}\right)\right] \exp\left(-\frac{\Sigma_{i=1}^{n} t_i^2}{2\theta^2}\right) \exp\left(-\frac{1}{\theta} \left(t_i^2 + b + \beta\right)\right)$$

$$P_1(\theta \mid t) = \frac{\prod_{i=1}^{n} \exp\left(-\frac{\Sigma_{i=1}^{n} t_i^2}{2}\right) \exp\left(-\frac{b}{\theta^2}\right) \exp\left(-\frac{1}{\theta} \left(t_i^2 + b + \beta\right)\right)}{\theta^{\alpha+1}}$$

We can rewrite $2a + 2n + \alpha + 2$ as $(2a + 2n + 0.5\alpha + 0.5 + 1)$, then by multiplying the integral in equation (A.8) by the quantity which equals to

$$\left(\frac{2(0.5\Sigma_{i=1}^{n} t_i^2 + b + \beta)}{\Gamma(a + n + 0.5\alpha + 0.5)}\right) \left(\frac{\Gamma(a + n + 0.5\alpha + 0.5)}{2(0.5\Sigma_{i=1}^{n} t_i^2 + b + \beta)}\right)$$

$\Gamma(.)$ is a gamma function. Then we get,
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

\[ P(\theta \mid t) = \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)}{\Gamma(a + n + 0.5\alpha + 0.5) A(t, \theta)} \exp\left(\frac{1}{\theta} (0.5\sum_{i=1}^{n} t_i^2 + b + \beta)\right) \]  \quad (A.9)

Where \( A(t, \theta) \) equals to

\[ A(t, \theta) = \int_0^{\infty} 2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta) \exp\left(\frac{1}{\theta} (0.5\sum_{i=1}^{n} t_i^2 + b + \beta)\right) d\theta = 1 \]

Be the integral of the pdf of SRIG distribution. Then we get the posterior distribution of \( \theta \) given the data \( t = (t_1, t_2, \ldots, t_n) \) is

\[ P(\theta \mid t) = \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)}{\Gamma(a + n + 0.5\alpha + 0.5)} \exp\left(\frac{1}{\theta} (0.5\sum_{i=1}^{n} t_i^2 + b + \beta)\right) \]

\[ \begin{align*}
a, b, \alpha, \beta, n \geq 0 \quad \text{(A.10)}
\end{align*} \]

It means that \( P(\theta \mid t) \sim \text{SRIG distribution with new parameters} \quad (a_{\text{new}} = a + n + 0.5\alpha + 0.5), b_{\text{new}} = (0.5\sum_{i=1}^{n} t_i^2 + b + \beta) \).

2- The posterior distribution using SRIG - non-informative distribution as double prior:

To find the posterior distribution using the SRIG - non-informative distribution, we follow these steps:

When the prior distribution of \( \theta \) take SRIG distribution, then the pdf is given by:

\[ f_{1}(\theta) = \frac{2b^2}{\Gamma a} \theta^{2a-1} \exp\left(-\frac{b}{\theta^2}\right) \quad \text{for} \quad a, b, \theta > 0 \]  \quad (A.1)

Again, when prior distribution of \( \theta \) take non-informative distribution, then the pdf is given by:

\[ f_{2}(\theta) = \frac{1}{\theta^C} \quad \text{for} \quad \theta, c > 0 \]  \quad (A.11)

We define the double prior for \( \theta \) by combining these two priors as follows:

\[ P_{e}(\theta) \propto f_{1}(\theta) f_{2}(\theta) \]  \quad (A.12)

\[ P_{1}(\theta) \propto \left[ \frac{2b^2}{\Gamma a} \theta^{2a-1} \exp\left(-\frac{b}{\theta^2}\right) \right] \left[ \frac{1}{\theta^C} \right] \]  \quad (A.13)

\[ P_{2}(\theta) \propto \left[ \frac{2b^2}{\Gamma a} \theta^{2a-1} \exp\left(-\frac{b}{\theta^2}\right) \right] \]  \quad for \( a, b, c, \theta > 0 \)  \quad (A.14)

Then the posterior distribution of \( \theta \) for the given the data \( t = (t_1, t_2, \ldots, t_n) \) is given by:

\[ P(\theta \mid t) = \frac{L(t \mid \theta) P(\theta)}{\int \theta L(t \mid \theta) P(\theta) d\theta} \]  \quad (A.6)

Substituting the equation (5) and (A.14) in equation (A.6), we get:

\[ \begin{align*}
P_{d}(\theta) = & \frac{\prod_{i=1}^{n} t_i}{\theta^{2n}} \exp\left(-\frac{\sum_{i=1}^{n} t_i^2}{2\theta^2}\right) \left[ \frac{2b^2}{\Gamma a} \theta^{2a-1} \exp\left(-\frac{b}{\theta^2}\right) \right] \quad \text{(A.15)}
\end{align*} \]
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

\[ P_2(\theta \mid \theta) = \frac{\theta^2 (2a + 2n + c + 1)}{a^{(a+n+0.5)c}} \exp\left[ -\frac{1}{\theta^2} \left( \frac{\sum_{i=1}^{n} t_i^2}{2} + b \right) \right] \]

... (A.16)

We can rewrite \((2a + 2n + c + 1)\) as \((2(a + n + 0.5c) + 1)\), then by multiplying the integral in equation (A.16) by the quantity which equals to

\[ \frac{\Gamma(a + n + 0.5c)}{2(0.5^a b_{n+0.5c})^a} \]

, where \(\Gamma(.)\) is a gamma function. Then we get,

\[ P_2(\theta \mid \theta) = \frac{\theta^2 (2a + 2n + c + 1)}{a^{(a+n+0.5)c} A_2 t_1^2 b_{n+0.5c}} \exp\left[ -\frac{1}{\theta^2} \left( \frac{\sum_{i=1}^{n} t_i^2}{2} + b \right) \right] \]

