Best approximation in metric space for contractive mapping of integral type and Applications of fixed point to Approximation theory

Sabah H. Malih
College of Education (Ibn – AL – Haitham)
University of Baghdad

Abstract:
We obtain generalization of best approximation for contractive condition of integral type and we next consider two results on invariant approximation also analyze the existence the application of fixed point and extend some known results of Habiniak [7], Sahab [10], Kumar [5] Branciari [1], Singh [12].

1. Introduction and preliminaries
The purpose of this paper is to study and analogous the invariant best approximation in the setting contractive condition of integral inequality in metric space and analyze the existence the application of fixed point. The generalization of applications on best approximation obtaining fixed point also common fixed point. Recently Sahab [10] have obtained some results on approximation theory in the setting contraction mapping without integral type. We prove the results on best approximation theory for mappings satisfying a general contractive condition of integral type. These results unify and extend some results in Habiniak [7], Sahab [10], Kumar [5] Branciari [1].

The following definition and results will be needed:
Let C be an nonempty subset of a metric space \(X, \delta\). A mapping \(T: C \to C\) is a compact mapping [4] if for every bounded subset \(K\) of \(C\), \(\overline{T(K)}\) (closure of \(T(K)\)) is compact. The restriction \(T_C: C \to X\) where \(T_C(x) = T(x)\) for all \(x \in C\). A mapping \(T: C \to C\) is called contraction mapping [3] if \(d(T_x, T_y) \leq \lambda d(x, y)\) for all \(x, y \in C\), \(\lambda \in (0, 1)\). If \(\lambda = 1\) then \(T\) is called non expansive mapping. Let \(T, f: C \to C\) be a mapping \(T\) is called \(f\) contraction [6], [11] if \(d(T_x, T_y) \leq \lambda d(f_x, f_y)\) for all \(x, y \in C\).
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\[ x \in [0, 1) \] if \( x = 1 \), then \( T \) is called \( f \)-nonexpansive mapping. If \( T(C) \subseteq C \) then \( C \) is called a \( T \)- invariant subset of \( X \).

**Definition 1.1** : [8], [4]

- Let \( C \) be a subset of a metric space \( (X, d) \), for any \( x \in X \) we denote
- \( d(x, C) = \inf \{d(x, y) : y \in C\} \)
- ii- If any \( y \in C : d(x, y) = d(x, C) = \inf \{d(x, z) : z \in C\} \) where \( x \in X \) is called \( y \) approximation to \( x \) by \( C \).
- iii- \( P_e(x) = \{y \in C : d(x, y) = d(x, C)\} \) such that \( P_e(x) \) is called the set of all best approximation of \( x \) from \( C \).
- iii- If for each \( x \in X \), \( P_e(x) \) is nonempty, then \( C \) is called proximinal. \( P_e(x) \) is bounded subset of \( X \) also if \( C \) is closed then \( P_e(x) \) is closed.

**Definition 1.2** : [8]

Let \( f \) and \( T \) be two self maps on a set \( X \). mapps \( f \) and \( T \) are said to be commuting if \( fTx = Tfx \) for all \( x \in X \). If \( fx = Tx \) for some \( x \in X \) then \( x \) is called coincidence point of \( f \) and \( T \).

**Definition 1.3** : [13]

Let \( (X, d) \) be a metric space and \( C \subseteq X \) and \( f, T : C \to C \) be a mappings, then :
- i- \( T \) and \( f \) are called compatible if \( \lim_{n \to \infty} d(f_{Tn}, fTx) = 0 \) for all \( n \) and \( \lim_{n \to \infty} (Tfx_n, fTnx) = 0 \), whenever \( (x_n) \) is a sequence such that: \( \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} f_{Tn} = t \) for some \( t \) in \( C \).
- ii- \( f \) and \( T \) are called weakly compatible if they commute at there coincidence points (i.e.) \( Tfx = fTx \) whenever \( Tx = fx \).

**Remark 1.4** : [2]

- i- Every compatible is weakly compatible but the converse is not true.
- ii- Every commute mappings is compatible mapping but the converse is not true.

