Best Approximation in Modular Spaces By Type of Nonexpansive Maps

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Abstract
This paper presents results about the existence of best approximations via nonexpansive type maps defined on modular spaces.

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1. Introduction and Preliminaries

Modular spaces are extensions of Lebesgue, Riesz, and Orlicz spaces of integrable functions [1]. A general theory of modular linear spaces was founded by Nakano 1950 [2]. Nakano’s modulars on real linear spaces are convex functionals. Nonconvex modulars and the corresponding modular linear spaces were constructed by Musielak and Orlicz (we refer to [2]). In 2006, Vyacheslav Chistyakov [3, 4] was introduced the concept of a metric modular on a set, inspired partly by the classical linear modulars on function spaces employed by Nakano and other in the sense of Chistyakov.

In the formulation given by Kowalskowskii[5], "a modular on a linear space $\mathcal{V}$ over the field $\mathcal{K}(= \mathbb{R}$ or $\mathbb{C}$) is a function $m: \mathcal{V} \to [0, \infty)$ such that

(i) $m(x) = 0 \iff x = 0$;
(ii) $m(\alpha x) = m(x)$ for $\alpha \in \mathcal{K}$ with $|\alpha| = 1$, for all $x \in \mathcal{V}$;
(iii) $m(\alpha x + \beta y) \leq m(x) + m(y)$ such that $\alpha, \beta \geq 0$, for all $x, y \in \mathcal{V}$.

Moreover, modular $m$ is called convex, if (iii) replaced by
(iii') $m(\alpha x + \beta y) \leq \alpha m(x) + \beta m(y)$ if $\alpha, \beta \geq 0, \alpha + \beta = 1$ for all $x, y \in \mathcal{V}.$"

"A sequence $\{v_n\} \subset \mathcal{V}$ is said to be $\gamma$-convergent to $v \in \mathcal{V}$ and write $v_n \to v$ if $m(v_n - v) \to 0$ as $n \to \infty$. A sequence $\{v_n\}$ is called Cauchy whenever $m(v_n - v_m) \to 0$ as $m,n \to \infty$. Also, $\mathcal{V}$ is called complete if any Cauchy sequence in $\mathcal{V}$ is convergent. A subset $B \subset \mathcal{V}$ is called closed if for any sequence $\{v_n\} \subset B$, convergent to $v \in \mathcal{V}$, we have $v \in B$" [6].

"A closed subset $B \subset \mathcal{V}$ is called compact if any sequence $\{v_n\} \subset B$ has a convergent subsequence" [7].

"A selfmap $J$ on $B \subset \mathcal{V}$ is called contraction mapping if $\exists h \in (0,1)$ for all $v, u$ in $\mathcal{V}$, $m(J(v) - J(u)) \leq h m(v - u)$ and if $h = 1$ then $J$ is called a non-expansive mapping" [7].

"A map $J$ is demi-continuous at $0$ if $\{v_n\} \subset B, v_n$ converges weakly to $v, w_n \in J(v_n)$ and $w_n \to 0 \Rightarrow 0 \in J(v)$.

$\mathcal{V}$ is said to be Opial if for every sequence $\{v_n\}$ in $\mathcal{V}$ weakly convergent to $v \in \mathcal{V}$ the inequality

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\[ \lim_{n \to \infty} \inf (v_n - v) < \lim_{n \to \infty} \inf (v_n - u) \]

holds for all \( u \neq v \) [7].

"Let \( V \) and \( W \) be two modular spaces, recall that a set-valued mapping \( f: V \to W \) is a subset of \( V \times W \) with domain \( \mathfrak{F} \); equivalently, \( f \) is a point to set mapping assigning to each \( u \in V \) a nonempty subset \( f(u) \) of \( W \).

let \( v \in V \), \( v \) is called a fixed point of \( S \) if \( v \in f(v) \) (when \( S \) is single valued, \( v \) is fixed point of \( S \) if \( v = f(v) \) A set-valued mapping \( f \) is upper semi continuous (shortly, u.s.c.) if and only if the set \( \{ u \in V : f(u) \cap B \neq \emptyset \} \) is closed for each closed subset \( B \) of \( W \)." See [8].

"Consider \( f \neq \emptyset \subset V \), the element \( y \in B \) is a best approximation for a given \( x \in V \); if

\[ m(x - y) = d_m(x, B) = \inf_{y \in B} m(x - y) \]

and \( P_B(x) \) or \( Px \) the set of all elements of best approximation of \( x \) by \( B \).

A subset \( B \) is called Chebysev if \( \forall x \in V, \exists ! y \in B \) such that

\[ m(x - y) = d_m(x, B). \]

Main Results.

First we start with the following definition:

**Definition 1:** A multivalued map \( f: B \to 2^B \) is called \(*\)-nonexpansive if \( \forall x, y \in B \) and \( a_x \in f(x) \) with

\[ m(x - a_x) = \sigma(x, f(x)), \]

\[ \exists a_y \in f(y) \text{ with } m(y - a_y) = \sigma(y, f(y)) \]

\[ \exists m(a_x - a_y) \leq m(x - y). \]

**Remark (2)** The concept of \(*\)-nonexpansive map coincides with a nonexpansive for a single valued map. Thus we have the result shown in [10].

Define \(*\)-nonexpansive map \( K: B \to 2^B \) by

\[ K(x) = \bigcup \{ P_y(y) : y \in f(x), \sigma(f(x), B) = \sigma(y, B) \} \]

For the first result, fix \( C(B) \) as the class of all nonempty compact subsets of \( B \) and \( b \)-starshaped mean starshaped with starcenter at \( b \). Then we have the following

**Theorem 2:** let \( B \) be a nonempty weakly compact \( b \)-starshaped subset of complete convex modular spaces \( V, K \) as in (1) and \( f: B \to C(B) \) is u.s.c such that \( 3x_0 \in B, \exists x_0 \in f(x_0), m(a_{x_0}) < \infty \). If \( \forall x, K(x) \) is compact Chebyshev and \( I - K \) is demi-closed at 0 then \( \exists z \in B \exists \sigma(z, f(z)) = \sigma(f(z), B) \).