... (A.17)

Where \( A_2(t, \theta) \) equals to

\[ A_2(t, \theta) = \int_0^\infty \frac{\theta^2 (2a + 2n + c + 1)}{a^{(a+n+0.5)c}} \exp\left[ -\frac{1}{\theta^2} \left( \frac{\sum_{i=1}^{n} t_i^2}{2} + b \right) \right] d \theta = 1. \]

Be the integral of the pdf of Gamma distribution. Then we get the posterior distribution of \( \theta \mid \) given the data \( y = (t_1, t_2, ..., t_n) \)

\[ P_2(\theta \mid \theta) = \frac{\theta^2 (2a + 2n + c + 1)}{a^{(a+n+0.5)c}} \exp\left[ -\frac{1}{\theta^2} \left( \frac{\sum_{i=1}^{n} t_i^2}{2} + b \right) \right] \]

... (A.18)

It means that \( P_2(\theta \mid \theta) \sim \text{SRIG distribution with new parameters} \]

\[ \begin{align*}
\alpha_{\text{new}} &= (a + n + 0.5c), \\
b_{\text{new}} &= (0.5^a \sum_{i=1}^{n} t_i^2 + b)
\end{align*} \]

3- The posterior distribution using the natural conjugate family of priors- non-informative distribution as double prior:

To find the posterior distribution using the natural conjugate family of priors- non-informative distribution, we follow these steps:

When the prior distribution of \( \theta \) take the natural conjugate family of priors distribution, then the pdf is given by:

\[ f_2(\theta) = \frac{1}{\theta^a+1} \exp\left[ -\frac{\beta}{2 \theta^a+1} \right] \text{ for } \theta, a, \beta > 0 \]

... (A.2)

Again, when prior distribution of \( \theta \) take Erlang distribution, then the pdf is given by:

\[ f_1(\theta) = \frac{1}{\theta^a} \text{ for } \theta, c > 0 \]

... (A.11)

We define the double prior for \( \theta \) by combining these two priors as follows:

\[ P_3(\theta) \propto f_2(\theta) f_1(\theta) \]

... (A.19)

\[ P_3(\theta) \propto \left[ \frac{1}{\theta^a+1} \exp\left[ -\frac{\beta}{2 \theta^a+1} \right] \right] \frac{1}{\theta^a} \]

... (A.20)
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

\[ P(\theta | \mathbf{t}) = \frac{\alpha \theta^{-(\alpha+\beta+1)} \exp\left(-\frac{\mathbf{t}^2}{2\theta}\right)}{L(\mathbf{t})} \quad \text{for } \alpha, \beta, c, \theta > 0 \quad \ldots \text{(A.21)} \]

Then the posterior distribution of \( \theta \) for the given data \( \mathbf{t} = (t_1, t_2, \ldots, t_n) \) is given by:

\[ P(\theta | \mathbf{t}) = \frac{L(\mathbf{t} | \theta) P(\theta)}{\int_{\theta} L(\mathbf{t} | \theta) P(\theta) d\theta} \quad \ldots \text{(A.6)} \]

Substituting the equation (5) and (A.21) in equation (A.6), we get:

\[ P(\theta | \mathbf{t}) = \frac{\theta^{-2n+\alpha+c+1} \exp\left[-\frac{1}{\theta^2} \sum_{i=1}^{n} t_i^2 + \frac{\beta}{2}\right]}{\int_{0}^{\infty} \theta^{-2n+\alpha+c+1} \exp\left[-\frac{1}{\theta^2} \sum_{i=1}^{n} t_i^2 + \frac{\beta}{2}\right] d\theta} \quad \ldots \text{(A.22)} \]

We can rewrite \((2n+\alpha+c+1)\) as \((2(n+0.5\alpha+0.5c)+1)\), then by multiplying the integral in equation (A.22) by the quantity which equals to \(2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)\) and \(\Gamma(n+0.5\alpha+0.5c)\), we get:

\[ \frac{\Gamma(n+0.5\alpha+0.5c)}{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)} \]

function. Then we get:

\[ P(\theta | \mathbf{t}) = \frac{\theta^{-2(n+0.5\alpha+0.5c)} \exp\left[-\frac{1}{\theta^2} (0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)\right]}{\Gamma(n+0.5\alpha+0.5c)} \quad \text{A}(A.23) \]

Where \(A(\theta, \mathbf{t})\) equals to

\[ A(\theta, \mathbf{t}) = \int_{0}^{\infty} \theta^{-2(n+0.5\alpha+0.5c)} \exp\left[-\frac{1}{\theta^2} (0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)\right] d\theta = 1 \]

Be the integral of the pdf of Gamma distribution. Then we get the posterior distribution of \( \theta \) given the data \( \mathbf{t} = (t_1, t_2, \ldots, t_n) \) is

\[ P(\theta | \mathbf{t}) = \frac{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)}{\Gamma(n+0.5\alpha+0.5c)} \exp\left[-\frac{1}{\theta^2} (0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)\right] \]

\[ \ldots \text{(A.24)} \]

It means that \( P(\theta | \mathbf{t}) \sim \text{SRIG distribution with new parameters} \) \( (a_{new} = n+0.5\alpha+0.5c, b_{new} = (0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)) \).
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

The following is the derivation of these estimators under the squared error loss function.