**Definition 1.5** : [9]

Let \( C \) be a subset of metric space \( (X, d) \) and \( \Delta = \{f_\alpha \} : \alpha \in C \) a family of functions from \([0, 1]\) into \( C \) such that \( f_\alpha(0) = \alpha \), for each \( \alpha \in C \). The family \( \Delta \) is said to be contractive if whenever there exists a function \( \Psi : (0, 1) \to (0, 1) \) such that for all \( \alpha, \beta \in C \) and \( s \in (0, 1) \) we have:

\[ d(f_\alpha(s), f_\beta(s)) \leq \Psi(s)d(\alpha, \beta) \]

The family is said to be jointly continuous if \( s \to s_0 \) in \([0, 1]\) and \( \alpha \to \alpha_0 \) in \( C \) imply that \( f_\alpha(s) \to f_{\alpha_0}(s) \) in \( C \).

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Hence property (w) on contractive jointly continuous family \( \Delta \) can now be defined as:

**Definition 1.6 : [9]**

Let \( T \) be a self map of the set \( C \) having a family of function \( \Delta = \{ f_x : x \in C \} \) as defined above then \( T \) is said to satisfy the property (w) if

\[
T(f_x(s)) = f_{T_x}(s), \quad \text{for all } x \in C \text{ and } s \in [0, 1].
\]

**Definition 1.7 : [1]**

Let \( (X, d) \) be a metric space, \( T : X \to X \) is called general contractive inequality of integral type if there exist a real number \( \lambda \in (0, 1) \) such that for each \( x, y \in X \), we have

\[
\int_0^d d(x, y)\, \phi(t)\, dt \leq \lambda \int_0^d \phi(t)\, dt
\]

where \( \phi : [0, +\infty) \to [0, +\infty) \) is a Lebesgue integrable mapping which is summable on each compact subset of \( [0, +\infty) \) nonnegative and such that for each \( \epsilon > 0 \),

\[
\int_0^\epsilon \phi(t)\, dt > 0.
\]

The generalization of definition 1.5 by

\[
\int_0^d d(x, y)\, \psi(t)\, dt \leq \int_0^d \phi(t)\, dt.
\]

The following result would also be used in the sequel:

**Theorem 1.8 : [5]**

Let \( T \) and \( f \) be compatible self maps of a complete metric space \( (X, d) \) satisfying the following conditions: \( T(X) = f(X) \), \( f \) is continuous

\[
\int_0^d d(T(y, x))\, \phi(t)\, dt \leq \lambda \int_0^d \phi(t)\, dt
\]

for each \( x, y \in X \), \( \lambda \in (0, 1) \), where \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue – integrable function, which is summable on each compact subset of \( \mathbb{R}^+ \), non-negative, and such that for each \( \epsilon > 0 \),

\[
\int_0^\epsilon \phi(t)\, dt > 0.
\]

Then \( T \) and \( f \) have a unique common fixed point.

2- Main Results

In this section we introduce two results best approximation and one result of fixed point we need the result fixed point the following.

**Theorem 2.1**
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Let T and f be weakly compatible self maps of a subset X of metric space (X, d) satisfying the following condition: \( \overline{T(X)} \subset f(X) \),

\[ d(Tx, Ty) \leq \lambda \int_0^1 \Theta(t) dt \leq \lambda \int_0^1 \Theta(t) dt \]

for each \( x, y \in X \), \( \lambda \in [0, 1) \), where \( \Theta: [0, +\infty) \rightarrow [0, +\infty) \) is a Lebesgue-integrable function which is summable on each compact subset \([0, +\infty)\) non-negative,

and such that for each \( \varepsilon > 0 \), \( \int_0^\varepsilon \Theta(t) dt > 0 \)

Then T and f have a unique common fixed point

Proof:

Since \( T(X) \subset f(X) \), let \( x_0 \in X \) such that \( f_{x_0} = T_{x_0} \), \( x_1 \in X \)

In general \( y_n = f_{x_{n+1}} = T_{x_n} \), \( n = 0, 1, 2, 3,... \)

Now

\[ \int_0^1 d(Tx_n, Tx_{n+1}) \Theta(t) dt \leq \lambda \int_0^1 d(Tx_n, Tx_{n+1}) \Theta(t) dt \leq \lambda \int_0^1 d(Tx_{n-1}, Tx_{n}) \Theta(t) dt \]

\[ \leq \lambda \int_0^1 d(Tx_{n-1}, Tx_{n}) \Theta(t) dt \]

\[ = \lambda \int_0^1 d(Tx_{n-2}, Tx_{n}) \Theta(t) dt \]

\[ = \lambda^n \int_0^1 d(Tx_0, Tx_n) \Theta(t) dt \rightarrow 0 \]