**Proof:**

The compactness of \( f(x), \forall x \) implies that \( K(x) \neq \emptyset \). Since \( K(x) \) is Chebyshev so by definition of \(*\)-nonexpansive, \( a_x \in K(x) \) is unique and \( \exists ! a_y \in K(y), \forall y \in B \exists \)

\[ m(a_x - a_y) \leq m(x - y) \]

Let \( f_n: B \to B \) such that \( f_n(x) = \theta_n a_x + (1 - \theta_n) b \), where \( 0 < \theta_n < 1, \forall n \) and \( \theta_n \to 1 \) as \( n \to \infty \).

By convexity of \( V \) and (2), we have \( \forall x, y \in B, \)

\[ m(f_n(x) - y) \leq \theta_n m(x - y). \]

So, \( \forall n, f_n \) is contraction and hence, by [6], has a fixed point \( z_n \in B \), the sequence \( \{ z_n \} \) has a subsequence, also say\( \{ z_n \} \), converging weakly to \( z \in B \). By definition of \( f_n, \exists a_n \in K(z_n) \exists \)

\[ z_n = f_n(z_n) = \theta_n a_n + (1 - \theta_n) b \]

And then

\[ y_n = a_n - z_n = (1 - \theta_n) (a_n - b) \to 0 \text{ as } n \to \infty \]

Since \( I - K \) is demi-closed at 0, the sequence \( \{ z_n \} \) converges weakly to \( z, y_n \to 0 \) where \( y_n = a_n - z_n \in K(z_n) - z_n \). Thus \( 0 \in (I - K)(z) \Rightarrow z \in K(z) \).

Therefore, for some \( w \in f(z) \) with

\[ m(f(z)) = \sigma(w, B), z \in P(w). \]

We have

\[ \sigma(z, f(z)) \leq m(z - w) = \sigma(w, B) = \sigma(f(z), B) \leq \sigma(z, f(z)) \]

\[ \Rightarrow \sigma(z, f(z)) = \sigma(f(z), B) \]

The proof is complete.

Now, we state the definition of weak nonexpansive map (shortly, called \( w \)-nonexpansive map)

**Definition 3:** A multivalued mapping \( f: B \to 2^B \) is called \( w \)-nonexpansive if \( \forall x \in B, a_x \in f(x) \) there is \( a_y \in f(y), \forall y \in B \exists m(a_x - a_y) \leq m(x - y) \).

**Theorem 4:** The result of Theorem (2) also hold if \( V \) satisfies Opial's condition instead of demi-closeness.
Proof: Since the *-nonexpansive mapping $K$ is weakly nonexpansive. So, $\forall n, a_n \in K(x_n), \exists b_n \in K(z) such that$

\begin{equation}
m(a_n - b_n) \leq m(x_n - z)
\end{equation}

As $K(z)$ is compact so $\langle b_n \rangle$ converges to some $u \in K(z)$. Combination of (4) with $b_n \to 0 \text{ and } z_n \to u \Rightarrow$

$$\liminf m(x_n - b_n) = \liminf m(x_n - u) \leq \liminf m(x_n - z)$$

By Opial's condition, we have

$$\liminf m(x_n - z) < \liminf m(x_n - u).$$

Thus $z = u \in K(z)$. Therefore, the final step of proof follows from previous argument.

About invariant best approximation we prove the following result

**Theorem (5):** Let $B$ be a closed subspace of a convex modular space $V$ and $J: B \to V$ be a continuous map. If $Pf:B \to B$ is linear nonexpansive map such that $\exists u_0 \in B$ with $(Pf)^2(u_0) - 2(Pf)(u_0) + u_0 = 0$ then $m(u_0 - J(u_0)) = \sigma(J(u_0), B)$. Moreover, if $J(u_0) \in B$, then $J$ has a fixed point.

**Proof:**

let $K = Pf$ then $K; B \to B$ is linear nonexpansive $\exists (K)^2(u_0) - 2(K)(u_0) + u_0 = 0$

From linearity of $K$, we have

$$(K - I)(K - I)(u_0) = 0$$

Let $K(u_0) = u$

$$\Rightarrow (K - I)(u) = 0 \Rightarrow (K - I)(u) = u.$$ 

$$\Rightarrow K(u_0) = u_0 + u \Rightarrow K^n(u_0) = nu, \forall n \geq 1.$$ 

Consider $nm(u) = m(K^n(u_0) - u_0)$

$$\leq m(K^n(u_0) - K(0)) + m(u_0)$$ 

Hence, $m(u) \leq \frac{2m(u_0)}{m}$, $\forall n \geq 1$. As $n \to \infty$, we get $u = 0 \Rightarrow K(u_0) = u_0$. Therefore, $(Pf)(u_0) = u_0 \Rightarrow m(u_0 - J(u_0)) = \sigma(J(u_0), B)$ done.

**Open problem**

Consider $J: B \to V$, where $B$ is convex set $J$ is midpoint concave (or convex) map if

$$\frac{1}{2} J(x) + \frac{1}{2} J(y) \subseteq J(\frac{x+y}{2}), \forall x, y \in B.$$ 

$$(or, J(\frac{x+y}{2}) \subseteq \frac{1}{2} J(x) + \frac{1}{2} J(y))$$ respectively. Is there $u_0 \in B \exists m(u_0 - J(u_0)) = \sigma(J(u_0), B)$?

**References**