The squared error loss function

To obtain the Bayes estimator, we minimize the posterior expected loss given by:

\[ L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \]

the risk function is:

\[ R(\hat{\theta}, \theta) = E[ L(\hat{\theta}, \theta) ] \]

\[ R(\hat{\theta}, \theta) = \int L(\hat{\theta}, \theta) P(\theta | x) \, d\theta \]

\[ R(\hat{\theta}, \theta) = \int (\hat{\theta} - \theta)^2 P(\theta | x) \, d\theta = R(\hat{\theta}, \theta) = (\hat{\theta}^2 - 2\hat{\theta} \theta + \theta^2) \int P(\theta | x) \, d\theta = \]

\[ R(\hat{\theta}, \theta) = \hat{\theta}^2 - 2 \hat{\theta} E(\theta | x) + E(\hat{\theta}^2 | x) \]

Let \( \hat{\theta} = R(\hat{\theta}, \theta) = 0 \), we get Bayes estimator of \( \theta \) denoted by \( \hat{\theta}_{Bayes} \) for the above prior as

\[ \hat{\theta}_{Bayes} = E(\theta | x) = \int \theta P(\theta | x) \, d\theta \]

\[ \hat{\theta}_{Bayes} = \int \theta P(\theta | x) \, d\theta , \; i = 1, 2, 3 \]

1. Bayes estimation using SRIG - natural conjugate family of priors distribution as double prior:

To obtain the Bayes' estimator under SRIG - natural conjugate family of priors distribution as double prior. Substituting the equation (A.10) in equation (B.4), we get:

\[ \hat{\theta}_{Bayes} = \int \theta P(\theta | x) \, d\theta , \; i = 1 \]

\[ \hat{\theta}_{Bayes} = \frac{2(0.5 \sum_{i=1}^{n} t_i^j + b + \beta)}{\Gamma(a + n + 0.5 \alpha + 0.5)} \exp[\frac{1}{\beta} (0.5 \sum_{i=1}^{n} t_i^j + b + \beta)] \]

\[ \hat{\theta}_{Bayes} = \frac{2(0.5 \sum_{i=1}^{n} t_i^j + b + \beta)}{\Gamma(a + n + 0.5 \alpha + 0.5)} \exp[\frac{1}{\beta} (0.5 \sum_{i=1}^{n} t_i^j + b + \beta)] \]
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

\[
\hat{\theta}_{11} = \int \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)}{\Gamma(a + n + 0.5\alpha + 0.5)} e^{-(2(\alpha+n+0.5c)\theta)} \exp\left[-\frac{1}{\theta^2} (0.5\sum_{i=1}^{n} t_i^2 + b + \beta) \right] d\theta \quad \ldots \ (B.7)
\]

By multiplying the integral in equation (B.7) by the quantity which equals to \( B_1 = \left( \frac{\Gamma(a+n+0.5\alpha)}{\Gamma(a+n+0.5\alpha+0.5)} \right) \), where \( \Gamma(\cdot) \) is a gamma function. Then we have

\[
\hat{\theta}_{11} - B_1 = \int \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)}{\Gamma(a + n + 0.5\alpha + 0.5)} e^{-(2(\alpha+n+0.5c)\theta)} \exp\left[-\frac{1}{\theta^2} (0.5\sum_{i=1}^{n} t_i^2 + b + \beta) \right] d\theta \quad \ldots \ (B.8)
\]

Then, we have

\[
\hat{\theta}_{11} = \frac{\Gamma(a+n+0.5\alpha)}{\Gamma(a+n+0.5\alpha+0.5)} \left( \frac{0.5\sum_{i=1}^{n} t_i^2 + b + \beta}{\Gamma(a+n+0.5\alpha+0.5)} \right)^{0.5} B_1(t,\theta) \quad \ldots \ (B.9)
\]

Where \( B_1(t,\theta) \) equals to

\[
B_1(t,\theta) = \int \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)}{\Gamma(a + n + 0.5\alpha)} e^{-(2(\alpha+n+0.5c)\theta)} \exp\left[-\frac{1}{\theta^2} (0.5\sum_{i=1}^{n} t_i^2 + b + \beta) \right] d\theta - 1
\]

Be the integral of the pdf of SRIG distribution. Then we get the Bayes estimator of \( \hat{\theta}_{11} \) as the following formula:

\[
\hat{\theta}_{11} = \frac{\Gamma(a+n+0.5\alpha)}{\Gamma(a+n+0.5\alpha+0.5)} \left( \frac{0.5\sum_{i=1}^{n} t_i^2 + b + \beta}{\Gamma(a+n+0.5\alpha+0.5)} \right)^{0.5} n, \alpha, \beta, b, a > 0 \quad \ldots \ (B.10)
\]

2. Bayes estimation using SRIG - non-informative distribution as double prior:

To obtain the Bayes' estimator under SRIG - non-informative distribution as double prior. Substituting the equation (A.18) in equation (B.4), we get:

\[
\hat{\theta}_{11} = \int \theta \ p (\theta | x) \ d\theta \quad , \quad \ldots \ (B.4)
\]

\[
\hat{\theta}_{11} = \int \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)}{\Gamma(a + n + 0.5\alpha)} e^{-(2(\alpha+n+0.5c)\theta)} \exp\left[-\frac{1}{\theta^2} (0.5\sum_{i=1}^{n} t_i^2 + b + \beta) \right] d\theta \quad \ldots \ (B.11)
\]

\[
\hat{\theta}_{11} = \int \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)}{\Gamma(a + n + 0.5\alpha)} e^{-(2(\alpha+n+0.5c+0.5)\theta)} \exp\left[-\frac{1}{\theta^2} (0.5\sum_{i=1}^{n} t_i^2 + b + \beta) \right] d\theta \quad \ldots \ (B.12)
\]

By multiplying the integral in equation (B.12) by the quantity which equals \( B_3 = \left( \frac{\Gamma(a+n+0.5\alpha+0.5)}{\Gamma(a+n+0.5\alpha+0.5+0.5)} \right) \), where \( \Gamma(\cdot) \) is a gamma function. Then we have
Appendix-B

The following is the derivation of these estimators under the squared error loss function.