Therefore \( \int_0^1 d(Tx_n, Tx_{n+1}) \Theta(t) dt \rightarrow 0 \) as \( n \to \infty \) hence \( d(Tx_n, Tx_{n+1}) \to 0 \)

We now to show that \( (Tx_n) \) is a Cauchy sequence suppose that \( (Tx_n) \) is not Cauchy sequence then there exist an \( \varepsilon > 0 \) and a sub sequence \( (m(v)_p) \) and \( (n(v)_p) \) such that for each positive integer \( p \); \( n(v)_p \) is minimal in the sense that

\[ d(Tx_m(v)_p, Tx_n(v)_p) \geq \varepsilon, \quad d(Tx_m(v)_p, Tx_n(v)_{p-1}) < \varepsilon \]

If \( d(Tx_m(v)_{p+1}, Tx_n(v)_{p+1}) \geq \varepsilon \) then:

\[ d(Tx_m(v)_{p+1}, Tx_n(v)_{p+1}) \leq d(Tx_m(v)_{p+1}, Tx_m(v)_p) + d(Tx_m(v)_p, Tx_n(v)_{p+1}) \]

Thus : \( d(Tx_m(v)_{p+1}, Tx_n(v)_{p+1}) \leq d(Tx_m(v)_{p+1}, Tx_m(v)_p) + \varepsilon \)

Since \( d(Tx_{n+1}, Tx_n) \to 0 \) then \( d(Tx_m(v)_{p+1}, Tx_n(v)_{p+1}) \to 0 \) as \( p \to \infty \) also

\[ \int_0^1 \Theta(t) dt \leq \int_0^1 d(Tx_m(v)_p, Tx_n(v)_p) \Theta(t) dt \leq \lambda \int_0^1 d(Tx_m(v)_p, Tx_n(v)_p) \Theta(t) dt \]

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\[
\int_0^\infty \left( \tau_{\chi_{3,m-1}}(x) \tau_{\chi_{3,m-1}}(x) \right)^2 \varphi(t) \, dt
\]

When \( p \to \infty \) then

\[
\int_0^\infty \varphi(t) \, dt \leq \lim_{p \to \infty} \int_0^\infty \left( \tau_{\chi_{3,m-1}}(x) \tau_{\chi_{3,m-1}}(x) \right)^2 \varphi(t) \, dt
\]

Which is contradiction thus:

\[
\int_0^\infty \varphi(t) \, dt \leq \int_0^\infty \left( \tau_{\chi_{3,m-1}}(x) \tau_{\chi_{3,m-1}}(x) \right)^2 \varphi(t) \, dt
\]

Which is contradiction thus \( \{ \chi_{n+1} \} \) is a Cauchy sequence. By definition of \( \{ \chi_{n+1} \} \) is also Cauchy sequence. Since \( f(X) \subset \overline{f(X)} \) is complete suppose that \( \overline{f(X)} \) is complete then \( \chi_{n+1} = u \in \overline{f(X)} \) and by definition of \( \{ \chi_{n+1} \} \) obtain \( f\chi_{n+1} = u \). Since \( u \in \overline{f(X)} = f(X) \), then \( v \in X \) such that \( u = T_v \)

Also \( u = f_z \) \( z \in X \)

Now: if \( n \to \infty \)

\[
\int_0^\infty \varphi(t) \, dt \leq \int_0^\infty \left( \tau_{\chi_{3,m-1}}(x) \tau_{\chi_{3,m-1}}(x) \right)^2 \varphi(t) \, dt
\]

Thus \( \tau_{\chi_{n+1}} \to T \tau = u = f_z \)

As \( \{ T \} \) is weakly compatible and \( T \tau = f_z \) then

\( T \tau = T \tau = f \tau = f \), therefore \( T \tau = f \tau \)

\[
\int_0^\infty \varphi(t) \, dt = \int_0^\infty \left( \tau_{\chi_{3,m-1}}(x) \tau_{\chi_{3,m-1}}(x) \right)^2 \varphi(t) \, dt = \int_0^\infty \left( \tau_{\chi_{3,m-1}}(x) \tau_{\chi_{3,m-1}}(x) \right)^2 \varphi(t) \, dt
\]

Then \( \int_0^\infty \varphi(t) \, dt \leq \int_0^\infty \left( \tau_{\chi_{3,m-1}}(x) \tau_{\chi_{3,m-1}}(x) \right)^2 \varphi(t) \, dt \)

Hence \( T \tau = u \)

Thus \( T \tau = u = f_z \)

If \( s \in C \) such that \( s = T \tau = f \), then

\[
\int_0^\infty \varphi(t) \, dt = \int_0^\infty \left( \tau_{\chi_{3,m-1}}(x) \tau_{\chi_{3,m-1}}(x) \right)^2 \varphi(t) \, dt
\]

Then \( \int_0^\infty \varphi(t) \, dt \leq \int_0^\infty \left( \tau_{\chi_{3,m-1}}(x) \tau_{\chi_{3,m-1}}(x) \right)^2 \varphi(t) \, dt \) Hence \( s = u \)

There for the unique common fixed point.

Theorem 2.2:
Let $(X,d)$ be a metric space and $T:X \to X$ with fixed point $u \in X$
 satisfying
$$\int_0^1 \varrho(t)dt \leq \frac{1}{2} \int_0^\infty \varrho(t)dt,$$
where $\varrho:(0,\infty) \to [0,\infty)$ is a Lebesgue-integrable function which
assumable on each compact subset of $(0,\infty)$, nonnegative, and such that
for each $\varepsilon > 0$: $\int_0^\infty \varrho(t)dt > \varepsilon$. If $C$ closed $T$
 invariant subset of $X$, further
the restriction $T|C$ is a compact, then the set $P_\varepsilon(u)$ of best a
approximation is nonempty.

**Proof:**

Let $x = d(u,C)$, $x > 0$ then there exist a minimizing sequence $(y_n)$ in
$C$ such that $d(x,y_n) = x$.

Since $(y_n)$ is bounded sequence. Since $T|C$ is compact
Then $(Ty_n)$ is a compact subset of $C$ and so $(Ty_n)$ has a convergent sub
sequence $(Ty_{n_k} : k \geq 1)$ with $\lim_{n \to \infty} Ty_{n_k} = x$ where $x \in C$

Now:

If $d(x,C) = x = \lim_{n \to \infty} d(y_n, x)$, then
$$\int_0^1 \varrho(t)dt = \lim_{n \to \infty} \int_0^1 \varrho(t)dt$$

Such that $\varepsilon > 0$ then $\int_0^r \varrho(t)dt = r, \quad r > 0$

Suppose that $d(x,C) > x$, now: either $d(u,x) = x$, then
$$d(x,C) = d(x,C) = x; \quad \int_0^r \varrho(t)dt = r$$

or $d(u,x) > x$, then:

$$r < \int_0^1 \varrho(t)dt = \lim_{n \to \infty} \int_0^1 \varrho(t)dt \leq \lim_{n \to \infty} \int_0^1 \varrho(t)dt$$

Therefore $r < \int_0^r \varrho(t)dt < r$ hence $d(u,x) = x$.

Thus $d(x,C) = x = d(x,C)$ then $x$ is best approximation to $u$ by $C$
Therefore $P_\varepsilon(u)$ is nonempty.

**Theorem 2.3:**

Let $(X,d)$ be a metric space, $X$ has family $\Delta$ and $T : X \to X$ be two
mappings and $C \subseteq X$ such that $C$ be a $T$-invariant subset of $X$ ;
$x_0 \in F(T) \cap F(g)$. Let $g$ satisfies property (w), $T_0 = gT$ on $D = P_\varepsilon(x_0)$. If $D$
is nonempty and $g(D) \subseteq D$ and $T \circ g$ satisfying

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\[ \int_0^\lambda \phi(t) \, dt \leq \lambda \int_0^\lambda \phi(t) \, dt, \quad \lambda \in (0, 1) \]

for each \( x, y \in D = D \cup \{ x_0 \} \). Where \( \phi(t) : [0, +\infty) \to [0, +\infty) \) is a Lebesgue-integrable function which is summable, nonnegative and such that for each \( \varepsilon > 0 \), \( \int_0^\varepsilon \phi(t) \, dt > 0 \) then

i- If \( \mathbb{T}(D) \) is compact and \( T \) is continuous and \( g(D) = D \), \( D \) is closed then \( D \cap F(T) \cap F(g) = \emptyset \)

ii- If \( D \) is compact and \( g \) is continuous also the range of \( f_n \) is contained in \( g(D) \) then \( D \cap F(T) \cap F(g) = \emptyset \)

iii- If \( \mathbb{T}(D) \) is compact on \( D \) and \( g \) is continuous \( g(D) = D \), \( D \) is closed then \( D \cap F(T) \cap F(g) = \emptyset \)