The squared error loss function
To obtain the Bayes' estimators, we minimize the posterior expected loss given by:

\[ L_i( \hat{\theta}, \theta) = (\hat{\theta} - \theta)^2, \]

the risk function is:

\[ R(\hat{\theta} - \theta) = \mathbb{E}[L_i(\hat{\theta}, \theta)] \]

\[ R(\hat{\theta} - \theta) = \int L_i(\hat{\theta}, \theta) P(\theta | x) d\theta \]

\[ R(\hat{\theta} - \theta) = \int (\hat{\theta} - \theta)^2 P(\theta | x) d\theta \]

\[ R(\hat{\theta} - \theta) = \frac{1}{2} \int (\theta^2 - 2\theta \hat{\theta} + \hat{\theta}^2) P(\theta | x) d\theta \]

\[ R(\hat{\theta} - \theta) = \hat{\theta}^2 - 2\hat{\theta} E(\theta | x) + E(\theta^2 | x) \]

Let \( R(\hat{\theta} - \theta) = 0 \), we get Bayes estimator of \( \theta \) denoted by \( \hat{\theta}_{\text{Bayes}} \) for the above prior as

\[ \hat{\theta}_{\text{Bayes}} = E(\theta | x) - \int \theta P(\theta | x) d\theta \]

\[ \hat{\theta}_{\text{Bayes}} = \frac{1}{n} \sum_{i=1}^{n} P(\theta | x), \quad i = 1, 2, 3 \]

1. Bayes estimation using SRIG - natural conjugate family of priors distribution as double prior
To obtain the Bayes' estimator under SRIG - natural conjugate family of priors distribution as double prior. Substituting the equation (A.10) in equation (B.4), we get:

\[ \hat{\theta}_{\text{SRIG}} = \int \theta P(\theta | x) d\theta, \quad i = 1, 2, 3 \]

\[ \hat{\theta}_{\text{SRIG}} = \int \frac{2(0.5 \sum_{i=1}^{n} t_i^2 + b + p)}{\Gamma(a + n + 0.5a + 0.5)} g(2(0.5 \sum_{i=1}^{n} t_i^2 + b + p)) \exp\left[-\frac{1}{b^2}(0.5 \sum_{i=1}^{n} t_i^2 + b + p)\right] d\theta \]

\[ \hat{\theta}_{\text{SRIG}} = \frac{2(0.5 \sum_{i=1}^{n} t_i^2 - b + p)}{\Gamma(a + n + 0.5a + 0.5)} g(2(0.5 \sum_{i=1}^{n} t_i^2 - b + p)) \exp\left[-\frac{1}{b^2}(0.5 \sum_{i=1}^{n} t_i^2 - b + p)\right] d\theta \]
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

\[ \hat{\theta}_{\text{max}} = \int_{0}^{\infty} \frac{2(0.5 \sum_{i=1}^{n} t_{i}^{2} + b + \beta)}{\Gamma(a + n + 0.5a + 0.5)} \exp\left[-\frac{1}{\theta^{2}}(0.5 \sum_{i=1}^{n} t_{i}^{2} + b + \beta)\right] d\theta \quad \ldots \quad (B.7) \]

By multiplying the integral in equation (B.7) by the quantity which equals to \( B_{1} = \frac{\Gamma(a + n + 0.5a)}{\Gamma(a + n + 0.5a)} \), where \( \Gamma(.) \) is a gamma function. Then we have

\[ \hat{\theta}_{\text{max}} = B_{1} \int_{0}^{\infty} \frac{2(0.5 \sum_{i=1}^{n} t_{i}^{2} + b + \beta)}{\Gamma(a + n + 0.5a + 0.5)} \exp\left[-\frac{1}{\theta^{2}}(0.5 \sum_{i=1}^{n} t_{i}^{2} + b + \beta)\right] d\theta \quad \ldots \quad (B.8) \]

Then, we have

\[ \hat{\theta}_{\text{max}} = \frac{\Gamma(a + n + 0.5a)}{\Gamma(a + n + 0.5a + 0.5)}(0.5 \sum_{i=1}^{n} t_{i}^{2} + b + \beta)^{0.5} B_{2}(t, \theta) \quad \ldots \quad (B.9) \]

Where \( B_{2}(t, \theta) \) equals to

\[ B_{2}(t, \theta) = \int_{0}^{\infty} \frac{2(0.5 \sum_{i=1}^{n} t_{i}^{2} + b + \beta)}{\Gamma(a + n + 0.5a + 0.5)} \exp\left[-\frac{1}{\theta^{2}}(0.5 \sum_{i=1}^{n} t_{i}^{2} + b + \beta)\right] d\theta = 1 \]

Be the integral of the pdf of SRIG distribution. Then we get the Bayes estimator of \( \hat{\theta}_{\text{max}} \), as the following formula:

\[ \hat{\theta}_{\text{max}} = \frac{\Gamma(a + n + 0.5a)}{\Gamma(a + n + 0.5a + 0.5)}(0.5 \sum_{i=1}^{n} t_{i}^{2} + b + \beta)^{0.5} n, a, b, a > 0 \quad \ldots \quad (B.10) \]

2. Bayes estimation using SRIG - non-informative distribution as double prior:

To obtain the Bayes’ estimator under SRIG - non-informative distribution as double prior. Substituting the equation (A.18) in equation (B.4), we get:

\[ \hat{\theta}_{\text{max}} = \int_{0}^{\infty} \theta P_{\text{SRIG}}(\theta | x) d\theta \quad , \quad i = 2 \quad \ldots \quad (B.4) \]

\[ \hat{\theta}_{\text{max}} = \int_{0}^{\infty} \frac{2(0.5 \sum_{i=1}^{n} t_{i}^{2} + b)}{\Gamma(a + n + 0.5a)} \exp\left[-\frac{1}{\theta^{2}}(0.5 \sum_{i=1}^{n} t_{i}^{2} + b)\right] d\theta \quad \ldots \quad (B.11) \]

\[ \hat{\theta}_{\text{max}} = \int_{0}^{\infty} \frac{2(0.5 \sum_{i=1}^{n} t_{i}^{2} + b)}{\Gamma(a + n + 0.5a)} \exp\left[-\frac{1}{\theta^{2}}(0.5 \sum_{i=1}^{n} t_{i}^{2} + b)\right] d\theta \quad \ldots \quad (B.12) \]

By multiplying the integral in equation (B.12) by the quantity which equals \( B_{3} = \frac{\Gamma(a + n + 0.5a - 0.5)}{\Gamma(a + n + 0.5a - 0.5)} \), where \( \Gamma(.) \) is a gamma function. Then we have
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

\[
\hat{\theta}_{\text{MSE}} = \frac{2(0.5 \Sigma_{i=1}^{n} t_i^2 + b)}{\Gamma(n + 0.5c)} \exp\left[\frac{1}{\theta} (0.5 \Sigma_{i=1}^{n} t_i^2 + b)\right] d\theta
\] ...

Then, we have

\[
\hat{\theta}_{\text{MSE}} = \frac{\Gamma(n + 0.5c - 0.5)}{\Gamma(n + 0.5c)} \left(0.5 \Sigma_{i=1}^{n} t_i^2 + b\right)^{0.5} \text{Be}(n, \theta)
\] ...

Where \( \text{Be}(t, \theta) \) equals to

\[
\text{Be}(t, \theta) = \int_{0}^{t} \frac{2(0.5 \Sigma_{i=1}^{n} t_i^2 + b)}{\Gamma(n + 0.5c - 0.5)} \exp\left[\frac{1}{\theta} (0.5 \Sigma_{i=1}^{n} t_i^2 + b)\right] d\theta = 1
\]

Be the integral of the pdf of SRIG distribution. Then we get the Bayes estimator of \( \hat{\theta}_{\text{MSE}} \) as the following formula:

\[
\hat{\theta}_{\text{MSE}} = \frac{\Gamma(n + 0.5c - 0.5)}{\Gamma(n + 0.5c)} \left(0.5 \Sigma_{i=1}^{n} t_i^2 + b\right)^{0.5} \text{Be}(t, \theta) \quad \text{for} \quad a, b, c > 0
\] ...

3. Bayes estimation using the natural conjugate family of prior distributions:

To obtain the Bayes estimator under the natural conjugate family of prior distributions as double prior. Substituting the equation (A.24) in equation (B.4), we get:

\[
\hat{\theta}_{\text{n}}, \quad \int_{0}^{\infty} \theta^n \exp\left[-\frac{1}{\theta} (0.5 \Sigma_{i=1}^{n} t_i^2 + 0.5b)\right] d\theta
\]

\[
\hat{\theta}_{\text{n}}, \quad \int_{0}^{\infty} \theta^n \exp\left[-\frac{1}{\theta} (0.5 \Sigma_{i=1}^{n} t_i^2 + 0.5b)\right] d\theta
\]

\[
\hat{\theta}_{\text{n}}, \quad \int_{0}^{\infty} \theta^n \exp\left[-\frac{1}{\theta} (0.5 \Sigma_{i=1}^{n} t_i^2 + 0.5b)\right] d\theta
\]

By multiplying the integral in equation (B.17) by the quantity which equals to

\[
\text{Be} = \left(\frac{\Gamma(n + 0.5c - 0.5)}{\Gamma(n + 0.5c)}\right), \text{ where } \Gamma(.) \text{ is a gamma function. Then we have}
\]

\[
\hat{\theta}_{\text{n}}, \quad \int_{0}^{\infty} \theta^n \exp\left[-\frac{1}{\theta} (0.5 \Sigma_{i=1}^{n} t_i^2 + 0.5b)\right] d\theta
\]

\[
\hat{\theta}_{\text{n}}, \quad \int_{0}^{\infty} \theta^n \exp\left[-\frac{1}{\theta} (0.5 \Sigma_{i=1}^{n} t_i^2 + 0.5b)\right] d\theta
\]

\[
\hat{\theta}_{\text{n}}, \quad \int_{0}^{\infty} \theta^n \exp\left[-\frac{1}{\theta} (0.5 \Sigma_{i=1}^{n} t_i^2 + 0.5b)\right] d\theta
\]

Then, we have

\[
\hat{\theta}_{\text{n}}, \quad \int_{0}^{\infty} \theta^n \exp\left[-\frac{1}{\theta} (0.5 \Sigma_{i=1}^{n} t_i^2 + 0.5b)\right] d\theta
\]

\[
\hat{\theta}_{\text{n}}, \quad \int_{0}^{\infty} \theta^n \exp\left[-\frac{1}{\theta} (0.5 \Sigma_{i=1}^{n} t_i^2 + 0.5b)\right] d\theta
\]

Where \( \text{Be}(t, \theta) \) equals to

\[
\text{Be}(t, \theta) = \int_{0}^{t} \frac{2(0.5 \Sigma_{i=1}^{n} t_i^2 + b)}{\Gamma(n + 0.5c - 0.5)} \exp\left[-\frac{1}{\theta} (0.5 \Sigma_{i=1}^{n} t_i^2 + 0.5b)\right] d\theta = 1
\]

Be the integral of the pdf of SRIG distribution. Then, we get the Bayes estimator of \( \hat{\theta}_{\text{MSE}} \) as the following formula:

\[
\hat{\theta}_{\text{n}}, \quad \int_{0}^{\infty} \theta^n \exp\left[-\frac{1}{\theta} (0.5 \Sigma_{i=1}^{n} t_i^2 + 0.5b)\right] d\theta
\]
Appendix-C: The following is the derivation of these estimators under the weighted squared error loss function.