**Proof:**

Let \( y \in D \) then \( \exists y' \in D \) since \( g(D) \subseteq D \). Further \( y \in C \) and then \( Ty \in C \) since \( C \) be \( T \)-invariant subset of \( X \)

\[ d(x_0, Ty') = d(x_0, C) = \varepsilon > 0 ; \quad \int_0^\varepsilon \phi(t) \, dt = r > 0 \]

let suppose that \( d(x_0, Ty) \geq \varepsilon \)

Now : either \( d(x_0, Ty) = \varepsilon \) then \( Ty \in D \)

or \( d(x_0, Ty) > \varepsilon \) , then

\[ r \leq \int_0^\varepsilon \phi(t) \, dt < \varepsilon \] hence \( d(x_0, Ty) = \varepsilon \)

Therefore

Hence \( Ty \in D \), then \( T : D \to D \). Choose \( \varepsilon_n \in (0, 1) \)

Such that \( \varepsilon_n \to 1 \), as \( n \to +\infty \), then define \( T_n \) as \( \mathbb{T}_n(x) = f_{Ty} (\varepsilon_n) \)

For all \( x \in D \) \( T_n \) is well defined map from \( D \) into \( D \) for each \( n \) and \( T_n(D) \subseteq D \).

Now \( T \) commutes with \( g \) and \( g \) satisfying property(w) then

\[ \mathbb{T}_n(gx) = g(f_{Ty} (\varepsilon_n)) = g(T_n(x)) \]

Thus \( T_n g = g T_n \) for all \( n \in N \) and for all \( x \in D \) , therefore

\[ \int_0^\varepsilon \phi(t) \, dt = \int_0^\varepsilon \phi(t) \, dt \leq \Psi(x) \int_0^\varepsilon \phi(t) \, dt \]

\[ \int_0^\varepsilon \phi(t) \, dt = \int_0^\varepsilon \phi(t) \, dt \leq \Psi(x) \int_0^\varepsilon \phi(t) \, dt \]

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\[ \int_0^\varepsilon \phi(t) \, dt = \int_0^\varepsilon \phi(t) \, dt \leq \Psi(x) \int_0^\varepsilon \phi(t) \, dt \]
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\[ \leq \mathcal{P}(t) \int_0^t \phi(x,y) \, dt \]

For all \( x, y \in D \)

\[ \int_0^{d(T_n x, T_n y)} \phi(t) \, dt \leq \mathcal{P}(t) \int_0^{d(x,y)} \phi(t) \, dt \]

Thus \( \int_0^{d(T_n x, T_n y)} \phi(t) \, dt \leq \mathcal{P}(t) \int_0^{d(x,y)} \phi(t) \, dt \) for all \( x, y \in D \)

Since \( T_n \) commutes with \( \mathcal{P} \) then \( T_n \) and \( \mathcal{P} \) are weakly compatible. If (i) \( T(D) \) is compact on \( D \), then \( T_n(D) \) is compact also \( T_n(D) \) is complete and \( T_n(D) \subseteq D = g(D) \); also \( D \) is closed \( \overline{T_n(D)} \subseteq D = g(D) \) thus \( \overline{T_n(D)} \subseteq \overline{g(D)} \) then by theorem (2.1) \( T_n \) and \( \mathcal{P} \) have common fixed point, \( x_n = T_n x_n = g x_n \) for all \( n \in \mathbb{N} \) and \( x_n \in D \).

Since \( T(D) \) is compact then the sequence \( \{ T_n(x) \} \) converging to \( y \in \overline{T(D)} \subseteq D = g(D) \) then \( y = g y \) where \( y \in D \) and \( x_n = T_n x_n = g x_n \).

Now : \( x_n = g x_n = T_n x_n = f_{T_n}(x_n) \rightarrow f_y(1) = y \) as \( n \rightarrow \infty \)

Thus \( x_n \rightarrow y \) and \( T_n x_n \rightarrow Ty \) by \( T \) is continuous then \( Ty = y \).

Also :

\[ \int_0^{d(T_n x, T_n y)} \phi(t) \, dt \leq \int_0^{d(x,y)} \phi(t) \, dt \]

\[ = \int_0^{d(x,y)} \phi(t) \, dt \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \]

Hence \( \int_0^{d(T_n x, T_n y)} \phi(t) \, dt \rightarrow 0 \) then \( T_n x_n \rightarrow Ty \)

Therefore \( Ty = g y \).

Hence \( y = Ty = g y \), then \( D \cap F(T) \cap F(g) = \emptyset \)

If (ii) \( (D) \) is compact and \( \mathcal{P} \) is continuous and range of \( f_n \) is contained in \( g(D) \) then \( T_n(D) \subseteq \overline{g(D)} \).

Since \( T_n \) commute with \( \mathcal{P} \) then \( (T_n, \mathcal{P}) \) are compatible then by theorem (1.8) \( x_n = T_n x_n = g x_n \)

Where \( x_n \in D \), but \( D \) is compact then \( \{ x_n \} \) has a subsequence \( \{ x_{n_k} \} \) converging to \( y \in D \)

Now : \( x_{n_k} = T_{n_k} x_{n_k} = g x_{n_k} \rightarrow y \) as \( n_k \rightarrow \infty \)

Since \( \mathcal{P} \) is continuous \( \mathcal{P} x_{n_k} \rightarrow \mathcal{P} y \) then \( \mathcal{P} x_{n_k} \rightarrow \mathcal{P} y \); but \( \mathcal{P} x_{n_k} \rightarrow y \)

then \( \mathcal{P} y = y \), also

\[ \int_0^{d(T_{n_k} x, T_{n_k} y)} \phi(t) \, dt \leq \int_0^{d(x,y)} \phi(t) \, dt \rightarrow 0 \quad \text{as} \quad n_k \rightarrow \infty \]

Thus :
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\[ \int_0^1 \phi(t)dt \to 0 \text{ then } Tx_n \to Ty \text{ as } n \to \infty \]

Now:
\[ x_n = T x_n x_n = I_{T x_n} (x_n) \to I_{T y} (y) = Ty \]

Therefore \( x_n \to Ty \) and \( x_n \to y \) then \( Ty = y \) thus \( Ty = gy = y \in D \)

Hence \( D \cap F(T) \cap F(g) \neq \emptyset \)

\( \text{iii-} \frac{d}{d(D)} \text{ is compact and } D \text{ is closed also } g(D) = D \text{, } g \text{ is continuous, then by} \)

prove part (i) if \( \frac{d}{d(D)} \) is compact we have
\[ x_n = T x_n x_n = g x_n \to y \text{ as } n \to \infty \]

Since \( g \) is continuous then \( g x_n \to gy \text{ also } g x_n \to y \text{ and } x_n \to y \)

Hence \( g y = y \), also \( g x_n \to y \text{ and } x_n \to y \)

Therefore
\[ \int_0^1 \phi(t)dt \to 0 \text{ then } Tx_n \to Ty \text{ as } n \to \infty \]

Now
\[ x_n = T x_n x_n = I_{T x_n} (x_n) \to I_{T y} (y) = Ty \text{ as } n \to \infty \]

Thus \( x_n \to Ty \) then \( Ty = y = gy \text{ and } y \notin g(D) = D \)

Then \( D \cap F(T) \cap F(g) \neq \emptyset \)

**Corollary 2.4:**

Let \((X, d)\) be a metric space, \( X \) having family \( \sigma \), let \( T : X \to X \) is continuous, \( C \subseteq X \) such that \( T(C) \subseteq C \) and \( x_0 \in F(T) \) if \( D \) nonempty, compact

\[ \int_0^1 \phi(t)dt \leq 1 \int_0^1 \phi(t)dt \text{ for all } \]

\[ x, y \in D', D' \cup \{x_0\}, \text{ where } \phi(x) : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is Lebesgue–integrable function} \]

which is summable, nonnegative and \( \int_0^1 \phi(t)dt > 0 \) then

\( D \cap F(T) \neq \emptyset \)

**Proof:**

If \( g \) is identity mapping in theorem (2.3) then \( D \cap F(T) \neq \emptyset \)

**References:**

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Best approximation in metric space for contractive mapping of integral type and Applications of fixed point to Approximation theory.............................................................. Sabah H. Malih