1. The weighted squared error loss function
   To obtain the Bayes estimator, we minimize the posterior expected loss given by:

   \[ \mathbb{R}_i \left( \frac{\hat{\theta} - \theta}{\sigma} \right) = E[\mathbb{L}_i(\hat{\theta}, \theta)] \quad \ldots \quad (C.1) \]

   \[ \mathbb{R}_i \left( \frac{\hat{\theta} - \theta}{\sigma} \right) = \int \mathbb{L}_i(\hat{\theta}, \theta) P(\theta \mid t) \, d\theta = \mathbb{R}_i \left( \frac{\hat{\theta} - \theta}{\sigma} \right) = \int \left( \frac{\hat{\theta} - \theta}{\sigma} \right)^2 P(\theta \mid t) \, d\theta \]

   \[ \mathbb{R}_i \left( \frac{\hat{\theta} - \theta}{\sigma} \right) = \int \left( \frac{\hat{\theta}^2 - 2\hat{\theta} \theta + \theta^2}{\sigma^2} \right) P(\theta \mid t) \, d\theta \]

   \[ \mathbb{R}_i \left( \frac{\hat{\theta} - \theta}{\sigma} \right) = \int \left( \frac{\hat{\theta}^2 - 2\hat{\theta} \theta + \theta^2}{\sigma^2} \right) P(\theta \mid t) \, d\theta \]

   \[ \mathbb{R}_i \left( \frac{\hat{\theta} - \theta}{\sigma} \right) = \hat{\theta} \int \frac{1}{\sigma} \, P(\theta \mid t) \, d\theta = \hat{\theta} \int \frac{1}{\sigma} \, P(\theta \mid t) \, d\theta \]

   \[ \mathbb{R}_i \left( \frac{\hat{\theta} - \theta}{\sigma} \right) = \hat{\theta} \int \frac{1}{\sigma} \, P(\theta \mid t) \, d\theta \]

   Let \( \mathbb{R}_i \left( \frac{\hat{\theta} - \theta}{\sigma} \right) = 0 \), we get Bayes estimator of \( \theta \) denoted by \( \hat{\theta}_{Bayes} \) for the above prior

   \[ \hat{\theta}_{Bayes} = \frac{1}{\hat{\theta}} \int \frac{1}{\sigma} \, P(\theta \mid t) \, d\theta \quad \ldots \quad (C.2) \]

   \[ \hat{\theta}_{Bayes} = \frac{1}{\hat{\theta}} \int \frac{1}{\sigma} \, P(\theta \mid t) \, d\theta \quad \ldots \quad (C.3) \]

2. Bayes estimation using SEIG - natural conjugate family of priors distribution as double prior:
   To obtain the Bayes estimator under SEIG - natural conjugate family of priors distribution as double prior. Substituting the equation (A.10) in the integral in equation (C.3), we get:

   \[ \mathbb{E} \left( \frac{1}{\sigma} \theta \mid t \right) = \frac{1}{\int \frac{1}{\sigma} \, P(\theta \mid t) \, d\theta} \]
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

\[
E_\theta \left[ \frac{1}{\theta} \right] = \frac{2(0.5\theta^n + \frac{1}{\alpha} + b + \beta)}{\Gamma(\alpha + n + 0.5\alpha + 0.5)} e^{\left(\frac{1}{\theta}(0.5\sum t_i^2 + b + \beta)\right)} \text{d}\theta 
\]

... (C.4)

\[
E_\theta \left[ \frac{1}{\theta} \right] = \frac{2(0.5\theta^n + \frac{1}{\alpha} + b + \beta)}{\Gamma(\alpha + n + 0.5\alpha + 0.5)} e^{\left(\frac{1}{\theta}(0.5\sum t_i^2 + b + \beta)\right)} \text{d}\theta 
\]

... (C.5)

\[
E_\theta \left[ \frac{1}{\theta} \right] = \frac{2(0.5\theta^n + \frac{1}{\alpha} + b + \beta)}{\Gamma(\alpha + n + 0.5\alpha + 0.5)} e^{\left(\frac{1}{\theta}(0.5\sum t_i^2 + b + \beta)\right)} \text{d}\theta 
\]

... (C.6)

By multiplying the integral in equation (C.6) by the quantity which equals to \(C_1 = \left(\frac{\Gamma(a + n + 0.5\alpha + 1)}{\Gamma(a + n + 0.5\alpha + 0.5)}\right)\), where \(\Gamma(\cdot)\) is a gamma function. Then we have

\[
E_\theta \left[ \frac{1}{\theta} \right] = C_1 \left(\frac{2(0.5\theta^n + \frac{1}{\alpha} + b + \beta)}{\Gamma(\alpha + n + 0.5\alpha + 0.5)} e^{\left(\frac{1}{\theta}(0.5\sum t_i^2 + b + \beta)\right)} \text{d}\theta \right) 
\]

... (C.9)

Then, we have

\[
E_\theta \left[ \frac{1}{\theta} \right] = \frac{\Gamma(a + n + 0.5\alpha + 1)}{\Gamma(a + n + 0.5\alpha + 0.5)} e^{\left(\frac{1}{\theta}(0.5\sum t_i^2 + b + \beta)\right)} \text{d}\theta 
\]

... (C.10)

Where \(C_2(t, \theta)\) equals to

\[
C_2(t, \theta) = \frac{2(0.5\theta^n + \frac{1}{\alpha} + b + \beta)}{\Gamma(\alpha + n + 0.5\alpha + 0.5)} e^{\left(\frac{1}{\theta}(0.5\sum t_i^2 + b + \beta)\right)} \text{d}\theta = 1.
\]

Be the integral of the pdf of SRIG distribution. Then we get the Bayes estimator of \(\theta\) as the following formula:

\[
E_\theta \left[ \frac{1}{\theta} \right] = \frac{\Gamma(a + n + 0.5\alpha + 1)}{\Gamma(a + n + 0.5\alpha + 0.5)} e^{\left(\frac{1}{\theta}(0.5\sum t_i^2 + b + \beta)\right)} \text{d}\theta
\]

... (C.11)

Substituting the equation (C.11) in equation (C.3), we get:

\[
\hat{\theta}_{WSE} = \frac{\Gamma(a + n + 0.5\alpha + 1)}{\Gamma(a + n + 0.5\alpha + 0.5)} e^{\left(\frac{1}{\theta}(0.5\sum t_i^2 + b + \beta)\right)} \text{d}\theta
\]

... (C.12)

2. Bayes estimation using SRIG - non-informative distribution as double prior:

To obtain the Bayes' estimator under SRIG - non-informative distribution as double prior. Substituting the equation (A.18) in the integral in equation (C.5), we get:

\[
E_\theta \left[ \frac{1}{\theta} \right] = \frac{1}{\theta} \text{P}_{\theta}(\theta) \text{d}\theta
\]

prior distribution for \(\theta\) with MSE. The same thing we can do it to compute Bayes estimators \(\hat{\theta}_{WSE}\).
Appendix-D: The following is the programs algorithm.

Algorithm (1): To compute Bayes estimators ($\hat{θ}_{SE_1}$) using SRIG - the natural conjugate family of priors Distribution as double prior distribution for $θ$ with MSE.

Start

Given values for the parameter $θ$ and sample size $n$, and ($a$, $b$) and ($α$, $β$) are the parameters for SRIG - the natural conjugate family of priors distribution and the number of replication ($r=1000$) for each sample size ($n$).

For $i=1$ to $r$

Generated data from Rayleigh distribution using Matlab

Compute $\hat{θ}_{SE_1}(i)$ using formula in table (4), also compute $MSE(\hat{θ}_{SE_1}(i))$ using formula (11) for $r=1$.

IF ($i \geq r$)

No

Yes

Compute the mean for $\hat{θ}_{SE_1}(i)$ for all $r=1000$, also compute the mean for $MSE(\hat{θ}_{SE_1}(i))$ for all $r=1000$.

Stop
A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

Note (1): we can reformulate the Algorithm (1) to compute Bayes estimators \( \hat{\theta}_{SEk}, k = 2,3 \) using other distributions as double

\[
\hat{\theta}_{SEk} = \frac{\sum_{i=1}^{n} t_i^2 + 0.5\beta}{\Gamma(n + 0.5\alpha + 0.5\beta)} \exp\left[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)\right] d\theta
\]

... (C.19)

\[
\hat{\theta}_{SEk} = \frac{\sum_{i=1}^{n} t_i^2 + 0.5\beta}{\Gamma(n + 0.5\alpha + 0.5\beta)} \exp\left[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)\right] d\theta
\]

... (C.20)

By multiplying the integral in equation (C.20) by the quantity which equals to

\[
C_5 = \frac{\Gamma(n + 0.5\alpha + 0.5\beta)}{\Gamma(n + 0.5\alpha + 0.5\beta + 0.5)}
\]

Then we have

\[
\hat{\theta}_{SEk} = \frac{\sum_{i=1}^{n} t_i^2 + 0.5\beta}{\Gamma(n + 0.5\alpha + 0.5\beta + 0.5)} \exp\left[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)\right] d\theta
\]

... (C.21)

Then, we have

\[
\hat{\theta}_{SEk} = \frac{\Gamma(n + 0.5\alpha + 0.5\beta + 0.5)}{\Gamma(n + 0.5\alpha + 0.5\beta)} \exp\left[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)\right] d\theta = 1.
\]

Where \( B_5(t, \theta) \) equals to

\[
B_5(t, \theta) = \frac{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)}{\Gamma(n + 0.5\alpha + 0.5\beta + 0.5)} \exp\left[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)\right] d\theta
\]

The integral of the pdf of SRIG distribution. Then we get the Bayes estimator of \( \theta \) as the following formula:

\[
\hat{\theta}_{SEk} = \frac{\sum_{i=1}^{n} t_i^2 + 0.5\beta}{\Gamma(n + 0.5\alpha + 0.5\beta + 0.5)} \exp\left[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)\right] d\theta
\]

... (C.22)

Substituting the equation (C.23) in equation (C.3), we get:

\[
\hat{\theta}_{SEk} = \frac{\sum_{i=1}^{n} t_i^2 + 0.5\beta}{\Gamma(n + 0.5\alpha + 0.5\beta + 0.5)} \exp\left[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)\right] d\theta
\]

... (C.24)
Appendix-E: The summarized and tabulated discussions and conclusions.

Table 5-1: Best Estimation according to the smallest value for $\text{MSE}(\hat{\theta})$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>The best estimation according to smallest value for $\text{MSE}(\hat{\theta})$ when the double prior distribution is</th>
<th>$\text{MSE}(\hat{\theta})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>(SRIG - the natural conjugate family of priors) $\text{dist}^a$, with $(a=5, b=0.5, \alpha=8, \beta=0.5)$</td>
<td>0.00276</td>
</tr>
<tr>
<td></td>
<td>(SRIG - the non-informative) $\text{dist}^a$, with $(a=5, b=0.5, c=5)$</td>
<td>0.00345</td>
</tr>
<tr>
<td></td>
<td>(the natural conjugate family of priors non-informative) $\text{dist}^a$, with $(a=0.5, \beta=5, c=1)$</td>
<td>0.02358</td>
</tr>
<tr>
<td>1</td>
<td>(SRIG - the natural conjugate family of priors) $\text{dist}^a$, with $(a=8, b=1, \alpha=5, \beta=1)$</td>
<td>0.01443</td>
</tr>
<tr>
<td></td>
<td>(SRIG - the non-informative) $\text{dist}^a$, with $(a=8, b=1, c=2)$</td>
<td>0.01559</td>
</tr>
<tr>
<td></td>
<td>(the natural conjugate family of priors non-informative) $\text{dist}^a$, with $(a=0.5, \beta=5, c=1)$</td>
<td>0.02103</td>
</tr>
<tr>
<td>1.5</td>
<td>(SRIG - the natural conjugate family of priors) $\text{dist}^a$, with $(a=8, b=1, \alpha=5, \beta=1)$</td>
<td>0.04622</td>
</tr>
<tr>
<td></td>
<td>(SRIG - the non-informative) $\text{dist}^a$, with $(a=8, b=1, c=2)$</td>
<td>0.03949</td>
</tr>
<tr>
<td></td>
<td>(the natural conjugate family of priors non-informative) $\text{dist}^a$, with $(a=0.5, \beta=5, c=1)$</td>
<td>0.03957</td>
</tr>
</tbody>
</table>

Table 5-2: Best Estimation according to the smallest value for $\text{MWSE}(\hat{\theta})$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>The best estimation according to smallest value for $\text{MWSE}(\hat{\theta})$ when the double prior distribution is</th>
<th>$\text{MWSE}(\hat{\theta})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>(SRIG - the natural conjugate family of priors) $\text{dist}^a$, with $(a=5, b=0.5, \alpha=8, \beta=0.5)$</td>
<td>0.00558</td>
</tr>
<tr>
<td></td>
<td>(SRIG - the non-informative) $\text{dist}^a$, with $(a=5, b=0.5, c=5)$</td>
<td>0.00643</td>
</tr>
<tr>
<td></td>
<td>(the natural conjugate family of priors non-informative) $\text{dist}^a$, with $(a=0.5, \beta=5, c=1)$</td>
<td>0.04115</td>
</tr>
<tr>
<td>1</td>
<td>(SRIG - the natural conjugate family of priors) $\text{dist}^a$, with $(a=8, b=1, \alpha=5, \beta=1)$</td>
<td>0.01526</td>
</tr>
<tr>
<td></td>
<td>(SRIG - the non-informative) $\text{dist}^a$, with $(a=8, b=1, c=2)$</td>
<td>0.01573</td>
</tr>
<tr>
<td></td>
<td>(the natural conjugate family of priors non-informative) $\text{dist}^a$, with $(a=0.5, \beta=5, c=1)$</td>
<td>0.01818</td>
</tr>
<tr>
<td>1.5</td>
<td>(SRIG - the natural conjugate family of priors) $\text{dist}^a$, with $(a=8, b=1, \alpha=5, \beta=1)$</td>
<td>0.03352</td>
</tr>
<tr>
<td></td>
<td>(SRIG - the non-informative) $\text{dist}^a$, with $(a=8, b=1, c=2)$</td>
<td>0.02782</td>
</tr>
<tr>
<td></td>
<td>(the natural conjugate family of priors non-informative) $\text{dist}^a$, with $(a=0.5, \beta=5, c=1)$</td>
<td>0.02442</td>
</tr>
</tbody>
</table>
مقارنة لمقدرات بيز لمعتلة توزيع رايلي باستعمال المحاكاة

د. جنان عباس ناصر العبدي - استاذ مساعد
الجامعة التقنية الإدارية - بغداد

المستخلص
مقارنة للتوزيعات الأولية المضاعفة مفترضة لممعتلة توزيع رايلي. فقد تم افتراض ثلاثة توزيعات أولية مضاعفة: توزيع الجذر التربيعي لمعكوس كاما (SRIG) - التوزيع غير المعلوماتي، توزيع العائلة المراقبة الطبيعية - التوزيع غير المعلوماتي، كدوال معلوماتية مضاعفة. البيانات ولدت من ثلاثة حالات لتوزيع رايلي لحجاب مختلفة من العينات (الصغيرة والمتوسطة والكبيرة). وتم إشتقاق مقدار بيز للممعتلة باستعمال حالة الخسارة التربوية والدالة التربيعية الموزونة مع ثلاثة دوال معلوماتية مضاعفة. استعملت المحاكاة للحصول على نتائج هذا البحث. وتحديد أفضل مقدر وفقاً لأقل قيمة لمعيار متوسط مربعات الخطأ ومتوسط مربعات الخطأ الموزونة. وجدنا أن أفضل تقدير للممعتلة (0) لكل حجوم العينات. عندما يكون التوزيع الأولي المضاعف توزيع الجذر التربيعي لمعكوس كاما (SRIG) - العائلة المراقبة الطبيعية عند القيم (a = 5, b = 0.5, α = 8, β = 0.5) أو (a = 8, b = 1, α = 5, β = 1) لقيم المحمولة الفعلية للممعتلة (1). حصلنا على أفضل تقدير لـ (0) عند يكون التوزيع الأولي المضاعف توزيع العائلة المراقبة الطبيعية - التوزيع غير المعلوماتي عند القيم (α = 0.5, β = 5, c = 1) لقيم المحمولة الفعلية (1.5).

المصطلحات الرئيسية: البحث، توزيع رايلي، طريقة بيز، التوزيع الأولي معلوماتية والغير المعلوماتية، المضاعفة، التوزيع الاحتمالي، حالة الخسارة التربوية، دالة الخسارة التربوية الموزونة.